[1]

ARITHMETIC OF CURVES OVER TWO DIMENSIONAL LOCAL FIELD

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ABSTRACT. We study the class field theory of curve defined over two dimensional local field. The approch used here is a combination of the work of Kato-Saito, and Yoshida where the base field is one dimensional

1. INTRODUCTION

Let k_1 be a local field with finite residue field and let X be a proper smooth geometrically irreducible curve over k_1. To study the fundamental group $\pi_1^{ab}(X)$, Saito in [9], introduced the groups $SK_1(X)$ and $V(X)$ and constructed the maps $\sigma : SK_1(X) \longrightarrow \pi_1^{ab}(X)$ and $\tau : V(X) \longrightarrow \pi_1^{ab}(X)^{g\acute{e}o}$ where $\pi_1^{ab}(X)^{g\acute{e}o}$ is defined by the exact sequence

$$0 \longrightarrow \pi_1^{ab}(X)^{g\acute{e}o} \longrightarrow \pi_1^{ab}(X) \longrightarrow Gal(k_1^{ab}/k_1) \longrightarrow 0$$

The most important results in this context are:

1) The quotient of $\pi_1^{ab}(X)$ by the closure of the image of σ and the cokernel of τ are both isomorphic to $\widehat{\mathbb{Z}}^r$ where r is the rank of the curve.
2) For this integer r, there is an exact sequence

$$0 \longrightarrow (\mathbb{Q}/\mathbb{Z})^r \longrightarrow H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \underset{v\in P}{\oplus} \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

where $K = K(X)$ is the function field of X and P designates the set of closed points of X.

These results are obtained by Saito in [9] generalizing the previous work of Bloch where he is reduced to the good reduction case [9, Introduction]. The method of Saito depends on class field theory for two-dimensional local ring having finite residue field. He shows these results for general curve except for the p -primary part in char$k = p > 0$ case [9, Section II-4]. The remaining p -primary part had been proved by Yoshida in [12].

There is another direction for proving these results pointed out by Douai in [3]. It consists to consider for all l prime to the residual characteristic, the group $Co\ker\sigma$ as the dual of the group W_0 of the monodromy weight filtration of $H^1(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$

$$H^1(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = W_2 \supseteq W_1 \supseteq W_0 \supseteq 0$$

where $\overline{X} = X \otimes_{k_1} \overline{k_1}$ and $\overline{k_1}$ is an algebraic closure of k_1. This allow him to extend the precedent results to projective smooth surfaces [3].

The aim of this paper is to use a combination of this approach and the theory of the monodromy-weight filtration of degenerating abelian varieties on local fields explained by Yoshida in his paper [12], to study curves over two-dimensional local fields (section 3).

Let X be a projective smooth curve defined over two dimensional local field k. Let K be its function field and P denotes the set of closed points of X. For each $v \in P$, $k(v)$ denotes the residue field at $v \in P$. A finite etale covering $Z \to X$ of X is called a c.s covering, if for any

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closed point x of X, $x \times_X Z$ is isomorphic to a finite sum of x. We denote by $\pi_1^{c.s}(X)$ the quotient group of $\pi_1^{ab}(X)$ which classifies abelian c.s coverings of X.

To study the class field theory of the curve X, we construct the generalized reciprocity map

$$\sigma/\ell : SK_2(X)/\ell \longrightarrow \pi_1^{ab}(X)/\ell$$

where $SK_2(X)/\ell = Co\ker\left\{K_3(K)/\ell \overset{\oplus\partial_v}{\longrightarrow} \underset{v\in P}{\oplus} K_2(k(v))/\ell\right\}$ and $\tau/l : V(X)/\ell \longrightarrow \pi_1^{ab}(X)^{géo}/\ell$ for all ℓ prime to residual characteristic. The group $V(X)$ is defined to be the kernel of the norm map $N : SK_2(X) \longrightarrow K_2(k)$ induced by the norm map $N_{k(v)/k^x} : K_2(k(v)) \longrightarrow K_2(k)$ for all v and $\pi_1^{ab}(X)^{géo}$ by the exact sequence

$$0 \longrightarrow \pi_1^{ab}(X)^{géo} \longrightarrow \pi_1^{ab}(X) \longrightarrow Gal(k^{ab}/k) \longrightarrow 0$$

The cokernel of σ/ℓ is the quotient group of $\pi_1^{ab}(X)/\ell$ that classifies completely split coverings of X ; that is ; $\pi_1^{c.s}(X)/\ell$.

