

THE SURFACE PARAMETRIZING CUBOIDS

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ABSTRACT. We study the surface \bar{S} parametrizing cuboids: it is defined by the equations relating the sides, face diagonals and long diagonal of a rectangular box. It is an open problem whether a ‘rational box’ exists, i.e., a rectangular box all of whose sides, face diagonals and long diagonal have (positive) rational length. The question is equivalent to the existence of nontrivial rational points on \bar{S} .

Let S be the minimal desingularization of \bar{S} (which has 48 isolated singular points). The main result of this paper is the explicit determination of the Picard group of S , including its structure as a Galois module over \mathbb{Q} . The main ingredient for showing that the known subgroup is actually the full Picard group is the use of the combined action of the Galois group and the geometric automorphism group of S (which we also determine) on the Picard group. This reduces the proof to checking that the hyperplane section is not divisible by 2 in the Picard group.

We use our explicit knowledge of the Picard group, together with that of a K3 surface obtained as a quotient of S , to study curves of low degree on \bar{S} . In this way, we completely classify all integral curves of degree at most 6 on \bar{S} .

1 Introduction

Let \mathbb{P}^6 be the projective space over \mathbb{Q} with homogeneous coordinates $a_1, a_2, a_3, b_1, b_2, b_3, c$; let \bar{S} be the surface in \mathbb{P}^6 defined by

$$(1.1) \quad \begin{cases} a_1^2 + b_1^2 = c^2 \\ a_2^2 + b_2^2 = c^2 \\ a_3^2 + b_3^2 = c^2 \\ a_1^2 + a_2^2 + a_3^2 = c^2 \end{cases}$$

and note that the equations in (1.1) are equivalent to the equations

$$(1.2) \quad \begin{cases} a_1^2 + a_2^2 = b_3^2 \\ a_1^2 + a_3^2 = b_2^2 \\ a_2^2 + a_3^2 = b_1^2 \\ a_1^2 + a_2^2 + a_3^2 = c^2. \end{cases}$$

These equations encode the relations between the three sides a_1, a_2, a_3 , the three face diagonals b_1, b_2, b_3 and the long diagonal c of a three-dimensional rectangular box.

The interest in this surface comes from a famous open problem:

Does there exist a ‘rational box’?

A rational box is a (non-degenerate) rectangular box all of whose sides, face diagonals and long diagonals have rational length. The existence of a rational box is therefore

Date: February 24, 2025.

2020 Mathematics Subject Classification. 14J29, 14C22, 14J50, 14G05.

equivalent to the existence of a rational point on \bar{S} with $a_1 a_2 a_3 \neq 0$. See van Luijk's undergraduate thesis [vL00] for a summary of the literature on this problem.

In this paper, we hope to make progress toward a better understanding of the rational box surface by proving some results on its geometry. This extends results obtained in [vL00].

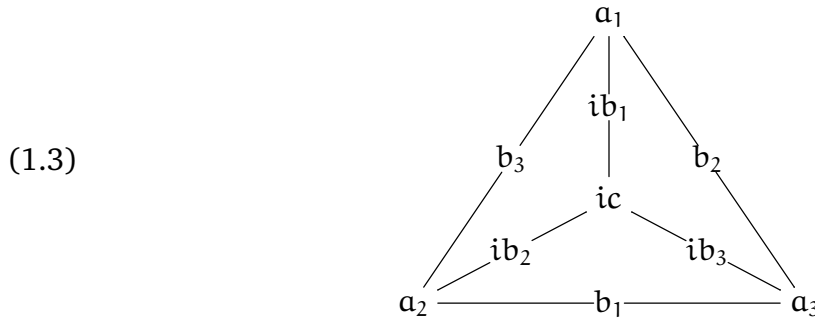
The surface \bar{S} has 48 isolated A_1 singularities. We let S denote the minimal desingularization of \bar{S} . Then we show the following.

Theorem 1. $\text{Aut}_{\mathbb{Q}}(S) = \text{Aut}_{\mathbb{Q}}(\bar{S})$ is an explicitly given group G of order 1536. Its action on \bar{S} extends to a linear action on the ambient \mathbb{P}^6 .

