Uniform Boundedness for Brauer Groups of K3 Surfaces

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Motivation: torsion on elliptic curves

Theorem (Merel, 1996)

Fix $d \in \mathbb{Z}_{>0}$. There is an integer $c = c(d)$ such that:
For all number fields k with $[k:\mathbb{Q}] = d$ and all elliptic curves E/k,

$$\#E(k)_{\mathrm{tors}} < c.$$

Question

Is there a Merel theorem for surfaces?

Elliptic curves: $\omega_E \simeq \mathscr{O}_E$.

Look at nice surfaces X with $\omega_X \simeq \mathscr{O}_X$.

Nice surfaces with $\omega_X \simeq \mathscr{O}_X$

Nice: smooth, projective, geometrically integral.

Two kinds:

- Geometrically abelian surfaces: $h^1(X, \mathscr{O}_X) = 2$.
- K3 surfaces: $h^1(X, \mathscr{O}_X) = 0$, e.g.
  1. $w^2 = x^6 + y^6 + z^6$ in $\mathbb{P}(1,1,1,3)$ (degree 2).
  2. $x^4 + y^4 = z^4 + w^4$ in $\mathbb{P}^3$ (degree 4).

Problem: K3 surfaces have no group structure.

Replacement for $E(k)_{\text{tors}}$?

Reinterpreting $E(k)_{\mathrm{tors}}$

$$
\begin{aligned}
E(k)_{\mathrm{tors}} &\simeq (\operatorname{Pic}^0 E)_{\mathrm{tors}} \\
&= (\operatorname{Pic} E)_{\mathrm{tors}} \\
&\simeq \mathrm{H}^1(E, \mathscr{O}_E^\times)_{\mathrm{tors}} \\
&\simeq \mathrm{H}^1_{\mathrm{et}}(E, \mathbb{G}_m)_{\mathrm{tors}} \\
&\simeq \operatorname{im}\left(\mathrm{H}^1_{\mathrm{et}}(E, \mathbb{G}_m)_{\mathrm{tors}} \to \mathrm{H}^1_{\mathrm{et}}(\overline{E}, \mathbb{G}_m)_{\mathrm{tors}}\right)
\end{aligned}
$$

Note: Hilbert 90 implies

$$
\ker\left(\mathrm{H}^1_{\mathrm{et}}(E, \mathbb{G}_m)_{\mathrm{tors}} \to \mathrm{H}^1_{\mathrm{et}}(\overline{E}, \mathbb{G}_m)_{\mathrm{tors}}\right) \simeq \mathrm{H}^1_{\mathrm{et}}(\operatorname{Spec} k, \mathbb{G}_m) = 0.
$$

Transcendental Brauer groups

For a K3 surface over a number field k, use

$$\mathrm{im}\left(\underbrace{\mathrm{H}^2_{\mathrm{et}}(X,\mathbb{G}_m)_{\mathrm{tors}}}_{\mathrm{Br}(X)} \to \underbrace{\mathrm{H}^2_{\mathrm{et}}(\overline{X},\mathbb{G}_m)_{\mathrm{tors}}}_{\mathrm{Br}(\overline{X})}\right)$$

First isomorphism theorem:

$$\mathrm{im}\left(\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})\right) \cong \mathrm{Br}(X)/\underbrace{\ker\left(\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})\right)}_{\mathrm{Br}_1(X)}$$

$\mathrm{Br}(X)/\mathrm{Br}_1(X)$ is the transcendental Brauer group of X.

Is $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ finite?

$E(k)_{\mathrm{tors}}$ is finite. Is the K3 analogue $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ finite?

Theorem (Skorobogatov–Zarhin 2008)

Let $k/\mathbb{Q}$ be a finitely generated field; let X/k be a K3 surface. Then $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ is finite.

Uniform boundedness conjectures

For $X/\mathbb{C}$ a K3 surface, always have $\mathrm{H}^2(X(\mathbb{C}),\mathbb{Z}) \simeq U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$.

