UNIVERSITÄT BAYREUTH

Simultaneous Torsion in the Legendre Family of Elliptic Curves

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Torsion groups and Galois representations of elliptic curves

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News Alert

On Wednesday, Peter Bruin, Maarten Derickx and I, motivated by Daeyeol Jeon's talk, proved the following.

Theorem.
Up to the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, there is exactly one elliptic curve E defined over a cyclic cubic extension K of $\mathbb{Q}$ such that E is not defined over $\mathbb{Q}$ and $E(K)$ contains a point of order 13.

The curve E is

$$y^2 + (1-r)xy - sy = x^3 - sx^2,$$

where

$$r = \frac{6\alpha^2 + 50\alpha - 208}{3^2 \cdot 13^2} \qquad \text{and} \qquad s = \frac{10\alpha^2 + 90\alpha - 1936}{3^2 \cdot 13^3}$$

and $\alpha^3 - \alpha^2 - 82\alpha + 64 = 0$ (disc$(K) = (13 \cdot 19)^2$; $K = 3.3.61009.1$).

And now for something completely different . . .

Introduction

Consider, for $\lambda \in \mathbb{C} \setminus \{0, 1\}$, the Legendre elliptic curve

$$E_\lambda \colon y^2 = x(x-1)(x-\lambda)\,.$$

For $\alpha \in \mathbb{C} \setminus \{0, 1\}$, let $P_\lambda(\alpha) \in E_\lambda$ be a point with x-coordinate α and define

$$T(\alpha) = \{\lambda \in \mathbb{C} \setminus \{0, 1\} : P_\lambda(\alpha) \in E_\lambda(\mathbb{C}) \text{ is torsion}\}.$$

Then $T(\alpha)$ is a countably infinite set
consisting of elements algebraic over $\mathbb{Q}(\alpha)$.

Now consider $\alpha, \beta \in \mathbb{C} \setminus \{0, 1\}$ with $\alpha \neq \beta$ and set $T(\alpha, \beta) = T(\alpha) \cap T(\beta)$.

Question.
What can we say about $T(\alpha, \beta)$?

Known Results

There are three cases:

- α and β are algebraic.
- $\mathrm{trdeg}_{\mathbb{Q}}\big(\mathbb{Q}(\alpha, \beta)\big) = 1$.
- α and β are algebraically independent. Then $T(\alpha, \beta) = \emptyset$.