We begin by proving the exactness of the Kato-Saito sequence (Proposition 4.3) :

$$\begin{aligned} 0 \longrightarrow \pi_1^{c.s}(X)/\ell &\longrightarrow H^4(K, \mathbb{Z}/\ell(3)) \\ &\longrightarrow \underset{v\in P}{\oplus} H^3(k(v), \mathbb{Z}/\ell(2)) \longrightarrow \mathbb{Z}/\ell \longrightarrow 0 \end{aligned}$$

To determinate the group $\pi_1^{c.s}(X)/\ell$, we need to consider a semi stable model of the curve X (see Section 5) and the weight filtration on its special fiber. In fact, we will prove in (Proposition 5.1) that $\pi_1^{c.s}(X) \otimes \mathbb{Q}_\ell$ admits a quotient of type $\mathbb{Q}_l^r$ where r is the rank of the first crane of this filtration.

Now, to investigate the group $\pi_1^{ab}(X)^{géo}$, we use class field theory of two-dimensional local field and prove the vanishing of the group $H^2(k, \mathbb{Q}/\mathbb{Z})$ (theorem 3.1). This yields the isomorphism

$$\pi_1^{ab}(X)^{géo} \simeq \pi_1^{ab}\left(\overline{X}\right)_{G_k}$$

Finally, by the Grothendick weight filtration on the group $\pi_1^{ab}\left(\overline{X}\right)_{G_k}$ and assuming the semi-stable reduction, we obtain the structure of the group $\pi_1^{ab}(X)^{géo}$ and information about the map $\tau : V(X) \longrightarrow \pi_1^{ab}(X)^{géo}$.

Our paper is organized as follows. Section 2 is devoted to some notations. Section 3 contains the proprieties which we need concerning two-dimensional local field: duality and the vanishing of the second cohomology group. In section 4, we construct the generalized reciprocity map and study the Bloch-Ogus complex associated to X. In section 5, we investigate the group $\pi_1^{c.s}(X)$.

2. Notations

For an abelian group M, and a positive integer $n \geq 1, M/n$ denotes the group M/nM.

For a scheme Z, and a sheaf $\mathcal{F}$ over the étale site of Z, $H^i(Z, \mathcal{F})$ denotes the i-th étale cohomology group. The group $H^1(Z, \mathbb{Z}/\ell)$ is identified with the group of all continues homomorphisms $\pi_1^{ab}(Z) \longrightarrow \mathbb{Z}/\ell$. If ℓ is invertible on $\mathbb{Z}/\ell(1)$ denotes the sheaf of l-th root of unity and for any integer i, we denote $\mathbb{Z}/\ell(i) = (\mathbb{Z}/\ell(1))^{\otimes i}$

For a field L, $K_i(L)$ is the i-th Milnor group. It coincides with the $i-$th Quillen group for $i \leq 2$. For ℓ prime to *char* L, there is a Galois symbol

$$h_{\ell,L}^i \quad K_iL/\ell \longrightarrow H^i(L, \mathbb{Z}/\ell(i))$$

which is an isomorphism for $i = 0, 1, 2$ ($i = 2$ is Merkur'jev-Suslin).

3. On two-dimensional local field

A local field k is said to be $n-$dimensional *local* if there exists the following sequence of fields k_i $(1 \leq i \leq n)$ such that
(i) each k_i is a complete discrete valuation field having k_{i-1} as the residue field of the valuation ring O_{k_i} of k_i, and
(ii) k_0 is a finite field.

For such a field, and for ℓ prime to $\mathrm{Char}(k)$, the well-known isomorphism

$$H^{n+1}\left(k, \mathbb{Z}/\ell\left(n\right)\right) \simeq \mathbb{Z}/\ell \tag{3.1}$$

and for each $i \in \{0, ..., n+1\}$ a perfect duality

$$H^i(k, \mathbb{Z}/\ell(j)) \times H^{n+1-i}(k, \mathbb{Z}/\ell(n-j) \longrightarrow H^{n+1}(k, \mathbb{Z}/\ell(n)) \simeq \mathbb{Z}/\ell \tag{3.2}$$

hold.