This group comes from the obvious independent sign changes on all coordinates and the equally obvious simultaneous permutations of the a and b coordinates, together with a less obvious automorphism. This leads to an exact sequence

$$1 \longrightarrow \mu_2^7 / \mu_2 \longrightarrow G \longrightarrow \mathfrak{S}_4 \longrightarrow 1,$$

where the kernel comes from the sign changes and the action of the \mathfrak{S}_4 quotient can be visualized as the action of the symmetry group of the tetrahedron in the diagram below (where an edge x — y — z corresponds to one of the rank 3 quadrics $x^2 + z^2 = y^2$ defining \bar{S}).



See Section 2 for details. The group was already known to van Luijk; we prove here that it is already the full automorphism group.

Theorem 2. *The geometric Picard group of S has maximal rank*

$$\text{rank Pic } S_{\mathbb{Q}} = \dim H^{1,1}(S(\mathbb{C})) = 64.$$

It is generated by an explicitly known set of curves on \bar{S} together with the exceptional divisors. The discriminant of the intersection pairing on the Picard group is -2^{28} .

We can take a suitable subset of 64 of the following curves (together with the 48 exceptional divisors) as generators (see Definition 6).

- The 32 strict transforms of the conics in the four hyperplanes $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $c = 0$;
- the 12 strict transforms of the genus 1 curves contained in the three hyperplanes $b_1 = 0$, $b_2 = 0$, $b_3 = 0$;
- the 48 strict transforms of the genus 1 curves contained in the twelve hyperplanes $a_j = \varepsilon a_{j+1}$ (where we set $a_4 := a_1$) or $a_j = \varepsilon ic$, where $j \in \{1, 2, 3\}$ and $\varepsilon \in \{1, -1\}$.

See Section 3. It is not hard to show that the geometric Picard rank is 64, since one easily finds enough curves to generate a group of that rank. These curves are already in [vL00]. The hard part is to show that the known curves generate the full Picard group

and not a proper subgroup of finite index. Since 2 is the only prime number dividing the discriminant of the known subgroup, it remains to show that no primitive element of the known subgroup is divisible by 2. We use the known action of the automorphism group together with the action of the absolute Galois group of \mathbb{Q} to reduce the proof of saturation to the statement that the single element corresponding to the hyperplane section is not divisible by 2. This claim is then fairly easily established.

This technique, especially the arguments in the proof of Theorem 8, may be helpful in similar situations, when one has a fairly large group acting on the Picard group. Indeed, A. Várilly-Alvarado and B. Viray [VAV11] use this technique to compute the Picard groups of various Enriques surfaces.

Several papers investigate restrictions that curves of geometric genus 0 or 1 on a surface must satisfy. The work of Bogomolov in [Bog77] highlighted the importance of symmetric differentials.

In [BTVA22], the authors study global sections of symmetric differentials on certain surfaces of general type. They establish what effect the singularities of the surface have on the sections that they determine. In turn, this allows them to deduce lower bounds for the number of singularities that the curves must contain, as well as information about dimensions of the linear spaces that the singularities must span. In particular, when they apply their methods to the surface of cuboids [BTVA22, Theorem 1.2], they conclude that any rational curve, other than the known conics, must pass through at least 7 nodes spanning the ambient \mathbb{P}^6 . See the corresponding paper for more details.

In [GFU20], the authors also study symmetric differentials, but, rather than finding several explicit sections, they focus on solving the implied algebraic differential equations that they impose on curves of genus at most 1. This method was pioneered by Vojta [Voj00]. When they apply their methods to the surface of cuboids [GFU20, Theorem 1.2], they conclude that all curves of genus at most 1 on \bar{S} must contain at least 2 singular points. Moreover, apart from the exceptional curves and the curves contained in $a_1 a_2 a_3 c = 0$, every rational curve on \bar{S} must contain at least 8 singular points (counted with multiplicity). See the corresponding paper for more details.

We obtain a similar result with our methods (see Lemma 21): Any rational curve C on \bar{S} that is not a conic must satisfy $C \cdot E \geq 8$, where E is the exceptional divisor (and we identify C with its strict transform), and any curve C of geometric genus 1 on \bar{S} must satisfy $C \cdot E \geq 4$ (this gives an improvement over the result in [GFU20]).