We call $\Lambda_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ the K3 lattice.

$NS(X)$ embeds primitively in $\mathrm{H}^2(X(\mathbb{C}),\mathbb{Z})$.

Conjecture (Weak uniform boundedness)

Fix a number field k and a primitive sublattice $\Lambda \subset \Lambda_{K3}$.
There is an integer $B = B(k,\Lambda)$ such that:
For all K3 surfaces X/k with $\Lambda \simeq \mathrm{NS}(\overline{X})$,

$$\#\mathrm{Br}(X)/\mathrm{Br}_1(X) < B.$$

Remarks

- Could ask for $B([k:\mathbb{Q}],\Lambda)$ instead of $B(k,\Lambda)$ (strong uniform boundedness).
- Weak Shafarevich conjecture (1994): for fixed number field k, there are only finitely many possibilities for $\mathrm{NS}(\overline{X})$.

  $\implies$ can dispense with Λ in the conjecture.
- Strong Shafarevich conjecture: for fixed $[k:\mathbb{Q}]$, there are only finitely many possibilities for $\mathrm{NS}(\overline{X})$.

  $\implies$ can dispense with Λ in the strong version of the conjecture.

Strongest form of the conjecture

Conjecture (Strong unif. boundedness + strong Shafarevich)
Fix $d \in \mathbb{Z}_{>0}$. There is an integer $B = B(d)$ such that:
For all number fields k with $[k : \mathbb{Q}] = d$ and all K3 surfaces X/k,

$$\# \mathrm{Br}(X)/\mathrm{Br}_1(X) < B.$$

This should be the K3 analogue of Merel's theorem.

ℓ-primary boundedness: an easier conjecture?

Conjecture (ℓ-primary boundedness)

Fix a number field k, a prime ℓ, and a primitive sublattice $\Lambda \subset \Lambda_{K3}$. There is an integer $B = B(k, \Lambda, \ell)$ such that for all K3 surfaces X/k with $\Lambda \simeq \mathrm{NS}(\overline{X})$,

$$\#(\mathrm{Br}(X)/\mathrm{Br}_1(X))[\ell^\infty] < B.$$

Strong version: replace $B(k, \Lambda, \ell)$ with $B([k : \mathbb{Q}], \Lambda, \ell)$.

After all, before Merel, there was Manin...

Theorem (Manin 1969)

Fix a number field k and a prime ℓ. There is an integer $c = c(k, \ell)$ such that for all elliptic curves E/k,

$$\#E(k)[\ell^\infty] < c.$$

Evidence

I. Kodaira dimension estimates for relevant moduli problem.

   Joint work with Tanimoto; Mckinnie, Sawon, and Tanimoto.

II. Conditional analogues in the case of full-level structures for abelian varieties.

   Joint work with Abramovich.

III. Special cases:

   i. Verification for some lattices Λ of rank 19.

      Joint work with Viray.

   ii. The CM case. (Gives Merel-type result for K3s with $\rho = 20$.)

      Orr/Skorobogatov

   iii. ℓ-primary boundedness for 1-dimensional families.

      Ambrosio/Cadoret/Charles (forthcoming)

I. Geometry: moduli of K3s with level structure

$\mathcal{K}_{2d}$ = coarse moduli space of projective K3 surfaces $/\mathbb{C}$ with polarization of degree $2d$.