The class field theory for such fields is summarized as follows: There is a map $h : K_2\left(k\right) \longrightarrow Gal(k^{ab}/k)$ which generalizes the classical reciprocity map for usually local fields. This map induces an isomorphism $K_2\left(k\right)/N_{L/k}K_2\left(L\right) \simeq Gal(L/k)$ for each finite abelian extension L of k. Furthermore, the canonical pairing

$$H^1\left(k, \mathbb{Q}_l/\mathbb{Z}_l\right) \times K_2(k) \longrightarrow H^3\left(k, \mathbb{Q}_l/\mathbb{Z}_l\left(2\right)\right) \simeq \mathbb{Q}_l/\mathbb{Z}_l \tag{3.3}$$

induces an injective homomorphism

$$H^1\left(k, \mathbb{Q}_l/\mathbb{Z}_l\right) \longrightarrow Hom(K_2(k), \mathbb{Q}_l/\mathbb{Z}_l) \tag{3.4}$$

It is well-known that the group $H^2\left(M, \mathbb{Q}/\mathbb{Z}\right)$ vanishes when M is a finite field or usually local field. Next, we prove the same result for two-dimensional local field

Theorem 3.1. *If k is a two-dimensional local field of characteristic zero, then the group $H^2\left(k, \mathbb{Q}/\mathbb{Z}\right)$ vanishes.*

Proof. We proceed as in the proof of theorem 4 of [11]. It is enough to prove that $H^2\left(k, \mathbb{Q}_l/\mathbb{Z}_l\right)$ vanishes for all l and when k contains the group μ_l of l-th roots of unity. For this, we prove that multiplication by l is injective. That is, we have to show that the coboundary map

$$H^1\left(k, \mathbb{Q}_l/\mathbb{Z}_l\right) \xrightarrow{\delta} H^2\left(k, \mathbb{Z}/l\mathbb{Z}\right)$$

is injective.

By assumption on k, we have

$$H^2\left(k, \mathbb{Z}/l\mathbb{Z}\right) \simeq H^2\left(k, \mu_l\right) \simeq \mathbb{Z}/\ell$$

The last isomorphism is well-known for one-dimensional local field and was generalized to non archimedian and locally compact fields by Shatz in [7]. The proof is now reduced to the fact that $\delta \neq 0$;

By class field theory of two dimensional local field, the cohomology group $H^1\left(k, \mathbb{Q}_l/\mathbb{Z}_l\right)$ may be identified with the group of continuous homomorphisms $K_2(k) \xrightarrow{\Phi} \mathbb{Q}_l/\mathbb{Z}_l$.

Now, $\delta(\Phi) = 0$ if and only if Φ is a $l-$th power, and Φ is a $l-$th power if and only if Φ is trivial on μ_l. Thus, it is sufficient to construct an homomorphism $K_2(k) \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l$ which is non trivial on μ_l.

Let i be the maximal natural number such that k contains a primitive l^i-th root of unity. Then, the image ξ of a primitive l^i-th root of unity under the composite map

$$k^x/k^{xl} \simeq H^1(k,\mu_l) \simeq H^1(k,\mathbb{Z}/l\mathbb{Z}) \longrightarrow H^1(k,\mathbb{Q}_l/\mathbb{Z}_l)$$

is not zero. Thus, the injectivity of the map

$$H^1(k,\mathbb{Q}_l/\mathbb{Z}_l) \longrightarrow Hom(K_2(k),\mathbb{Q}_l/\mathbb{Z}_l)$$

gives rise to a character which is non trivial on μ_l. □

Remark 3.2. This proof is inspired by the proof of Proposition 7 of Kato [5]

4. Curves over two dimensional local field

Let k be a two dimensional local field of characteristic zero and X a smooth projective curve defined over k.
We recall that we denote:
$K = K(X)$ its function field,
P : set of closed points of X, and for $v \in P$,
$k(v)$: the residue field at $v \in P$
The residue field of k is one-dimensi6nal local field. It is denoted by k_1
Let $\mathcal{H}^n(\mathbb{Z}/\ell(3))$, $n \geq 1$, the Zariskien sheaf associated to the presheaf $U \longrightarrow H^n(U,\mathbb{Z}/\ell(3))$. Its cohomology is calculated by the Bloch-Ogus resolution. So, we have the two exact sequences:

$$H^3(K,\mathbb{Z}/\ell(3)) \longrightarrow \underset{v\in P}{\oplus} H^2(k(v),\mathbb{Z}/\ell(2)) \longrightarrow H^1\left(X_{Zar},\mathcal{H}^3(\mathbb{Z}/\ell(3))\right) \longrightarrow 0 \tag{4.1}$$

$$0 \longrightarrow H^0(X_{Zar},\mathcal{H}^4(\mathbb{Z}/\ell(3))) \longrightarrow H^4(K,\mathbb{Z}/\ell(3)) \longrightarrow \underset{v\in P}{\oplus} H^3(k(v,\mathbb{Z}/\ell(2))) \tag{4.2}$$