In [FSM16], the authors study the surface of cuboids using theta-functions. They exploit the fact that the surface is a divisor in a Siegel modular variety and determine explicitly a modular group associated to the surface. In particular, they find an explicit product of modular curves that covers the surface of cuboids. This allows them to find bounds for the degree of a unibranch curve on the surface in terms of the genus of the curve: see [FSM16, Theorem 3.1]. For instance, unibranch rational curves on \bar{S} must have degree at most 176. See the corresponding paper for more details.

This paper contains a number of computational results. We provide a Magma [BCP97] script at [Sto25] that contains code that checks most of the computational claims we make (set `quick := false` to also run checks that take more time). We also provide a transcript of a Magma session where we have computed equations for the fibrations given in Section 5.

2 The Automorphism Group

Throughout the paper, unless stated explicitly, all objects will be considered over $\bar{\mathbb{Q}}$ (or \mathbb{C}). In particular, $\text{Pic } S$ denotes the geometric Picard group and $\text{Aut}(S)$ the geometric automorphism group of S .

In this section, we determine the automorphism group of S and \bar{S} . We begin with some basic geometric properties of \bar{S} .

Lemma 3. *The scheme \bar{S} is a geometrically integral complete intersection of dimension two and multidegree $(2, 2, 2, 2)$ in \mathbb{P}^6 with 48 isolated A_1 singularities.*

Proof. Every irreducible component of \bar{S} has dimension at least two, since \bar{S} is defined by four equations. If \bar{S} had a component of dimension at least three, then the intersection of \bar{S} with any hyperplane would have a component of dimension at least two. Let C be the hyperplane defined by the vanishing of c . The scheme $\bar{S} \cap C$ is the union of eight smooth conics defined over the field $\mathbb{Q}(i)$ by

$$c = 0, \quad b_1 = \varepsilon_1 i a_1, \quad b_2 = \varepsilon_2 i a_2, \quad b_3 = \varepsilon_3 i a_3, \quad a_1^2 + a_2^2 + a_3^2 = 0$$

for $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{\pm 1\}$. In particular it is pure of dimension one, so that every irreducible component of \bar{S} has dimension two; note also that $\bar{S} \cap C$ is reduced. It follows that \bar{S} is the complete intersection of the equations in (1.1).

Thus \bar{S} is Cohen-Macaulay of dimension two, and to prove that it is integral it suffices to show that the singular locus of \bar{S} has codimension at least two. Since $\bar{S} \cap C$ is reduced, no component of dimension one of the singular locus of \bar{S} is contained in C . Let $p = [\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma]$ be a point in $\bar{S} \setminus C$. Examining the equations of \bar{S} we see immediately that if $\alpha_1 \alpha_2 \alpha_3 \neq 0$ or $\beta_1 \beta_2 \beta_3 \neq 0$, then the rank of the Jacobian of the equations at p is four and such points are therefore smooth. We conclude at once that the singular points of \bar{S} are the points for which all three coordinates appearing in one of the six rank three quadrics in (1.1) or (1.2) vanish; in particular, the surface \bar{S} has only finitely many singular points. The fact that the singular points are 48 and that they are of type A_1 is immediate from the equations. \square

As a corollary, we deduce that \bar{S} is Cohen-Macaulay, Gorenstein, reduced, normal and projectively normal. By the adjunction formula, the canonical sheaf on \bar{S} is the sheaf $\mathcal{O}_{\bar{S}}(1)$. Since \bar{S} is projectively normal we deduce also that $p_g(\bar{S}) = 7$; it is an easy calculation to see that $\chi(\bar{S}, \mathcal{O}_{\bar{S}}) = 8$. Let $b: S \rightarrow \bar{S}$ be the blow-up of \bar{S} at its 48 singular points; thus S is smooth and it is the minimal desingularization of \bar{S} . Denote by K_S a canonical divisor of S ; we have $\mathcal{O}_S(K_S) \simeq b^* \mathcal{O}_{\bar{S}}(1)$ and $(K_S)^2 = 16$. Since the singularities of \bar{S} are rational double points, we have $\chi(S, \mathcal{O}_S) = \chi(\bar{S}, \mathcal{O}_{\bar{S}}) = 8$ and also $p_g(S) = 7$ and $q(S) = 0$. Using Noether's formula we finally deduce that the Hodge diamond of S is

$$\begin{array}{ccccc} & & & & 1 \\ & & & 0 & 0 \\ & & 7 & 64 & 7 \\ & & 0 & 0 & \\ & & & & 1 \end{array}$$

