THE ARITHMETIC OF DEL PEZZO SURFACES

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Let X be a smooth projective variety over a field k. Classify X according to dimension.

1. Dimension 1

Suppose $\dim X = 1$. Work over $\overline{k}$. There is one discrete invariant, namely the genus g. If $g = 0$, the canonical divisor and its multiples have no nonzero sections If $g = 1$, the canonical divisor is trivial, as are its multiples; there is a lot to say. If $g = 2$, the canonical divisor is ample, and its multiples acquire more and more sections; also $X(k)$ is finite.

Proposition 1.1. *For a genus 0 curve, the following are equivalent:*

(1) $X \simeq \mathbb{P}^1_k$
(2) $X(k) \neq \emptyset$
(3) *There exists* $\mathcal{L} \in \operatorname{Pic} X$ *of degree* 1.
(4) *There exists* $\mathcal{L} \in \operatorname{Pic} X$ *of odd degree.*

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4). Now suppose (4): we are given $\mathcal{L}$ with $\deg \mathcal{L} = 2n+1$. The canonical bundle ω_X has degree -2. Then $\mathcal{L} \otimes \omega_X^n$ has degree 1. By Riemann-Roch, it has two independent sections; it gives a morphism $X \to \mathbb{P}^{2-1} = \mathbb{P}^1$ of degree 1, so it is an isomorphism. □

2. Dimension 2

For $m > 0$, the line bundle $\omega_X^{\otimes m}$ defines a rational map $\phi_m \colon X \dashrightarrow \mathbb{P}\left(H^0(X, \omega_X^{\otimes m})\right)$ if it has at least one nonzero global section. The Kodaira dimension is the eventual value (as m becomes large and sufficiently divisible) of the dimension of $\phi_m(X)$; if there are no global sections of any $\omega_X^{\otimes m}$, the Kodaira dimension is said to be negative.

Kodaira dimension 0: e.g., K3 surfaces, abelian varieties.

Kodaira dimension 1: some elliptic surfaces.

Kodaira dimension 2: surface of general type.

We work over $\overline{k}$. If Kodaira dimension is negative, then X is ruled: any point of X is contained in a rational curve. Moreover there is a rational map $X \dashrightarrow C$ whose fibers are genus 0 curves. If moreover, C is of genus 0, then X is a rational surface, which means that is birational to $\mathbb{P}^2$.

We now study rational varieties over a perfect field k.

Theorem 2.1 (Iskovskikh)**.** *Given a rational variety X of dimension 2 over a perfect field k, at least one of the following happens:*

(a) *X is birational to a conic bundle over a conic.*
(b) *X is k-birational to a del Pezzo surface.*

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(Conic means a twist of $\mathbb{P}^1$.)

Definition 2.2. X is a *del Pezzo surface* if $\omega_X^\vee$ is ample.

Over $\overline{k}$, every del Pezzo surface is $\mathbb{P}^1 \times \mathbb{P}^1$ or the blow-up of $\mathbb{P}^2$ at δ points in general position, where $0 \leq \delta \leq 8$. The degree of the del Pezzo surface is $\deg X = (K_X)^2$, which equals $9 - \delta$ for the blow-up of $\mathbb{P}^2$ at δ points.

When does the existence of one rational point on X imply that X is isomorphic to $\mathbb{P}^2$, or has a Zariski dense set of rational points.

Theorem 2.3 (Segre-Manin)**.** *Suppose that X is a del Pezzo surface (always over a perfect field k), and $p \in X(k)$ is a general point (the "general" hypothesis is needed currently only when* $\deg X = 2$*). If* $\deg f \leq 2$ *implies that $X(k)$ is Zariski dense.*

"General" means "not on any exceptional curve". (But the theorem might be true for all p on all del Pezzo surfaces.)

Exceptional curves (also called (-1)-curves) are curves that can be blown down: numerically, they are curves C such that $C^2 = -1$ and $K.C = -1$. The first condition implies that C does not move, and second implies that C cannot be broken down into pieces.

Example 2.4. Let X be $\mathbb{P}^2$ blown up at 5 general points. Above each point, we get an exceptional curve. There is also an exceptional curve above the line through any pair of the 5 points, and above the conic through the 5 points.

Every del Pezzo surface has only finitely many exceptional curves, and their structure is independent of the location of the points blown up, provided that they are general.

$\deg X$	property	# of (-1)-curves	dual graph	automorphism group
9	$\mathbb{P}^2$ (over $\overline{k}$)	0	empty	
8	$\mathrm{Bl}_p(\mathbb{P}^2)$ (over $\overline{k}$)	1	one point	
7	$\mathrm{Bl}_{\{p_1,p_2\}}(\mathbb{P}^2)$ (over $\overline{k}$)	3	chain	$A_1 = \mathbb{Z}/2\mathbb{Z}$
6	$\mathrm{Bl}_{\{p_1,p_2,p_3\}}(\mathbb{P}^2)$ (over $\overline{k}$) last to be a toric variety	6 $A_4 = S_5$	hexagon	$A_2 \times A_1 = D_6 = D_3 \times \mathbb{Z}/2\mathbb{Z}$
5	$\overline{\mathcal{M}}_{0,5}$	10	Petersen graph	E_6
4	complete intersection of 2 quadrics in $\mathbb{P}^4$	16	Clebsch graph	D_5
3	cubic surface in $\mathbb{P}^3$	27		E_6
2	double cover of $\mathbb{P}^2$ branched over a smooth quartic	56		E_7
1	rational elliptic surfaces	240		E_8

In the last two column, we give the dual graph of the (-1)-curves, and the automorphism group of this configuration, which is often a Weyl group, in which case we label it with the corresponding root system name.