4.1. **The reciprocity map.**

We introduce the group $SK_2(X)/\ell$:

$$SK_2(X)/\ell = Co\ker\left\{K_3(K)/\ell \overset{\oplus\partial_v}{\longrightarrow} \underset{v\in P}{\oplus} K_2(k(v))/\ell\right\}$$

where $\partial_v : K_3(K) \longrightarrow K_2(k(v))$ is the boundary map in K-Theory. It will play an important role in class field theory for X as pointed out by Saito in the introduction of [9]. In this section, we construct a map

$$\sigma/\ell : SK_2(X)/\ell \longrightarrow \pi_1^{ab}(X)/\ell$$

which describe the class field theory of X.
By definition of $SK_2(X)/\ell$, we have the exact sequence

$$K_3(K)/\ell \longrightarrow \underset{v\in P}{\oplus} K_2(k(v))/\ell \longrightarrow SK_2(X)/\ell \longrightarrow 0$$

On the other hand, it is known that the following diagram is commutative:

$$\begin{array}{ccc} K_3(K)/\ell & \longrightarrow \underset{v\in P}{\oplus} & K_2(k(v))/\ell \\ \downarrow h^3 & & \downarrow h^2 \\ H^3(K,\mathbb{Z}/\ell(3)) & \longrightarrow \underset{v\in P}{\oplus} & H^2(k(v),\mathbb{Z}/\ell(2)) \end{array}$$

where h^2, h^3 are the Galois symbols. This yields the existence of a morphism

$$h : SK_2(X)/\ell \longrightarrow H^1\left(X_{Zar}, \mathcal{H}^3(\mathbb{Z}/\ell(2))\right)$$

taking in account the exact sequence (4.1). This morphism fit in the following commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & K_3(K)/\ell & \longrightarrow \underset{v\in P}{\oplus} & K_2(k(v))/\ell & \rightarrow & SK_2(X)/\ell & \longrightarrow 0 \\ & \downarrow h^3 & & \downarrow h^2 & & \downarrow h & \\ 0 \longrightarrow & H^3(K, \mathbb{Z}/\ell(2)) & \longrightarrow \underset{v\in P}{\oplus} & H^2(k(v), \mathbb{Z}/\ell(2)) & \rightarrow & H^1\left(X_{Zar}, \mathcal{H}^3(\mathbb{Z}/\ell(2))\right) & \longrightarrow 0 \end{array}$$

By Merkur'jev-Suslin, the map h^2 is an isomorphism, which imply that h is surjective. On the other hand the spectral sequence

$$H^p(X_{Zar}, \mathcal{H}^q(\mathbb{Z}/\ell(3))) \Rightarrow H^{p+q}(X, \mathbb{Z}/\ell(3))$$

induces the exact sequence

$$\begin{aligned} (4.3) \qquad & 0 \longrightarrow H^1\left(X_{Zar}, \mathcal{H}^3(\mathbb{Z}/\ell(3))\right) \xrightarrow{e} H^4(X, \mathbb{Z}/\ell(3)) \\ & \longrightarrow H^0\left(X_{Zar}, \mathcal{H}^4(\mathbb{Z}/\ell(3))\right) \longrightarrow H^2\left(X_{Zar}, \mathcal{H}^3(\mathbb{Z}/\ell(3))\right) = 0 \end{aligned}$$

Composing h and e, we get the map

$$SK_2(X)/\ell \longrightarrow H^4(X, \mathbb{Z}/\ell(3))$$

Finally the group $H^4(X, \mathbb{Z}/\ell(3))$ is identified to the group $\pi_1^{ab}(X)/\ell$ by the duality [4,II, th 2.1]

$$H^4(X, \mathbb{Z}/\ell(3)) \otimes H^1(X, \mathbb{Z}/\ell) \longrightarrow H^5(X, \mathbb{Z}/\ell(3)) \simeq H^3(k, \mathbb{Z}/\ell(2)) \simeq \mathbb{Z}/\ell$$