It follows from the above that the rational map associated to the canonical divisor K_S on S is a morphism and that it is the contraction of the 48 exceptional curves of b , followed by the inclusion of \bar{S} into \mathbb{P}^6 . Therefore the canonical divisor on S is big

and nef, so that S is a minimal surface of general type and \bar{S} is its canonical model: there are no curves on S with negative intersection with the canonical divisor, and, in particular, there are no (-1) -curves on S . Moreover, the 48 exceptional curves of b are the only (-2) -curves on S . Since the morphism $S \rightarrow \mathbb{P}^6$ is the morphism associated to the canonical divisor, the automorphism groups of S and \bar{S} coincide; denote this group by G . The group G is naturally identified with the subgroup of $\text{Aut}(\mathbb{P}^6)$ preserving the subscheme \bar{S} , since the canonical divisor class of S is G -invariant.

The symmetric group \mathfrak{S}_3 on $\{1, 2, 3\}$ acts on \mathbb{P}^6 and \bar{S} by permuting simultaneously the indices of a_1, a_2, a_3 and b_1, b_2, b_3 and fixing c . Note that also the linear automorphism of order two

$$\sigma: \begin{cases} a_1 \mapsto a_1 & b_1 \mapsto -ib_2 \\ a_2 \mapsto a_2 & b_2 \mapsto ib_1 \\ a_3 \mapsto -ic & b_3 \mapsto b_3 \\ c \mapsto ia_3 \end{cases}$$

of \mathbb{P}^6 preserves \bar{S} and therefore induces an automorphism of S . Let $G' \subset G$ be the subgroup generated by \mathfrak{S}_3 , σ and all the sign changes of the variables.

Proposition 4. *The group $G = \text{Aut}(S)$ is equal to G' .*

Proof. First, we show that the rank three quadrics vanishing on \bar{S} are the six rank three quadrics appearing in (1.1) and (1.2). For $j \in \{1, 2, 3\}$, let

$$q_j := a_j^2 + b_j^2 - c^2 \quad \text{and} \quad r_j := a_1^2 + a_2^2 + a_3^2 - a_j^2 - b_j^2,$$

and let $Q_j := V(q_j)$ and $R_j := V(r_j)$. Let

$$q_4 := a_1^2 + a_2^2 + a_3^2 - c^2,$$

and let q be an equation of a quadric vanishing on \bar{S} . Since the ideal of \bar{S} is generated by the equations in (1.1), we have $q = \lambda_1 q_1 + \lambda_2 q_2 + \lambda_3 q_3 + \lambda_4 q_4$, for some $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \bar{\mathbb{Q}}$.

If at least two of the coefficients $\lambda_1, \lambda_2, \lambda_3$ are non-zero, then we easily see that q has rank at least four. Thus at most one of the coefficients $\lambda_1, \lambda_2, \lambda_3$ is non-zero, and we conclude that the rank three quadrics vanishing on \bar{S} are $q_1, q_2, q_3, r_1, r_2, r_3$. It follows that the set $\mathcal{Q} := \{Q_1, Q_2, Q_3, R_1, R_2, R_3\}$ is fixed by the induced action of G .

Second, we show that G' acts transitively on \mathcal{Q} . This is immediate: the group \mathfrak{S}_3 acts transitively on $\{Q_1, Q_2, Q_3\}$ and $\{R_1, R_2, R_3\}$, while σ acts transitively on $\{Q_1, R_2\}$ and $\{Q_2, R_3\}$ (and it fixes Q_3 and R_1).