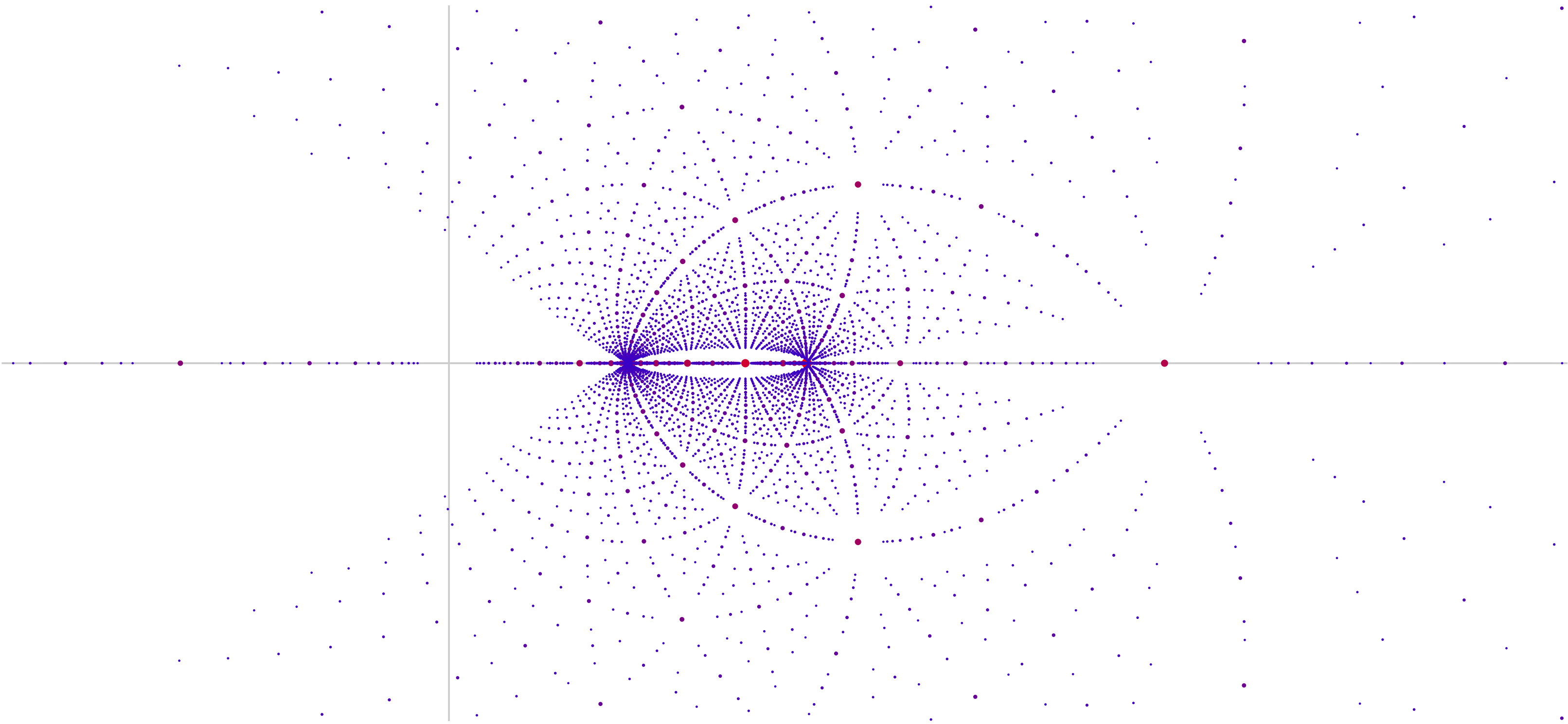
Masser and Zannier showed that $T(2, 3)$ is finite
and then proved the following more general result.

Theorem (Masser and Zannier).
$T(\alpha, \beta)$ is always finite; when $\mathrm{trdeg}_{\mathbb{Q}}\big(\mathbb{Q}(\alpha, \beta)\big) = 1$, this is effective.

Goals of this talk:
(1) Get effectivity for some algebraic α, β.
(2) Get optimal result for transcendence degree 1.
(3) Use this to get more information on the algebraic case.