$\mathscr{Y}_0(2d,p)$ = coarse moduli space of pairs $(X, \langle\alpha\rangle)$, where
$X/\mathbb{C}$ = projective K3 surface of degree $2d$, and
$0 \neq \langle\alpha\rangle \subseteq (\operatorname{Br} X)[p]$ is a p-torsion subgroup

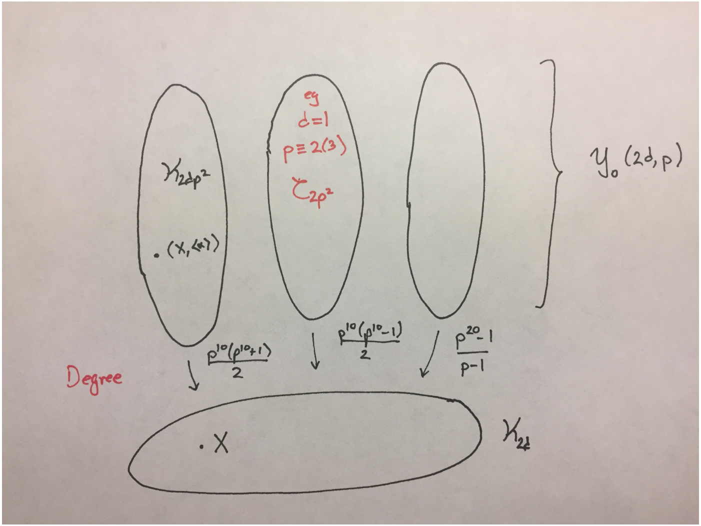
There is a forgetful map $\mathscr{Y}_0(2d,p) \to \mathcal{K}_{2d}$

Theorem (Gritsenko, Hulek, Sankaran 2007)

$\mathcal{K}_{2d}$ is of general type for $d > 61$.

Joint work with McKinnie, Sawon, and Tanimoto (2017)

Let $p \nmid d$ be prime. Very general picture:



I. Geometry: moduli of K3s with level structure

$\mathscr{C}_D :=$ coarse moduli of special cubic fourfolds of discriminant D;

Theorem (Hassett 2000)

$\mathscr{C}_D$ is non-empty if and only if $D > 6$ and $D \equiv 0, 2 \pmod 6$. When non-empty, it is an irreducible algebraic variety of dimension 19.

Theorem (Tanimoto, V.-A. to appear)

The non-empty $\mathscr{C}_D$ are of general type for all $D > 198$.

Theorem (Ma, 2017)

Up to isomorphism, there are only finitely many Noether-Lefschetz cycles in $\mathscr{K}_{2d}$ and $\mathscr{C}_D$ of dimension ≥ 9 that are not of general type.

II. Full-level structures on abelian varieties, assuming Lang

Let A be a g-dimensional abelian variety over a number field k.

A full-level m structure on A is an isomorphism of k-group schemes

$$A[m] \xrightarrow{\sim} (\mathbb{Z}/m\mathbb{Z})^g \times (\mu_m)^g$$

(not necessarily compatible with the Weil pairing).

Theorem (Abramovich, V.-A. 2016)

Assume Lang's conjecture.

Fix $g \in \mathbb{Z}_{>0}$, a prime ℓ and a number field k.

There is an integer $r = r(k, g, \ell)$ such that no (pp) abelian variety A/k of dimension g has full-level ℓ^r structure.

II. Full-level structures on abelian varieties, assuming Vojta

Theorem (Abramovich, V.-A. 2017)

Assume Vojta's conjecture.

Fix $g \in \mathbb{Z}_{>0}$ and a number field k.

There is an integer $m_0 = m_0(k, g)$ such that:

For any $m > m_0$ there is no (pp) abelian variety A/k of dimension g with full-level m structure.

III. Special cases

Theorem (Orr, Skorobogatov 2017)

Fix $d \in \mathbb{Z}_{>0}$. There is an integer $c = c(d)$ such that:
For all number fields k with $[k : \mathbb{Q}] = d$ and all CM K3 surfaces X/k,

$$\# \mathrm{Br}(X)/\mathrm{Br}_1(X) < c.$$

CM K3 surface: $\mathrm{End}(\mathrm{NS}(\overline{X})^{\perp}) \otimes \mathbb{Q}$ is a CM field.

III. Special cases

Fix a number field k, as well as non-CM elliptic curves E, E' with a cyclic isogeny of minimal degree d between them.

