RATIONAL POINTS ON SURFACES

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Let $X \subset \mathbb{P}^n$ be a surface over $\mathbb{Q}$. Let $H\colon \mathbb{P}^n(\mathbb{Q}) \to \mathbb{R}_{>0}$ be defined by $H(x) = \prod_v H_v(x)$ where $H_v(x) := \max_j |x_j|_v$. Let $N(X, B) = \#\{x \in X(\mathbb{Q}) : H(x) \le B\}$ as $B \to \infty$.

Step 1: Classification over $\mathbb{C}$. Rational surfaces (e.g., del Pezzo surfaces), K3 surfaces etc., general type.

Step 2: Geometric invariants. One of the invariants would be the degree. Consider

$$x_1^2 x_2^2 + x_2^2 x_3^2 + x_3^2 x_1^2 = x_0 x_1 x_2 x_3.$$

This has $N(X, B) \sim B^{3/2}$ instead of B^ϵ as one might guess from the degree. The explanation is that this is a singular surface, and the singularities change the behavior. Therefore we reduce to smooth models: it will be helpful to consider all ample line bundles at once, and corresponding height functions. Let U be an open subset such that $X - U$ consists of accumulating curves containing many rational points. We may assume that the inverse image of $X - U$ in the resolution of singularities is a normal crossings divisor.

In our examples, we will have $\operatorname{Pic} X \simeq \mathbb{Z}^r$, and the effective cone $\Lambda_e ff(X)$ will be finitely generated. We also have the anticanonical class $-K_X$ and the ample line bundle L. Let $a(L) := \inf\{a : aL + K_X \in \Lambda_{\text{eff}}(X)\}$. Let $b(L)$ be the codimension of the face of $\Lambda_{\text{eff}}(X)$ containing $a(L)L + K_X$. The constant $c(\mathcal{L})$ is defined by Peyre for a metrized line bundle.

Universal torsors: $\mathbb{A}^3 - \{0\} \to \mathbb{P}^2$ reduces counting rational points on $\mathbb{P}^2$ to counting (primitive) integral points in $\mathbb{A}^3 - \{0\}$. The lattice point count is approximated by a volume of some domain on a torsor.

Examples: de la Bretèche did $\mathrm{Gr}(2, 5)$ over a dP5, de la Bretèche and Browning did $x_0x_1 - x_2^2 = x_0^2 - x_1x_4 + x_3^2 = 0$ (a singular dP4 with a D_4 singularity) and $x_0x_1 - x_2^2 = x_0x_4 - x_1x_2 + x_3^2 = 0$ (a singular dP4 with a D_5 singularity).

Today: Harmonic analysis approach to these questions.

Setup: Let G be a linear algebraic group of dimension 2. E.g., $G = \mathbb{G}_a^2$ or $G = \mathbb{G}_m^2$ (or non-split tori), or $G = \mathbb{G}_a \times \mathbb{G}_m$, or $G = \mathbb{G}_a \rtimes \mathbb{G}_m$. Let X be an equivariant compactification of G. Choose a faithful representation $\rho\colon G \to \mathrm{PGL}_{n+1}$, to get an action on $\mathbb{P}^n$. Let X be the closure of $\rho(G)$. Reduce to X smooth with $X - G = \bigcup_\alpha D_\alpha$.

Example 0.1. Let $G = \mathbb{G}_a^2$. Then the group of boundary divisors $\operatorname{Div}^b(X) = \bigoplus_\alpha \mathbb{Z} \cdot D_\alpha$ is isomorphic to $\operatorname{Pic} X$. We have $-K_X = \sum \kappa_\alpha D_\alpha$ with $\kappa_\alpha \ge 2$. Also $\Lambda_{\text{eff}}(X) = \bigoplus_{\alpha \ge 0} \mathbb{R}_{\ge 0} D_\alpha$.

Example 0.2. Let $G = \mathbb{G}_m^2$. We have

$$0 \to \mathcal{X}^\times(G) \to \operatorname{Div}^b X \to \operatorname{Pic} X \to 0$$

where $\mathcal{X}^\times(G)$ is the group of characters. We have $-K_X = \sum \kappa_\alpha D_\alpha$ with $\kappa_\alpha = 1$.

The other cases are the same, except that the character group $\mathcal{X}^\times(G)$ changes.

Example 0.3. Take $\mathbb{G}_m^2 \subset \mathbb{P}^2$. Blow up the three fixed points to get a dP6.

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Example 0.4. Take $\mathbb{G}_a^2 \subset \mathbb{P}^2$. There is a pointwise-fixed line, so blowing up any set of points on this line gives an equivariant compactification.