Structure of $T(\alpha)$

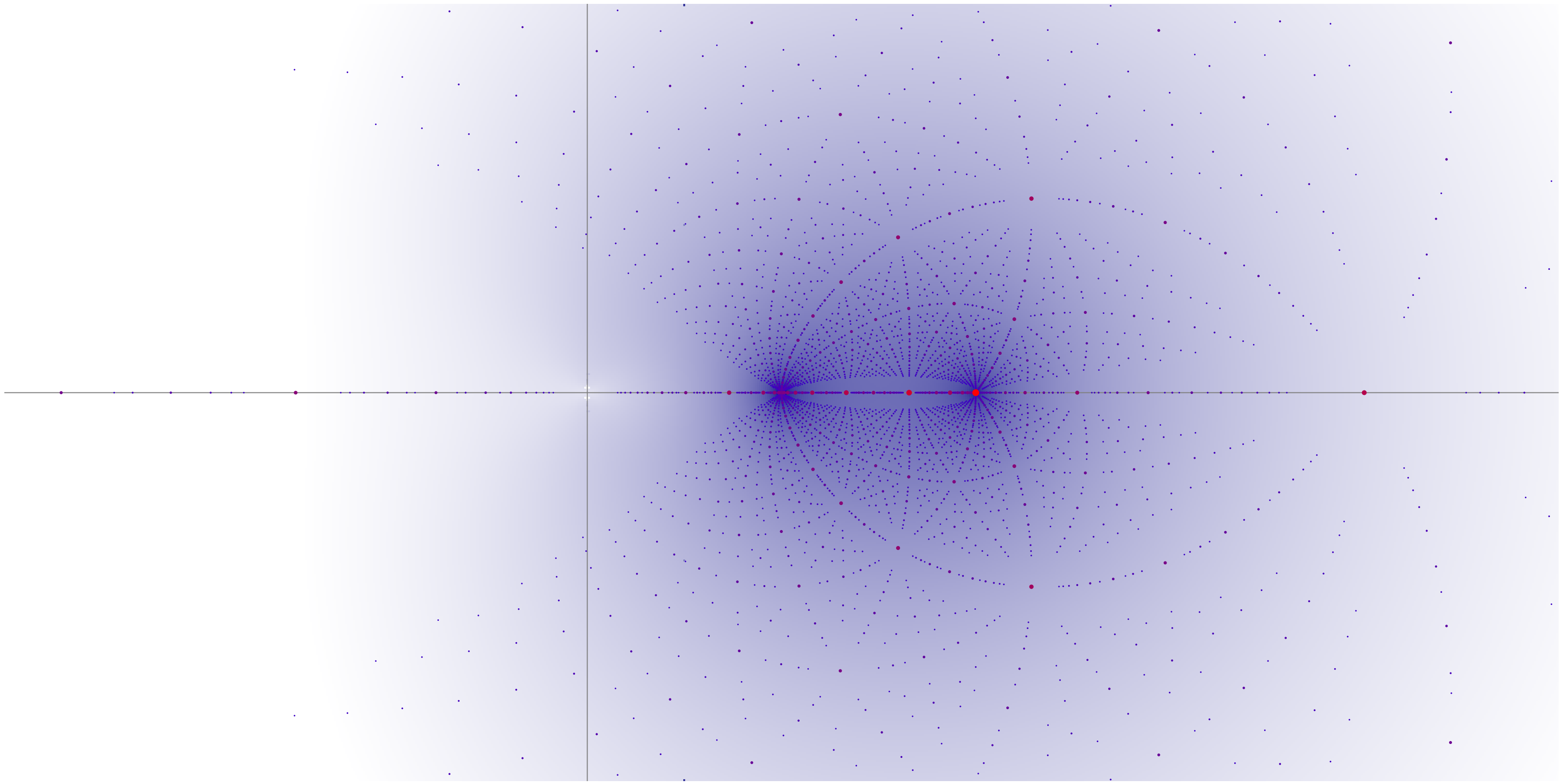
In $\mathbb{C}$, $T(\alpha)$ is all over the place,
reflecting the fact that E_{tors} is dense in $E(\mathbb{C})$:



This shows $T_{40}(2)$, where $T_n(\alpha) = \{\lambda \in T(\alpha) : P_\lambda(\alpha) \in E_\lambda \text{ has order } \leq n\}$.

Aside

DeMarco, Wang and Ye show
that there is actually a limiting distribution μ_α
and that $\mu_\alpha \neq \mu_\beta$ when $\alpha \neq \beta$.



Aside

DeMarco, Wang and Ye show
that there is actually a limiting distribution μ_α
and that $\mu_\alpha \neq \mu_\beta$ when $\alpha \neq \beta$.

So when $T(\alpha, \beta)$ is infinite,
we can approximate both μ_α and μ_β with the same sequence of points,
implying $\mu_\alpha = \mu_\beta$ and therefore $\alpha = \beta$.

This gives an alternative proof of the Masser-Zannier result.

Structure of $T(\alpha)$, p-adically

Fix a prime p.
In contrast to the situation over $\mathbb{C}$, E_{tors} is discrete in $E(\mathbb{C}_p)$.
This translates into $T(\alpha)$ being discrete in $\mathbb{C}_p \setminus \{0, 1\}$.