Hence, we obtain the map

$$\sigma/\ell : SK_2(X)/\ell \longrightarrow \pi_1^{ab}(X)/\ell$$

Remark 4.1. By the exact sequence (4.2) the group $H^0\left(X_{Zar}, \mathcal{H}^4(\mathbb{Z}/\ell(3))\right)$ coincides with the kernel of the map

$$H^4(K, \mathbb{Z}/\ell(3)) \longrightarrow \underset{v\in P}{\oplus} H^3(k(v), \mathbb{Z}/\ell(2))$$

and by localization in étale cohomology

$$\underset{v\in P}{\oplus} H^2(k(v), \mathbb{Z}/\ell(2)) \longrightarrow H^4(X, \mathbb{Z}/\ell(3)) \longrightarrow H^4(K, \mathbb{Z}/\ell(3)) \longrightarrow \underset{v\in P}{\oplus} H^3(k(v), \mathbb{Z}/\ell(2))$$

and taking in account (4.3), we see that $H^1\left(X_{Zar}, \mathcal{H}^4(\mathbb{Z}/\ell(3))\right)$ is the cokernel of the Gysin map

$$\underset{v\in P}{\oplus} H^2(k(v), \mathbb{Z}/\ell(2)) \xrightarrow{g} H^4(X, \mathbb{Z}/\ell(3))$$

and consequently the morphism g factorize through $H^1\left(X_{Zar}, \mathcal{H}^4(\mathbb{Z}/\ell(3)\right)$

$$\begin{array}{ccc} \underset{v\in P}{\oplus} H^2(k(v),\mathbb{Z}/\ell(2)) & \overset{g}{\longrightarrow} & H^4(X,\mathbb{Z}/\ell(3)) \\ \searrow & & \nearrow \\ & H^1\left(X_{Zar},\mathcal{H}^4(\mathbb{Z}/\ell(3))\right) & \end{array}$$

Then, we deduce the following commutative diagram

$$\begin{array}{ccccc} K_3(K)/\ell & \to \underset{v\in P}{\oplus} & K_2(k(v))/\ell & \to & SK_2(X)/\ell \longrightarrow 0 \\ \downarrow h^3 & & \downarrow h^2 & & \downarrow h \\ H^3(K,\mathbb{Z}/\ell(3)) & \to \underset{v\in P}{\oplus} & H^2(k(v),\mathbb{Z}/\ell(2)) & \to & H^1\left(X_{Zar},\mathcal{H}^4(\mathbb{Z}/\ell(3))\right) \longrightarrow 0 \\ & & \downarrow g & \swarrow e & \\ & & \pi_1^{ab}(X)/l = H^4(X,\mathbb{Z}/\ell(3)) & & \end{array}$$

The surjectivity of the map h implies that the cokernel of

$$\sigma/\ell : SK_2(X)/\ell \longrightarrow \pi_1^{ab}(X)/\ell$$

coincides with the cokernel of e which is $H^0\left(X_{Zar},\mathcal{H}^4(\mathbb{Z}/\ell(3))\right)$. Hence $Co\ker\sigma/\ell$ is the dual of the kernel of the map

$$H^1(X,\mathbb{Z}/\ell) \longrightarrow \prod_{v\in P} H^1(k(v),\mathbb{Z}/\ell) \tag{4.4}$$

4.2. **The Kato-Saito exact sequence.**

Definition 4.2. Let Z be a Noetherian scheme. A finite etale covering $f : W \to Z$ is called a c.s covering if for any closed point z of Z , $z \times_Z W$ is isomorphic to a finite scheme-theoretic sum of copies of z We denote $\pi_1^{c.s}(Z)$ the quotient group of $\pi_1^{ab}(Z)$ which classifies abelian c.s coverings of Z.

Hence, the group $\pi_1^{c.s}(X)/\ell$ is the dual of the kernel of the map

$$H^1(X,\mathbb{Z}/\ell) \longrightarrow \prod_{v\in P} H^1(k(v),\mathbb{Z}/\ell)$$

as in [9, section 2, definition and sentence just below]. Now, we are able to calculate the homologies of the Bloch-Ogus complex associated to X.