Third, we show that the stabilizer of Q_1 in G is the same as the stabilizer of Q_1 in G' ; from this the result follows since G and G' act transitively on \mathcal{Q} . Let $\tau \in G$ be an automorphism of \mathbb{P}^6 stabilizing \bar{S} and Q_1 . In particular τ stabilizes Q_1^{sing} , the singular locus of Q_1 , and $Q_1^{\text{sing}} \cap \bar{S}$. The (underlying set of the) intersection $Q_1^{\text{sing}} \cap \bar{S}$ is the set of eight points $\{[0, 1, \varepsilon_1, 0, \varepsilon_2 i, \varepsilon_3 i, 0] : \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{1, -1\}\}$. Clearly the actions of the stabilizers in G and G' are transitive on this set, since both groups contain arbitrary sign changes of the variables, and if an automorphism of $V(a_1, b_1, c) \simeq \mathbb{P}^3$ fixes the eight points above, then it is the identity (the points for which $\varepsilon_1 \varepsilon_2 \varepsilon_3 = 1$ are essentially the characters of $(\mathbb{Z}/2\mathbb{Z})^2$, and therefore the corresponding points span \mathbb{P}^3). Thus, we may assume that τ acts as the identity on $V(a_1, b_1, c) \subset \mathbb{P}^6$. Hence τ fixes the variable c , up to a sign, since it fixes a_2, b_2 ; similarly it fixes a_1 and b_1 up to signs, since it fixes a_2, b_3 and a_2, a_3 respectively, and we conclude that $G' = \text{Aut}(\bar{S}) = \text{Aut}(S)$. \square

To compute the size of the group G , note that each element of $G = G'$ induces a permutation of the set $\{a_1^2, a_2^2, a_3^2, -c^2\}$ and every permutation is obtained. This gives a surjective group homomorphism $G \rightarrow \mathfrak{S}_4$. The subgroup $\langle \mathfrak{S}_3, \sigma \rangle$ permutes $\{a_1, a_2, a_3, -ic\}$. One can check that the kernel of $\langle \mathfrak{S}_3, \sigma \rangle \rightarrow \mathfrak{S}_4$ consists of maps that fix b_1^2, b_2^2, b_3^2 . So the kernel of $G \rightarrow \mathfrak{S}_4$ consists exactly of the automorphisms that change the signs of a subset of the variables. This shows that

$$\#G = 2^{7-1} \# \mathfrak{S}_4 = 64 \cdot 24 = 1536.$$

The group G' is described in [vL00, p. 25] as a subgroup of the automorphism group of \bar{S} . The new statement here is that it is already the full automorphism group.

3 The Picard Group

In this section, we determine the (geometric) Picard group of S . We first show that the hyperplane section of \bar{S} is not divisible in the Picard group. Recall that the canonical class of S is the pull-back of the hyperplane class of \bar{S} .

Lemma 5. *The canonical divisor class of S is a primitive vector in $\text{Pic } S$.*

Proof. Let K_S be a canonical divisor of S ; since $(K_S)^2 = 16$, it suffices to show that there is no divisor R on S such that $K_S \sim 2R$. We argue by contradiction and suppose that such a divisor R exists. By the Riemann-Roch formula we deduce that

$$h^0(S, \mathcal{O}_S(R)) + h^2(S, \mathcal{O}_S(R)) \geq 6,$$

and from Serre duality it follows that $h^2(S, \mathcal{O}_S(R)) = h^0(S, \mathcal{O}_S(R))$; thus

$$h^0(S, \mathcal{O}_S(R)) \geq 3.$$

Note also that $2 \dim |R| \leq \dim |2R| = 6$, and therefore $h^0(S, \mathcal{O}_S(R)) \in \{3, 4\}$. The image S' of S under the rational map determined by the linear system $|R|$ is an irreducible, non-degenerate subvariety of $|R|^\vee$. Let k be the number of independent quadratic equations vanishing along S' . The image of $\text{Sym}^2 H^0(S, \mathcal{O}_S(R))$ in $H^0(S, \mathcal{O}_S(K_S))$ is an $\text{Aut}(S)$ -invariant subspace. We easily see that the non-trivial $\text{Aut}(S)$ -invariant subspaces of \mathbb{P}^6 are $V(a_1, a_2, a_3, c)$ and $V(b_1, b_2, b_3)$. It follows that the image of $\text{Sym}^2 H^0(S, \mathcal{O}_S(R))$ in $H^0(S, \mathcal{O}_S(K_S))$ has dimension

$$\binom{h^0(S, \mathcal{O}_S(R)) + 1}{2} - k$$

and that this number must be in $\{0, 3, 4, 7\}$.