Since $T(\alpha)$ moves continuously with α,
we can show that $T(\alpha, \beta)$ is empty if α and β are p-adically close:

Proposition.
Let $\alpha, \beta \in \mathbb{C}_p$ with $0 < |\alpha(\alpha - 1)|_p \leq 1$ and $0 < |\beta - \alpha|_p < |\alpha(\alpha - 1)|_p \, |p|_p^{2/(p-1)}$.
Then $T(\alpha, \beta) = \emptyset$.

We also get that $T(\alpha, \beta) = \emptyset$ when $|\alpha|_p < |p|_p^{2/(p-1)}$ and $|\beta - 1|_p < |p|_p^{2/(p-1)}$.

There are slightly better results when $p = 2$.

Application

If $\alpha \in \mathbb{Z}$, then there are only finitely many $\beta \in \mathbb{Z} \setminus \{0,1\}$ with $T(\alpha,\beta) \neq \emptyset$.

Example.
Consider $\alpha = 2$ and $\beta \in \mathbb{Z} \setminus \{0,1\}$.
We will see in a moment that $T(2,\beta) = \emptyset$ when β is odd.
From the above, we get that $T(2,\beta) = \emptyset$ when
$\beta - 2$ is divisible by 8, 9 or a prime $p \geq 5$.
This leaves only $\beta = -10, -4, -2, 4, 6, 8, 14$.

It turns out that the sets $T(2,\beta)$ for these β can all be determined explicitly with the methods discussed later in this talk.

We obtain that $T(2,\beta) = \emptyset$ except for $\beta \in \{-2, 4\}$
and that $T(2,-2) = T(2,4) = \{4\}$.

Idea for Effectivity

If we can show that $T(\alpha) \subset \mathbb{C}_p$ is sufficiently localized,
then we get a handle on $T(\alpha, \beta)$ when α and β are not p-adically close.

Easy Lemma.

For $\alpha, \lambda \in \mathbb{C}_p \setminus \{0, 1\}$ the following are equivalent:

- $\lambda \in T(\alpha)$.
- $\lambda = \alpha$, or $\psi_n(\lambda, \alpha) = 0$ for some $n \geq 3$,
  where $\psi_n(\lambda, x)$ is the nth division polynomial of E_λ.
- α is preperiodic for the Lattès map $f_\lambda \colon x \longmapsto \dfrac{(x^2 - \lambda)^2}{4x(x-1)(x-\lambda)}$ on $\mathbb{P}^1$.
  (This point of view was used by Mavraki.)

2-adic Localization

We look specifically at $p = 2$. $|\cdot|$ denotes the 2-adic absolute value.

It is easy to see that $T(1/\alpha) = \{1/\lambda : \lambda \in T(\alpha)\}$, so we can assume that $|\alpha| \le 1$.
Then for all $\lambda \in T(\alpha)$, we have $|\lambda| \le 1$ as well
(as can be seen from the division polynomials or from the Lattès map).

If $|\lambda| \le 1$ and $x \in \mathbb{C}_2$ has $|x| > 1$, then $|f_\lambda(x)| = 4|x|$,
and x cannot be preperiodic.

So if $\lambda \in T(\alpha)$, we must have that $\lambda = \alpha$ ($\iff f_\lambda(\alpha) = \infty$) or $|f_\lambda(\alpha)| \le 1$.
The latter means $|\lambda - \alpha^2|^2 \le |4\alpha(\alpha - 1)(\alpha - \lambda)| \le |4|$, which says that

$$\lambda \equiv \alpha^2 \bmod 2\,.$$

Corollary. $T(2,3) = \emptyset$.