Generalizing [10, Theorem 7], we obtain :

Proposition 4.3. *Let X be a projective smooth curve defined over k Then for all ℓ, we have the following exact sequence*

$$\begin{array}{rl} 0 \longrightarrow \pi_1^{c.s}(X)/\ell & \longrightarrow H^4(K,\mathbb{Z}/\ell(3)) \\ & \longrightarrow \underset{v\in P}{\oplus} H^3(k(v),\mathbb{Z}/\ell(2)) \longrightarrow \mathbb{Z}/\ell \longrightarrow 0. \end{array}$$

Proof. Consider the localization sequence on X

$$\begin{array}{l} \underset{v\in P}{\oplus} H^2(k(v),\mathbb{Z}/\ell(2)) \overset{g}{\longrightarrow} H^4(X,\mathbb{Z}/\ell(3)) \longrightarrow H^4(K,\mathbb{Z}/\ell(3)) \\ \qquad \longrightarrow \underset{v\in P}{\oplus} H^3(k(v),\mathbb{Z}/\ell(2)) \longrightarrow H^5(X,\mathbb{Z}/\ell(3)) \longrightarrow 0 \end{array}$$

We know that the cokernel of the Gysin map g coincides with $\pi_1^{c.s}(X)/\ell$ and we use the isomorphism $H^5(X,\mathbb{Z}/\ell(3)) \simeq \mathbb{Z}/\ell$. □

5. The group $\pi_1^{c.s}(X)$

In his paper [9], Saito don't prove the $p-$ primary part in the char $k = p > 0$ case. This case was developed by Yoshida in [12]. His method is based on the theory of monodromy-weight filtration of degenerating abelian varieties on local fields. In this work, we use this approach to investigate the group $\pi_1^{c.s}(X)$. As mentioned by Yoshida in [12, section 2] Grothendieck's theory of monodromy-weight filtration on Tate module of abelian varieties are valid where the residue field is arbitrary perfect field.

We assume the semi-stable reduction and choose a regular model $\mathcal{X}$ of X over $SpecO_k$, by which we mean a two dimensional regular scheme with a proper birational morphism $f : \mathcal{X} \longrightarrow SpecO_k$ such that $\mathcal{X} \otimes_{O_k} k \simeq X$ and if $\mathcal{X}_s$ designates the special fiber $\mathcal{X} \otimes_{O_k} k_1$, then $Y = (\mathcal{X}_s)_{réd}$ is a curve defined over the residue field k_1 such that any irreducible component of Y is regular and it has ordinary double points as singularity.

Let $\overline{Y} = Y \otimes_{k_1} \overline{k_1}$, where $\overline{k_1}$ is an algebraic closure of k_1 and
$\overline{Y}^{[p]} = \bigsqcup_{i_/ < i_1 < \cdots < i_p} \overline{Y_{i_/}} \cap \overline{Y_{i_1}} \cap \cdots \cap \overline{Y_{i_p}}$, $(\overline{Y}_i)_{i\in I}$ = collection of irreducible components of $\overline{Y}$.

Let $|\overline{\Gamma}|$ be a realization of the dual graph $\overline{\Gamma}$, then the group $H^1\left(|\overline{\Gamma}|, \mathbb{Q}_l\right)$ coincides with the group $W_0(H^1\left(\overline{Y}, \mathbb{Q}_l\right))$ constituted of elements of weight 0 for the filtration

$$H^1(\overline{Y}, \mathbb{Q}_\ell) = W_1 \supseteq W_0 \supseteq 0$$

of $H^1(\overline{Y}, \mathbb{Q}_\ell)$ deduced from the spectral sequence

$$E_1^{p,q} = H^q(\overline{Y}^{[p]}, \mathbb{Q}_\ell) \Longrightarrow H^{p+q}(\overline{Y}, \mathbb{Q}_\ell)$$

For details see [2], [3] and [6]

Now, if we assume further that the irreducible components and double points of $\overline{Y}$ are defined over k_1, then the dual graph $\overline{\Gamma}$ of $\overline{Y}$ go down to k_1 and we obtain the injection

$$W_0(H^1\left(\overline{Y}, \mathbb{Q}_l\right)) \subseteq H^1\left(Y, \mathbb{Q}_l\right) \hookrightarrow H^1\left(X, \mathbb{Q}_l\right)$$

