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# How many rational points can a curve have?

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# The Basic Trichotomy

Let  $C$  be a **nice\*** curve over a **number field**  $K$ .

We are interested in its set of **rational points**,  $C(K)$ .

If  $C(K) = \emptyset$ , there is nothing else to say.

So we assume that we have a **pointed curve**  $(C, P_0)$  over  $K$ ,  
i.e., a nice curve  $C$  over  $K$  together with a **point**  $P_0 \in C(K)$ .

The **structure** of the set  $C(K)$  depends on the **geometry** of the curve,  
which is governed by its **genus**  $g$ .

- $g = 0$ :  $(C, P_0) \cong (\mathbb{P}^1, \infty)$ ; **lots** of points.
- $g = 1$ :  $(C, P_0) \cong (E, 0)$  is an **elliptic curve**;  
 $C(K)$  is a **finitely generated abelian group** (Mordell-Weil).
- $g \geq 2$ :  $C(K)$  is **finite** (Faltings).

\*smooth, projective, geometrically irreducible

# Questions

- $g \geq 2$ :  $C(K)$  is finite.

So for curves of “higher genus”  $g \geq 2$ , it makes sense to ask about the size  $\#C(K)$  of the set of rational points.

- Fixing  $g$  and  $K$  (or  $[K : \mathbb{Q}]$ ), is  $\#C(K)$  bounded?
- If so, how does the bound grow with  $g$ ?

Caporaso-Harris-Mazur, Pacelli (1997):

Weak Lang Conjecture<sup>†</sup> implies  $\#C(K) \leq B(g, [K : \mathbb{Q}])$

Mestre (published in C-H-M 1994):  $B(g, 1) \geq 8g + 12$

Stoll (2008):  $B(2, 1) \geq 642$

<sup>†</sup>rational points are not Zariski-dense on varieties of general type

# The Jacobian

Basically all results on  $C(K)$  are obtained via the **Jacobian variety**  $J$  of  $C$ .  
 $J$  is an **abelian variety** of **dimension**  $g$ .

By the (Mordell-)Weil Theorem,  
its **group**  $J(K)$  of  **$K$ -rational points** is **finitely generated**.

When  $g > 0$ , there is an **embedding**  $i: (C, P_0) \rightarrow (J, 0)$ ,  
so that  $C(K) \cong \text{im}(i) \cap J(K)$ .

A generalization of Faltings's theorem:

( $\star$ ) If  $\Gamma \subset J(\mathbb{C})$  is of **finite rank**<sup>‡</sup>, then  $\text{im}(i) \cap \Gamma$  is **finite**.

So Weil + ( $\star$ )  $\implies \#C(K) < \infty$ .

<sup>‡</sup> $\dim_{\mathbb{Q}} \Gamma \otimes_{\mathbb{Z}} \mathbb{Q} < \infty$

## Another Question

(★) If  $\Gamma \subset J(\mathbb{C})$  is of finite rank, then  $\text{im}(i) \cap \Gamma$  is finite.

This raises another uniformity question:

- Can  $\#(\text{im}(i) \cap \Gamma)$  be bounded in terms of  $g$  and  $\text{rk } \Gamma$ ?

This was answered fairly recently, building on Vojta's (1991) approach to Faltings's Theorem, by Dimitrov-Gao-Habegger and Kühne (2021) and by Yuan (2023):

### **Theorem.**

For  $g \geq 2$ , there are  $c_1(g), c_2(g) > 0$  such that  $\#(\text{im}(i) \cap \Gamma) \leq c_1(g) \cdot c_2(g)^{\text{rk } \Gamma}$  for all curves  $C$  of genus  $g$  (and  $i$  as above) and all subgroups  $\Gamma \subset J(\mathbb{C})$ .

# A Very Recent Result

## Theorem.

For  $g \geq 2$ , there are  $c_1(g), c_2(g) > 0$  such that  $\#(\text{im}(i) \cap \Gamma) \leq c_1(g) \cdot c_2(g)^{\text{rk } \Gamma}$  for all curves  $C$  of genus  $g$  (and  $i$  as above) and all subgroups  $\Gamma \subset J(\mathbb{C})$ .

The obvious next question is

- Can we give **explicit values** for  $c_1(g)$  and  $c_2(g)$ ?

In an arXiv preprint from February 2026, Yuan and coauthors show:

## Theorem.

We can take  $c_1(g) = 10^{13}g^8$  and  $c_2(g) = 1 + \min\left\{\frac{5}{4\sqrt{g}}, \frac{3 \log g}{g}\right\}$ .

Note that  $c_2(g) \rightarrow 1$  as  $g \rightarrow \infty$ ; this was expected.