A Slightly More Precise Result

Note that we have

$$\lambda \in T(\alpha) \iff f_\lambda(\alpha) \in \{0, 1, \lambda, \infty\} \quad \text{or} \quad \lambda \in T(f_\lambda(\alpha)) .$$

The first condition is

$$\lambda \in S(\alpha) := \left\{\alpha, \alpha^2, \alpha(2-\alpha), \tfrac{\alpha^2}{2\alpha-1}\right\} .$$

We can easily show that for $|\alpha| \leq 1$ (similarly for $|\alpha| > 1$),

$$T(f_\lambda(\alpha)) \subset R(\alpha) := \{\alpha^2 + 2u\alpha(1-\alpha) : u \in \mathbb{C}_2, |u^2 - \alpha| < 1\} .$$

So if $R(\alpha) \cap R(\beta) = \emptyset$, then we can determine $T(\alpha, \beta)$:

$$T(\alpha, \beta) \subset S(\alpha) \cup S(\beta) .$$

This will be the case when α and β are 2-adically sufficiently distinct.

Examples

The result applies to show the following.

- $T(2,3) = \emptyset$.
- $T(2,4) = \{4\}$.
- $T(3,-3) = \{-3, 9\}$.
- $T(\omega, \omega^2) = \{\omega, \omega^2\}$, where ω is a primitive cube root of unity.

Let $\boldsymbol{\mu}$ be the set of all roots of unity.
Then $\#(T(\alpha) \cap \boldsymbol{\mu}) \leq 3$ for all α, and

$$\#(T(\alpha) \cap \boldsymbol{\mu}) = 3 \iff \alpha \in \boldsymbol{\mu} \quad \text{and} \quad \mathrm{ord}(\alpha) \in \{3, 6, 12\}.$$

Further Refinement

We can extend this line of argument.
Assume that $|\alpha|, |\beta| \leq 1$ and that $\lambda \in T(\alpha, \beta)$.
Then the x-coordinate of any point $mP_\lambda(\alpha) + nP_\lambda(\beta)$ with $m, n \in \mathbb{Z}$ must be either infinite or of absolute value ≤ 1.

This translates into conditions of the form

$$p(\lambda) = 0 \qquad \text{or} \qquad |p_1(\lambda)| \leq |p_2(\lambda)|$$

for certain polynomials p, p_1, p_2.

If, for some choice of pairs (m, n),
the conditions of the second type are contradictory,
then we have effectively bounded $T(\alpha, \beta)$ by a finite set.

More Examples

- $T(-3,9) = \{9, -\frac{27}{5}\}$ (with $(m,n) = (6,0),\ (0,4)$).
- $T\left(\frac{-3}{5}, \frac{9}{5}\right) = \{\frac{9}{25}, -\frac{27}{5}\}$ (with $(m,n) = (4,0),\ (0,6)$).
- $T\left(\frac{9}{25}, \frac{9}{5}\right) = \{\frac{9}{25}, \frac{189}{125}\}$ (with $(m,n) = (2,0),\ (0,3)$).

Not successful so far for:

- $T\left(-\frac{27}{5}, -\frac{3}{5}\right)$ (another representative in $\mathbb{Q} \times \mathbb{Q}$ with $\#T_{50} = 2$).
- $T\left(-\frac{3}{5}, \frac{9}{25}\right)$ (the essentially only rational pair with $\#T_{50} = 3$).

Question.

Can we always determine $T(\alpha, \beta)$ in this way?

Transcendence Degree 1

Assume that $\mathrm{trdeg}_{\mathbb{Q}}\big(\mathbb{Q}(\alpha,\beta)\big) = 1$
and let $F \in \mathbb{Z}[a,b]$ be irreducible such that $F(\alpha,\beta) = 0$.
Assume that $\lambda \in T(\alpha,\beta)$. Then

$$\big(\lambda = \alpha \text{ or } \exists n \geq 3\colon \psi_n(\lambda,\alpha) = 0\big) \quad \text{and} \quad \big(\lambda = \beta \text{ or } \exists n \geq 3\colon \psi_n(\lambda,\beta) = 0\big).$$

Eliminating λ, we see that F divides
$\psi_n(a,b)$ or $\psi_n(b,a)$ or $R_n(a,b) := \mathrm{Res}_t(\psi_n(t,a),\psi_n(t,b))/(a-b)^{\deg_t \psi_n(t,x)}$,
for some $n \geq 3$.