Proposition 5.1. *The group $\pi_1^{c.s}(X) \otimes \mathbb{Q}_l$ admits a quotient of type $\mathbb{Q}_l^r$, where r is the $\mathbb{Q}_l-$rank of the group $H^1\left(|\overline{\Gamma}|, \mathbb{Q}_l\right)$*

Proof. We know (4.4) that $\pi_1^{c.s}(X) \otimes \mathbb{Q}_l$ is the dual of the kernel of the map

$$\alpha : H^1\left(X, \mathbb{Q}_l\right) \longrightarrow \prod_{v\in P} H^1\left(k\left(v\right), \mathbb{Q}_l\right)$$

We will prove that $W_0(H^1\left(\overline{Y}, \mathbb{Q}_l\right)) \subseteq Ker\alpha$. The group $W_0 = W_0(H^1\left(\overline{Y}, \mathbb{Q}_l\right))$ is calculated as the homology of the complex

$$H^0(\overline{Y}^{[0]}, \mathbb{Q}_\ell) \longrightarrow H^0(\overline{Y}^{[1]}, \mathbb{Q}_\ell) \longrightarrow 0$$

Hence $W_0 = H^0(\overline{Y}^{[1]}, \mathbb{Q}_\ell)/\operatorname{Im}\{H^0(\overline{Y}^{[0]}, \mathbb{Q}_\ell) \longrightarrow H^0(\overline{Y}^{[1]}, \mathbb{Q}_\ell)\}$. Thus, it suffices to prove the vanishing of the composing map
$H^0(\overline{Y}^{[1]}, \mathbb{Q}_\ell) \longrightarrow W_0 \subseteq H^1\left(Y, \mathbb{Q}_l\right) \hookrightarrow H^1\left(X, \mathbb{Q}_l\right) \longrightarrow H^1\left(k\left(v\right), \mathbb{Q}_l\right)$
for all $v \in P$.

Let z_v be the $0-$ cycle in $\overline{Y}$ obtained by specializing v, which induces a map $z_v^{[1]} \longrightarrow \overline{Y}^{[1]}$. Consequently, the map $H^0(\overline{Y}^{[1]}, \mathbb{Q}_\ell) \longrightarrow H^1\left(k\left(v\right), \mathbb{Q}_l\right)$ factors as follows

$$\begin{array}{ccc} H^0(\overline{Y}^{[1]},\mathbb{Q}_\ell) & \longrightarrow & H^1\left(k\left(v\right),\mathbb{Q}_l\right) \\ \searrow & & \nearrow \\ & H^0(z_v^{[1]},\mathbb{Q}_\ell) & \end{array}$$

But the trace $z_v^{[1]}$ of $\overline{Y}^{[1]}$ on z_v is empty. This implies the vanishing of $H^0(z_v^{[1]},\mathbb{Q}_\ell)$. □

Let $V(X)$ be the kernel of the norm map $N: SK_2\left(X\right) \longrightarrow K_2(k)$ induced by the norm map $N_{k(v)/k^x}: K_2\left(k(v)\right) \longrightarrow K_2(k)$ for all v . Then, we obtain a map $\tau/l : V(X)/\ell \longrightarrow \pi_1^{ab}\left(X\right)^{g\acute{e}o}/\ell$ and a commutative diagram

$$\begin{array}{ccccc} V(X)/\ell & \longrightarrow & SK_2\left(X\right)/\ell & \rightarrow & K_2(k)/\ell \\ \downarrow \tau/l & & \downarrow \sigma/\ell & & \downarrow h/l \\ \pi_1^{ab}\left(X\right)^{g\acute{e}o}/\ell & \longrightarrow & \pi_1^{ab}\left(X\right)/\ell & \rightarrow & Gal(k^{ab}/k)/l \end{array}$$

where the map $h/l : K_2\left(k\right)/l \longrightarrow Gal(k^{ab}/k)/l$ is the one obtained by class field theory of k (section 3). From this diagram we see that the group $Co\ker\tau/l$ is isomorphic to the group $Co\ker\sigma/\ell$. Next, we investigate the map τ/l.