Height pairing: $H\colon G(\mathbf{A}_F)\times \mathrm{Div}^b(X)_{\mathbb{C}} \to \mathbb{C}$. Define $H = \prod_\alpha H_{D_\alpha}$ where $H_{D_\alpha} = \prod_v H_{D_\alpha,v}$, where $H_{D_\alpha,v}(g_v)$ is the v-adic distance from g_v to the boundary D_α.

Properties: invariance under the action of $K_v \subset G(k_v)$ with $K_v = G(\mathcal{O}_v)$ for almost all v follows from equivariance of the compactification.

If $g \in G(\mathbf{A}_f)$, then

$$Z(s,g) = \sum_{\gamma \in G(F)} H(\gamma g, s)^{-1} \in L^2(G(F)\backslash G(\mathbf{A}_F))$$

For $G = \mathbb{G}_a^2$ the quotient on the right is compact; for $G = \mathbb{G}_m^2$ it is not compact.

Poisson:

$$\sum_{\gamma\in G(F)} H(\gamma, s)^{-1} = \sum_{\psi \in \left(G(F)\backslash G(\mathbf{A}_f)\right)^\perp} \hat{H}(\psi, s).$$

where ψ ranges over unitary characters and

$$\hat{H}(\psi,s) := \prod_v \int_{G(F_v)} H_v(g_v,s)^{-1}\psi_v(g_v)\,dg_v$$

Pointwise convergence follows from continuity. For $G = \mathbb{G}_a^2$, we have $(G(F)\backslash G(\mathbf{A}_f))^\perp = F^2$, but K_v-invariance of H implies that we need only sum over ψ_a with $a \in \mathcal{O}^2$ instead of $a \in F^2$. We have

$$Z(s,g) = \int_{G(\mathbf{A}_f)} H(g',s)^{-1}\,dg' + \sum_{a\in\mathcal{O}^2-\{0\}} \hat{H}(\psi_a, s).$$

The first term is the main term, and the second term is the error term. The first term is a Denef/Loeser/Igusa-type integral. The outcome is

$$\prod_{v\in S}\prod_{v\notin S}\left(1 + \sum_{\alpha\in\mathcal{A}} \frac{\#D_\alpha^0(\mathbb{F}_q)}{q^2}\cdot\frac{q-1}{q^{S_\alpha-K_\alpha+1}-1} + \sum_{\alpha\neq\alpha'}\frac{1}{q^2}\frac{(q-1)^2}{(q^{S_\alpha-K_\alpha+1}-1)(q^{S_{\alpha'}-K_{\alpha'}+1}-1)}\right)$$

for a finite set, where $D_\alpha^0 := D_\alpha - \bigcup_{\alpha'} D_\alpha\cap D_{\alpha'}$. Get

$$\int_{G(\mathbf{A}_F)} H(g,s)^{-1}\,ds = \prod_{\alpha\in\mathcal{A}} \zeta_F(s_\alpha - K_\alpha + 1)\times Q(s)$$

where $Q(s)$ is holomorphic.

$$\hat{H}(\psi_\alpha, s) = \prod_{\alpha\in\mathcal{A}_0(a)} \zeta_F(S_\alpha - K_\alpha + 1)\times Q_a(s).$$

For all N we have $|Q_a(s)| \le 1/\|a\|^N$.

For $G = \mathbb{G}_m^2$,

$$L(s,g) = \int_\chi \hat{H}(\chi,s)\,d\chi$$

where χ ranges over characters $KG(F)\backslash G(\mathbf{A}_F) \to \mathbb{S}\subset\mathbb{C}^\times$. This equals

$$\int_{\chi=\chi_m\in M=\mathcal{X}^*(G)_{\mathbb{R}}=\mathbb{R}^r} \prod_v \int_{G(F_v)} H_v(s,g)^{-1}|g|^{im}\,dg\,d\chi_m.$$

$$0 \to M \to \mathrm{Div}^b X \to \mathrm{Pic}\, X.$$
$$\hat{H}(\chi_m, s) = \prod \zeta_F(s_\alpha - k_\alpha + 1 + im_\alpha) \times Q(s + im).$$
$$\chi_\Lambda(s) := \frac{1}{2\pi i} \int \frac{dm}{\prod(s_\alpha - k_\alpha + im_\alpha)}.$$

Let $G = \mathbb{G}_m \ltimes \mathbb{G}_a$. Let $\mathcal{H} = L^2(G(F)\backslash G(\mathbf{A}))$. We have $\mathcal{H} = \bigoplus \mathcal{H}_\psi$ with $\psi \in (\mathbb{G}_a(F)\backslash \mathbb{G}_a(\mathbf{A}_F))^\perp$.

$$\mathcal{H}_{\psi_0} = \int_{\chi_m} H\chi \, dg$$