Proposition 1.
For all $n \geq 3$, the polynomial $\psi_n(a,b)\psi_n(b,a)R_n(a,b)$ is squarefree in $\mathbb{Q}[a,b]$.

Sketch of proof. Write the possible b near $a = 0$ as Puiseux series in a (using Tate parameterization) and check that they are distinct.

Result

Let, for $n \geq 3$, C_n be the curve in $\mathbb{P}^1_a \times \mathbb{P}^1_b$ given by

$$\psi_n(a,b)\psi_n(b,a)R_n(a,b) = 0$$

and let $C = \bigcup_n C_n$ be the filtered union (by divisibility) of the C_n.

By Proposition 1, C is reduced.
This implies that each component of C corresponds
to a family of triples (α, β, λ) with $\lambda \in T(\alpha, \beta)$, where λ is unique.
This gives

Proposition 2.
Let $\alpha, \beta \in \mathbb{C} \setminus \{0, 1\}$ with $\alpha \neq \beta$. Then
$\#T(\alpha, \beta) \leq$ the number of branches of C passing through (α, β).

Consequences

- If $(\alpha, \beta) \notin C$, then $T(\alpha, \beta) = \emptyset$.
  This applies when α and β are algebraically independent.
- If (α, β) is a smooth point on C, then $\#T(\alpha, \beta) \leq 1$.
  This applies when $\operatorname{trdeg}_{\mathbb{Q}}\big(\mathbb{Q}(\alpha, \beta)\big) = 1$ and $T(\alpha, \beta) \neq \emptyset$.
- If $\#T(\alpha, \beta) \geq 2$, then (α, β) is a singular point on a component of C or an intersection point of two or more components of C.

If $F = 0$ describes a component of C, we can bound n in terms of $\deg F$.
This gives effectivity in the $\operatorname{trdeg} = 1$ case.
Note that we have to know F: we can't say whether $T(e, \pi)$ is empty or not!

(Masser and Zannier show $\#T(\alpha, \beta) \leq 6(12 \deg F)^{32}$ when $\operatorname{trdeg} = 1$.)

Computations

We have computed all $F \in \mathbb{Q}[a, b]$ giving irreducible components of C satisfying $\deg_{ab} F := \deg_a F + \deg_b F \leq 192$.

Based on this,
we computed all singularities on components with $(\deg_{ab} F)^2 \leq 384$
and all intersections of components with $(\deg_{ab} F_1)(\deg_{ab} F_2) \leq 384$.
We then computed $T_{50}(\alpha, \beta) = T_{50}(\alpha) \cap T_{50}(\beta)$ for these points (α, β),
leading to $> 2 \cdot 10^6$ pairs with $\#T_{50}(\alpha, \beta) \geq 2$.

558 of these have $\#T_{50}(\alpha, \beta) \geq 3$ (with all torsion orders ≤ 18),
15 of these have $\#T_{50}(\alpha, \beta) \geq 4$,
and 3 of these have $\#T_{50}(\alpha, \beta) = 5$; a representative is $(i, -i)$ with

$$T_{100}(i, -i) = \{-1, 3 \pm 2\sqrt{2}, \tfrac{1}{3} \pm \tfrac{2}{3}\sqrt{-2}\}.$$

Conjectures

Conjecture 1.

$T(i,-i) = \{-1, 3 \pm 2\sqrt{2}, \frac{1}{3} \pm \frac{2}{3}\sqrt{-2}\}$.

Conjecture 2 (Uniform boundedness).

$\#T(\alpha,\beta)$ is uniformly bounded (perhaps by 5).

Conjecture 3 (Finiteness).

There are only finitely many (α,β) with $\#T(\alpha,\beta) \geq 3$.

Conjecture 4 (Bounded height).

The height of (α,β) such that $\#T(\alpha,\beta) \geq 2$ is uniformly bounded.