We start by the following result which is a consequence of the structure of the two-dimensional local field k

Lemma 5.2. *There is an isomorphism*

$$\pi_1^{ab}\left(X\right)^{g\acute{e}o} \simeq \pi_1^{ab}\left(\overline{X}\right)_{G_k},$$

where $\pi_1^{ab}\left(\overline{X}\right)_{G_k}$ *is the group of coinvariants under* $G_k = Gal(k^{ab}/k)$.

Proof. As in the proof of Lemma 4.3 of [12], this is an immediate consequence of (Theorem 3.1). □

Finally, we are able to deduce the structure of the group $\pi_1^{ab}\left(X\right)^{g\acute{e}o}$

Theorem 5.3. *The group* $\pi_1^{ab}\left(X\right)^{g\acute{e}o}\otimes\mathbb{Q}_l$ *is isomorphic to* $\widehat{\mathbb{Q}_l}^{r}$ *and the map*

$\tau : V(X) \longrightarrow \pi_1^{ab}\left(X\right)^{g\acute{e}o}$ *is a surjection onto* $(\pi_1^{ab}\left(X\right)^{g\acute{e}o})_{tor}$.

Proof. By the preceding lemma, we have the isomorphism $\pi_1^{ab}\left(X\right)^{g\acute{e}o} \simeq \pi_1^{ab}\left(\overline{X}\right)_{G_k}$. On the other hand the group $\pi_1^{ab}\left(\overline{X}\right)_{G_k}\otimes\mathbb{Q}_\ell$ admits the filtration [12,Lemma 4.1 and section 2]

$$W_0(\pi_1^{ab}\left(\overline{X}\right)_{G_k}\otimes\mathbb{Q}_l) = \pi_1^{ab}\left(\overline{X}\right)_{G_k}\otimes\mathbb{Q}_l \supseteq W_{-1}(\pi_1^{ab}\left(\overline{X}\right)_{G_k}\otimes\mathbb{Q}_l) \supseteq W_{-2}(\pi_1^{ab}\left(\overline{X}\right)_{G_k}\otimes\mathbb{Q}_l)$$

But; by assumption; the curve X admits a semi-stable reduction, then the group $Gr_0(\pi_1^{ab}\left(\overline{X}\right)_{G_k}\otimes\mathbb{Q}_l) = W_0(\pi_1^{ab}\left(\overline{X}\right)_{G_k}\otimes\mathbb{Q}_l)/W_{-1}(\pi_1^{ab}\left(\overline{X}\right)_{G_k}\otimes\mathbb{Q}_l)$ has the following structure

$$0 \longrightarrow Gr_0(\pi_1^{ab}\left(\overline{X}\right)_{G_k}\otimes\mathbb{Q}_l)_{tor} \longrightarrow Gr_0(\pi_1^{ab}\left(\overline{X}\right)_{G_k}\otimes\mathbb{Q}_l) \longrightarrow \widehat{\mathbb{Q}_l}^{r'} \longrightarrow 0$$

where r' is the $k-rank$ of X. This is confirmed by Yoshida [12, section 2], independently of the finitness of the residue field of k considered in his paper. The integer r' is equal to the integer $r = H^1\left(\left|\overline{\Gamma}\right|,\mathbb{Q}_l\right) = H^1\left(\left|\Gamma\right|,\mathbb{Q}_l\right)$ by assuming that the irreducible components and double points of $\overline{Y}$ are defined over k_1.

On the other hand, the exact sequence

$$0 \longrightarrow W_{-1}(\pi_1^{ab}\left(\overline{X}\right)_{G_k}) \longrightarrow \pi_1^{ab}\left(\overline{X}\right)_{G_k} \longrightarrow Gr_0(\pi_1^{ab}\left(\overline{X}\right)_{G_k}) \longrightarrow 0$$

and (Proposition 5.1) allow us to conclude that the group $W_{-1}(\pi_1^{ab}\left(\overline{X}\right)_{G_k})$ is finite and the map $\tau : V(X) \longrightarrow \pi_1^{ab}(X)^{g\acute{e}o}$ is a surjection onto $(\pi_1^{ab}(X)^{g\acute{e}o})_{tor}$ as established by Yoshida in [12] for curve over usually local fields. □

Remark 5.4. If we apply the same method of Saito to study curves over two-dimensional local fields, we need class field theory of two-dimensional local ring having one-dimensional local field as residue field. This is done by myself in [1]. Hence, one can follow Saito 's method to obtain the same results.

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