Case 1: $h^0(S, \mathcal{O}_S(R)) = 3$, so that $S' \subset \mathbb{P}^2$.

Since S' is irreducible and non-degenerate, we deduce that S' cannot be contained in two independent quadrics, or it would be degenerate. Thus S' must be defined by $k \leq 1$ quadrics, and this is incompatible with the previous constraints.

Case 2: $h^0(S, \mathcal{O}_S(R)) = 4$, so that $S' \subset \mathbb{P}^3$.

Thus $k \geq 3$. Since S' is irreducible and non-degenerate, any quadric vanishing on S' must be irreducible. Hence, any two independent quadrics vanishing along S' intersect in a scheme of pure dimension one and degree four. Since S' is non-degenerate, its degree is at least three. Because there are at least three independent quadrics vanishing on S' , this implies $k = 3$, and S' is a twisted cubic curve. But then the map

$\text{Sym}^2 H^0(S, \mathcal{O}_S(\mathbb{R})) \rightarrow H^0(S, \mathcal{O}_S(K_S))$ is surjective, which implies that the image of S under K_S factors through the image under R . So \bar{S} would have to be a curve, contradicting the fact that it is a surface.

Thus the class of the canonical divisor K_S is not the double of a divisor class on S , and the proof is complete. \square

Next we prove that $\text{Pic } S$ is a free abelian group of rank 64 and find a set of generators of a subgroup of finite index.

Definition 6. We list some sets of curves on S (we let $a_4 := a_1$, to simplify the notation).

- (1) \mathcal{G}_0 — the 48 exceptional curves of the resolution $S \rightarrow \bar{S}$ (24 defined over \mathbb{Q} , 24 defined over $\mathbb{Q}(i)$);
- (2) \mathcal{G}_1 — the 32 strict transforms of the conics in the four hyperplanes $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $c = 0$ (24 defined over \mathbb{Q} , 8 defined over $\mathbb{Q}(i)$);
- (3) \mathcal{G}_2 — the 12 strict transforms of the genus 1 curves contained in the three hyperplanes $b_1 = 0$, $b_2 = 0$, $b_3 = 0$ (defined over $\mathbb{Q}(i)$);
- (4) \mathcal{G}_3 — the 48 strict transforms of the genus 1 curves contained in the twelve hyperplanes $a_j = \varepsilon a_{j+1}$ or $a_j = \varepsilon ic$, where $j \in \{1, 2, 3\}$ and $\varepsilon \in \{1, -1\}$ (24 defined over $\mathbb{Q}(\sqrt{2})$, 24 defined over $\mathbb{Q}(i, \sqrt{2})$).

These curves are already described in [vL00, p. 48]. Denote by $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ the set consisting of the above 140 curves.

Proposition 7. *The geometric Picard group of S is a free abelian group of rank 64, and the curves in \mathcal{G} generate a subgroup of finite index.*

Proof. Since $h^1(S, \mathcal{O}_S) = 0$, the Picard group is finitely generated. Moreover, each torsion class in the Picard group determines an unramified connected cyclic covering $\pi: \tilde{S} \rightarrow S$ that is trivial if and only if the class in the Picard group is trivial. Any such cover induces a similar cover on \bar{S} : the inverse image under π of each (-2) -curve in S consists of a disjoint union of (-2) -curves in \tilde{S} , which can be contracted to rational double points, thus obtaining an unramified connected cyclic cover of \bar{S} . By [Dim86, Thm. 2.1] we have $H_1(\bar{S}, \mathbb{Z}) = 0$, and hence the cyclic covering is trivial. We obtain that all torsion classes in $\text{Pic } S$ are trivial, and $\text{Pic } S$ is torsion free.