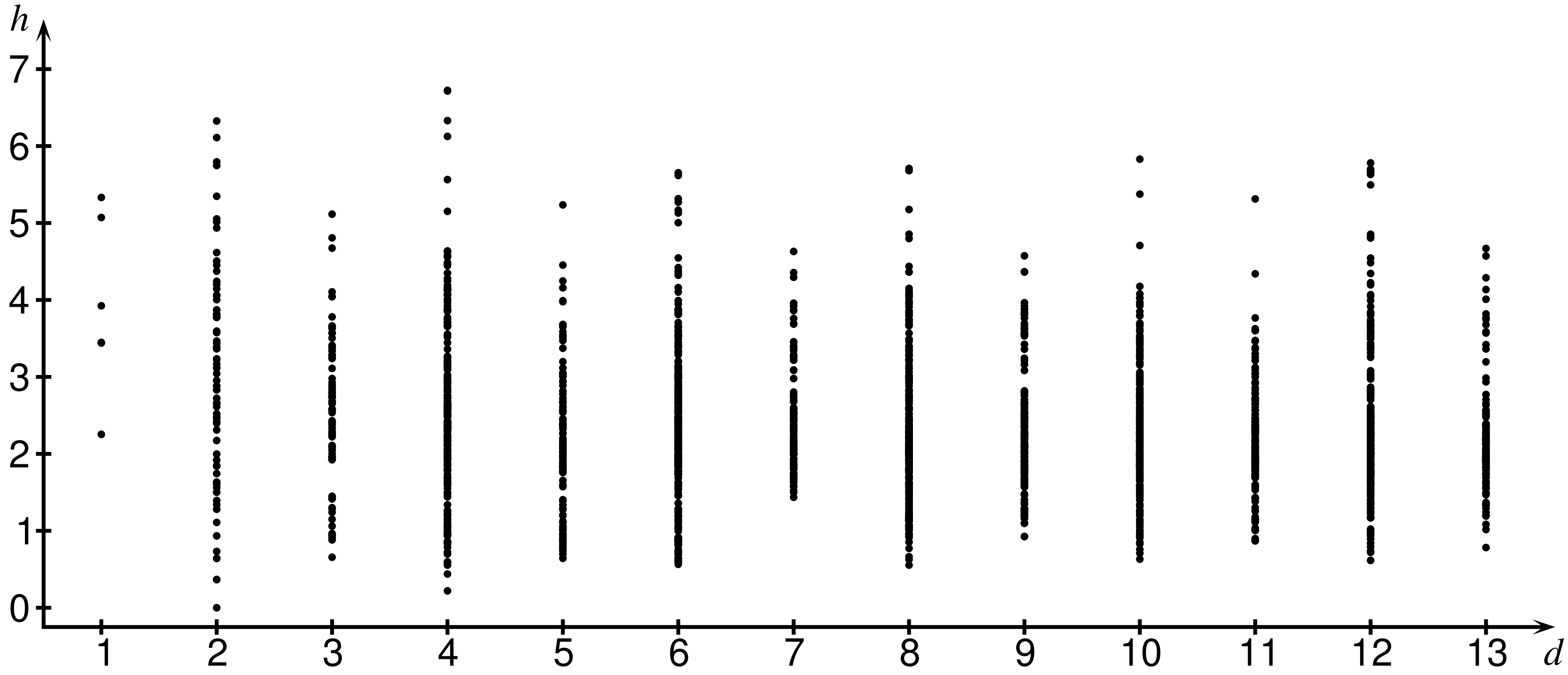
Conjecture 5 (Bounded degree).

There is a uniform bound for $\left[\mathbb{Q}(\alpha,\beta,\lambda) : \mathbb{Q}(\alpha,\beta)\right]$ when $\lambda \in T(\alpha,\beta)$.
The bound might even be 2.

Conjecture 5 would imply effectivity of $T(\alpha,\beta)$.

Heights

This shows the (symmetrized) heights h of pairs (α, β) with $\#T(\alpha, \beta) \geq 2$, ordered according to the degree d of $\mathbb{Q}(\alpha, \beta)$.



Thank You!