# A Uniform Bound

## Theorem.

For  $g \geq 2$ , there are  $c_1(g), c_2(g) > 0$  such that  $\#(\text{im}(i) \cap \Gamma) \leq c_1(g) \cdot c_2(g)^{\text{rk } \Gamma}$  for all curves  $C$  of genus  $g$  (and  $i$  as above) and all subgroups  $\Gamma \subset J(\mathbb{C})$ , where  $c_1(g) = 10^{13}g^8$  and  $c_2(g) = 1 + \min\left\{\frac{5}{4\sqrt{g}}, \frac{3 \log g}{g}\right\}$ .

We can apply this with  $\Gamma = J(K)$  to obtain  $\#C(K) \leq c_1(g) \cdot c_2(g)^{\text{rk } J(K)}$ .

The example that gives  $B(2, 1) \geq 642$  has  $\text{rk } J(\mathbb{Q}) = 22^{\S}$ ; compare with  $c_1(2) \cdot c_2(2)^{22} \approx 2.88 \cdot 10^{21}$ .

Assuming that  $\text{rk } A(\mathbb{Q}) \leq B'(g)$

for all abelian varieties  $A$  of dimension  $g$  over  $\mathbb{Q}$ ,

we obtain the uniform bound  $\#C(K) \leq c_1(g) \cdot c_2(g)^{B'(g[K:\mathbb{Q}])}$ .

<sup>§</sup>under GRH

# What Should We Expect?

Considerations based on conjectures around **unlikely intersections** suggest that perhaps  $\#C(K) \ll g + \text{rk } J(K)$ .

(We get  $\#C(K) \geq 2g + 2 + 2 \text{rk } J(K)$  easily from hyperelliptic curves, so this is best possible.)

For **small rank**, there are results getting quite close:

- Stoll (2019):

$$\#C(K) \ll_{[K:\mathbb{Q}]} (\text{rk } J(K) + 1) \cdot g \quad \text{if } C \text{ is hyperelliptic and } \text{rk } J(K) \leq g - 3.$$

- Katz-Rabinoff-Zureick-Brown (2016):

$$\#C(K) \ll_{[K:\mathbb{Q}]} g^2 \quad \text{if } \text{rk } J(K) \leq g - 3.$$

The constants involved are **explicit**, and **small** ( $< 100$ ) for  $K = \mathbb{Q}$ .

Note, however, that these results are **not uniform in K**.

# Isotrivial Families

A result with a slightly different flavor:

**Theorem** (Stoll).

Let  $\mathcal{C} \rightarrow S$  be an **isotrivial family** of nice curves of genus  $g \geq 2$  defined over  $K$ , with induced **family of Jacobians**  $\mathcal{J} \rightarrow S$ .

Then, away from **finitely many  $K$ -isomorphism classes**,

$$\#\mathcal{C}_s(K) \leq 82(g-1) + 2 \operatorname{rk} \mathcal{J}_s(K)$$

for all  $s \in S(K)$  such that  $\operatorname{rk} \mathcal{J}_s(K) \leq g-1$ .

# Comparison And Open Questions

Note that  $c_2(g)^g \leq \left(1 + \frac{3 \log g}{g}\right)^g \leq g^3$ .

If we **assume** that  $\text{rk } A(\mathbb{Q}) \leq \gamma \cdot \dim A$  for abelian varieties  $A$  over  $\mathbb{Q}$ , then this gives  $\#C(K) \leq 10^{13} \cdot g^{8+3\gamma[K:\mathbb{Q}]}$ .

This leads to the first of the following open questions:

- What can we say about  $\limsup_{\dim A \rightarrow \infty} \frac{\text{rk } A(\mathbb{Q})}{\dim A} = \sup_A \frac{\text{rk } A(\mathbb{Q})}{\dim A}$ ?  
( $A$  abelian variety over  $\mathbb{Q}$ .) This is  $\geq 29$  (Elkies-Klagsbrun).
- What can we say about  $\limsup_{g(C) \rightarrow \infty} \frac{\#C(\mathbb{Q})}{g(C)}$  and/or  $\sup_C \frac{\#C(\mathbb{Q})}{g(C)}$ ?  
( $C$  nice curve over  $\mathbb{Q}$  with  $g(C) \geq 2$ .) They are  $(\geq 8, \geq 321)$ .

# Summary

- The study of **uniformity questions** related to **rational points on curves** or **points mapping into finite-rank subgroups** of the Jacobian has made stunning success in the recent past.
- The **explicit bounds** we now have are still **much larger** than what we **expect to be true**.
- There are **better bounds** that apply in **restricted settings**.
- A **very interesting** (but certainly **extremely hard**) question is whether certain quantities ( **$\text{rk } A(\mathbb{Q})$** ,  **$\#C(\mathbb{Q})$** ) grow **linearly** or **superlinearly** with the dimension/genus.

Thank You!