From the calculation of the Hodge numbers of S , we deduce that the rank of $\text{Pic } S$ is at most 64. Thus, to conclude, it suffices to show that the intersection matrix of the curves in \mathcal{G} has rank 64. By the adjunction formula, we easily see that the self-intersection of the divisor class of a conic or a genus one curve in our list is -4 (in both cases, we mean the strict transform in S of the corresponding curve). The evaluation of the remaining pairwise intersection numbers of the curves above is straightforward, tedious, and preferably done by computer. We check using a computer that the rank of the intersection matrix of the 140 curves in \mathcal{G} is 64, concluding the proof of the proposition. \square

Theorem 8. *The Picard group of S is a free abelian group of rank 64, and it is generated by the classes of curves in \mathcal{G} .*

The discriminant of the intersection pairing on $\text{Pic } S$ is -2^{28} .

Proof. By Proposition 7, the Picard group is free abelian of rank 64, and the lattice L generated by the classes of the curves in \mathcal{G} is of finite index in the Picard group. It remains to show that L is already the full Picard group.

The computation of the intersection matrix shows that the discriminant of L is -2^{28} . Thus the cokernel of the inclusion $L \rightarrow \text{Pic } S$ is a finite abelian 2-group. Hence, to prove the equality $L = \text{Pic } S$ it suffices to prove that the natural morphism $L/2L \rightarrow \text{Pic } S/2\text{Pic } S$ is injective; denote by L_2 its kernel. The Galois group $\text{Gal}(\mathbb{Q}(i, \sqrt{2})/\mathbb{Q})$ acts on the lattice $\text{Pic } S$, since the divisors in \mathcal{G} are defined over $\mathbb{Q}(i, \sqrt{2})$, while S is defined over \mathbb{Q} . Denote by \tilde{G} the group of automorphisms of $\text{Pic } S$ generated by G and $\text{Gal}(\mathbb{Q}(i, \sqrt{2})/\mathbb{Q})$. The action of \tilde{G} on $\text{Pic } S$ induces an action of \tilde{G} on L , since \mathcal{G} is stable under the action of both G and $\text{Gal}(\mathbb{Q}(i, \sqrt{2})/\mathbb{Q})$, and hence there is an action of \tilde{G} on $L/2L$. The \mathbb{F}_2 -vector space $L_2 \subset L/2L$ is invariant under \tilde{G} (it is the kernel of a \tilde{G} -equivariant homomorphism). Let $\tilde{G}_2 \subset \tilde{G}$ be a Sylow 2-subgroup. Any representation of a 2-group on an \mathbb{F}_2 -vector space of positive dimension has a non-trivial fixed subspace. In particular, there is a non-trivial \tilde{G}_2 -invariant subspace in L_2 . Using a computer we check that the subspace of $(L/2L)^{\tilde{G}_2}$ coming from classes in L that have even intersection with L has dimension 1, spanned by the reduction modulo 2 of the canonical class (note that the canonical class is fixed by the action of \tilde{G} on $\text{Pic } S$). We deduce that if $L_2 \neq 0$, then the canonical divisor class is divisible by two in $\text{Pic } S$. By Lemma 5 we know that this is not the case. It follows that $\text{Pic } S$ is generated by the classes of the curves in \mathcal{G} . \square

Corollary 9. *The intersection pairing on $\text{Pic } S$ is even; in particular, there are no curves of odd degree on the surface S .*

Proof. By Theorem 8 the Picard group of S is generated by the elements of \mathcal{G} ; since all the elements of \mathcal{G} have even self-intersection, we deduce that the pairing on $\text{Pic } S$ is even. By the adjunction formula, the degree of any curve on S has the same parity as its self-intersection; since the pairing on $\text{Pic } S$ is even, we conclude that every curve on S has even degree. (This latter statement would also follow from the fact that all curves in \mathcal{G} have even degree.) \square

As a consequence, also the surface \bar{S} contains no curves of odd degree; van Luijk had already shown that there are no lines on \bar{S} , see [vL00, Prop. 3.4.11].

Using the explicitly known structure of $\text{Pic } S$ as a Galois module, we obtain the following result.