Let $X = \mathrm{Kum}(E \times E') = (\widetilde{E \times E'})/\iota$, where $\iota \colon x \mapsto -x$.

Let $\Lambda_d = \mathrm{NS}(\overline{X})$.

Λ_d has rank 19, discriminant $2d$, + indep. of E, E' and isogeny.

Theorem (V.-A., Viray 2016)

Fix a positive integer r, and a prime ℓ.

There is a positive integer $B = B(r, d, \ell)$ such that

for all K3 surfaces X/k with $[k : \mathbb{Q}] = r$ and $\mathrm{NS}(\overline{X}) \simeq \Lambda_d$,

$$\#\left(\mathrm{Br}(X)/\mathrm{Br}_1(X)\right)[\ell^\infty] < B.$$

End of Part I

End of Talk

II. Full-level structures on abelian varieties

Theorem (Abramovich, V.-A. 2016)

Assume Lang's conjecture.

Fix $g \in \mathbb{Z}_{>0}$, a prime ℓ and a number field k.

There is an integer $r = r(k, g, \ell)$ such that no (pp) abelian variety A/k of dimension g has full-level ℓ^r structure.

Engine behind proof:

Let $\pi_m : \mathscr{A}_g^{[m]} \to \mathscr{A}_g$ be the (finite) 'forget' map.

Theorem (Abramovich, V.-A. 2016; Brunebarbe 2016)

Let $X \subset \mathscr{A}_g$ be a closed subvariety.

There is an integer m_X such that, for all $m > m_X$,

every irreducible component of $\pi_m^{-1}(X) \subset \mathscr{A}_g^{[m]}$ is of general type.

II. Full-level structures on abelian varieties, assuming Vojta

Conjecture (Vojta c. 1984)

X a smooth projective variety over a number field K.

D a normal crossings divisor on X; H a big line bundle on X.

Fix a positive integer r and $\delta > 0$.

There is a proper Zariski closed $Z \subset X$ containing D such that

$$N_X^{(1)}(D, x) + d_K(K(x)) \geq h_{K_X + D}(x) - \delta h_H(x) - O(1)$$

for all $x \in X(\overline{K}) \smallsetminus Z(\overline{K})$ with $[K(x) : K] \leq r$.

What about $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$?

Lemma

Let X be a variety over a field k of characteristic 0. Assume that $\mathrm{Pic}(\overline{X}) \simeq \mathbb{Z}^r$. Then there is an integer $M = M(r)$, independent of X, such that $\#\mathrm{Br}_1(X)/\mathrm{Br}_0(X) < M$.

Idea of the proof.

1. Pass to a finite Galois extension K/k such that $\mathrm{Pic}(X_K) \cong \mathbb{Z}^r$.
2. Hochschild–Serre $\implies \mathrm{Br}_1(X)/\mathrm{Br}_0(X) \simeq \mathrm{H}^1(\mathrm{Gal}(K/k), \mathbb{Z}^r)$.
3. $$\mathrm{H}^1(G, \mathbb{Z}^r) \simeq \frac{(\mathbb{Z}^r/|G|)^G}{(\mathbb{Z}^r)^G/(|G|)} \text{ where } G = \mathrm{Gal}(K/k).$$

   $\implies \#\mathrm{H}^1(G, \mathbb{Z}^r)$ divides $|G|^r$, regardless of action.
4. G acts through a finite subgroup of $\mathrm{GL}_r(\mathbb{Z})$ (only finitely many possibilities). $\square$

What about $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$?

Lemma

Let X be a variety over a field k of characteristic 0. Assume that $\mathrm{Pic}(\overline{X}) \simeq \mathbb{Z}^r$. Then there is an integer $M = M(r)$, independent of X such that $\#\mathrm{Br}_1(X)/\mathrm{Br}_0(X) < M$.

Corollary

There is an absolute constant M such that, for all K3 surfaces X over a field of characteristic 0, we have

$$\#\mathrm{Br}_1(X)/\mathrm{Br}_0(X) < M.$$