SINGULAR DEL PEZZO SURFACES AND ROOT SYSTEMS

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Let X be a smooth del Pezzo surface of degree ≤ 7. On X we have the (-1)-curves: their classes in $\operatorname{Pic} X$ are characterized by $(D, D) = -1$ and $(D, -K) = 1$. The symmetry group of the configuration of (-1)-curves is given by the Weyl group $W(R)$ of the root system R whose roots are the (-2)-classes: $E \in \operatorname{Pic} X$ with $(E, E) = -2$ and $(E, -K) = 0$.

d	7	6	5	4	3	2	1
R_d	A_1	$A_2 + A_1$	A_4	D_5	E_6	E_7	E_8

A smooth del Pezzo surface of degree $d \leq 7$ may be obtained as an iterated blow-up of $\mathbb{P}^2$ in $9-d$ points in general position, meaning that one never blows up a point on a (-1)-curve.

Let $\tilde{Y}$ be the blow-up of $\mathbb{P}^2$ in $9-d$ points in almost general position: don't blow up points on (-2)-curves. Contract the (-2)-curves to obtain a singular del Pezzo surface Y.

Then $\operatorname{Pic} X$ and $\operatorname{Pic} \tilde{Y}$ can be identified if $\deg X = \deg \tilde{Y}$. We have

$$\{(-2)\text{-curves}\} \subset \{\text{effective } (-2)\text{-classes}\} \subset \{(-2)\text{-classes}\}.$$

The effective (-2)-classes are the positive roots of a root system $R_{\tilde{Y}}$ (consisting of those and their negatives), and is contained in the root system R_d of all (-2)-classes. The set of (-2)-curves is the set of simple roots of $R_{\tilde{Y}}$.

Any root system $R \subset R_d$ occurs as $R_{\tilde{Y}}$ of a singular del Pezzo Y, except for the following four: $7A_1$ in degree 2, and $8A_1$, $7A_1$, $D_4 + 4A_1$ in degree 1.

Let Y be a singular del Pezzo surface over $\mathbb{Q}$ (i.e., a surface that after base extension to $\overline{\mathbb{Q}}$ becomes a singular del Pezzo surface over $\overline{\mathbb{Q}}$. Then $\operatorname{Pic} \tilde{Y}_{\overline{\mathbb{Q}}}$ contains a root system $R_{\tilde{Y}_{\overline{\mathbb{Q}}}} \subset R_d$.

Let $G = \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We assume $\tilde{Y}(\mathbb{Q}) \neq \emptyset$ from now on; then

$$\operatorname{Pic} \tilde{Y} = (\operatorname{Pic} \tilde{Y}_{\overline{\mathbb{Q}}})^G.$$

Proposition 0.1. *The set of $\sum_{D \in G \cdot E} D$ such that E is a root of $R_{\tilde{Y}_{\overline{\mathbb{Q}}}}$ form a root system $R_{\tilde{Y}}$ in $\operatorname{Pic} \tilde{Y}$. The effective (-2)-classes map to positive roots of $R_{\tilde{Y}}$. The (-2)-curves map to simple roots of $R_{\tilde{Y}}$.*

Example 0.2. Consider A_5 with Galois acting as an involution: then $R_{\tilde{Y}} = B_3$: two long roots and one short root in a chain, with a single arrow from long root 1 to long root 2, and a double arrow from long root 2 to short root 1.

More generally, $A_{2n+1} \rightsquigarrow B_{n+1}$, $A_{2n} \rightsquigarrow B_n$, $D_n \rightsquigarrow C_{n-1}$, $E_6 \rightsquigarrow F_4$, and $D_4 \rightsquigarrow G_2$.

Application: Manin's conjecture.

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Let Y be a singular or smooth del Pezzo surface of degree $d \geq 3$. We have $Y \hookrightarrow \mathbb{P}^d$. Let $U = Y - \{\text{lines of } Y\}$. If $x = (x_0 : \cdots : x_d) \in Y(\mathbb{Q})$ with $x_0, \ldots, x_d \in \mathbb{Z}$ relatively prime, define $H(x) = \max_i |x_i|$. Let $N_{U,H}(B)$ be the number of $x \in U(\mathbb{Q})$ with $H(x) \leq B$. It is conjectured that

$$N_{U,H}(B) \sim cB(\log B)^k$$

where $c > 0$ (if there is a rational point) and $k = \operatorname{rk} \operatorname{Pic} \tilde{Y} - 1$. There is a prediction for c, namely $c = \alpha(\tilde{Y})\beta(\tilde{Y})\gamma(\tilde{Y})$ where $\alpha(\tilde{Y})$ is the volume of the set of $x \in (\operatorname{Pic} \tilde{Y})_{\mathbb{R}}$ such that $(x, -K) = 1$ and $(x, D) \geq 0$ for all effective divisors.

Theorem 0.3 (Derenthal, Joyce, Teitler)**.** $\alpha(\tilde{Y}) = \alpha(X)/\#W(R_{\tilde{Y}})$ *where* X *is any smooth del Pezzo surface of the same degree as* Y*, and the same action of Galois on the Picard group.*