Theorem 10. *The algebraic part of the Brauer group of S is the isomorphic image of the Brauer group of \mathbb{Q} in the Brauer group of S .*

Proof. It is well-known that the cokernel of the inclusion of the Brauer group of \mathbb{Q} in the algebraic part of the Brauer group of S is isomorphic to the Galois cohomology group $H^1(\mathbb{Q}, \text{Pic } S)$. Since $\text{Pic } S$ is torsion free and we found a set of generators of the Picard group of S defined over $\mathbb{Q}(i, \sqrt{2})$, we have

$$H^1(\mathbb{Q}, \text{Pic } S) = H^1(\text{Gal}(\mathbb{Q}(i, \sqrt{2})/\mathbb{Q}), \text{Pic } S).$$

A computation in Magma [BCP97] shows that the latter cohomology group vanishes, establishing the result. \square

This means that there is no ‘algebraic Brauer-Manin obstruction’ to weak approximation on S . It would be interesting to investigate the transcendental quotient of the Brauer group.

4 The cuboid surface as a modular surface

In the paper [Bea17], Beauville gives a construction of \bar{S} as a quotient of a product of curves. We consider the curve X in \mathbb{P}^4 given by the equations

$$u^2 = 2xy, \quad v^2 = x^2 - y^2, \quad w^2 = x^2 + y^2.$$

This is a smooth curve of genus 5, which is a model of the modular curve $X(8)$ whose non-cuspidal points correspond to elliptic curves E together with a symplectic isomorphism $\mathbb{Z}/8\mathbb{Z} \times \mu_8 \rightarrow E[8]$. The geometric automorphism group of X is $\mathrm{PSL}(2, \mathbb{Z}/8\mathbb{Z})$, a group of order $192 = 3 \cdot 2^6$. The canonical epimorphism $\mathrm{PSL}(2, \mathbb{Z}/8\mathbb{Z}) \rightarrow \mathrm{PSL}(2, \mathbb{Z}/4\mathbb{Z}) \cong \mathfrak{S}_4$ has kernel G_0 isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$; in terms of our model, the automorphisms in the kernel are given by sign changes of u, v and w .

We consider the diagonal action of G_0 on $X \times X$. Denoting the coordinates on the first factor by u_1, v_1, w_1, x_1, y_1 and on the second factor by u_2, v_2, w_2, x_2, y_2 , invariants of the action are given by $U = u_1 u_2, V = v_1 v_2, W = w_1 w_2, X = x_1 x_2, Y = y_1 y_2$ and $T = x_1 y_2, Z = x_2 y_1$. We obtain the relations

$$XY = TZ, \quad U^2 = 4XY, \quad V^2 = X^2 + Y^2 - T^2 - Z^2, \quad W^2 = X^2 + Y^2 + T^2 + Z^2.$$

Setting

$$\begin{aligned} U &= 2b_1, & V &= 2b_2, & W &= 2b_3, \\ X &= a_1 + c, & Y &= -a_1 + c, & T &= a_2 + ia_3, & Z &= a_2 - ia_3, \end{aligned}$$

this gives the equations

$$\begin{aligned} a_1^2 + a_2^2 + a_3^2 &= c^2 \\ a_1^2 + b_1^2 &= c^2 \\ a_1^2 - a_2^2 + a_3^2 + c^2 &= 2b_2^2 \\ a_1^2 + a_2^2 - a_3^2 + c^2 &= 2b_3^2, \end{aligned}$$

which are equivalent to the equations defining \bar{S} . Note that the isomorphism $(X \times X)/G_0 \xrightarrow{\cong} \bar{S}$ is defined over $\mathbb{Q}(i)$ and not over \mathbb{Q} .

We have $X/G_0 \cong X(4) \cong \mathbb{P}^1$. The action of $(G_0 \times G_0)/G_0 \cong G_0$ (with G_0 embedded diagonally) on the quotient $(X \times X)/G_0 \cong \bar{S}$ is via sign changes on $u_1 u_2 = 2b_1, v_1 v_2 = 2b_2$ and $w_1 w_2 = 2b_3$. Therefore the quotient \bar{S}/G_0 is obtained by projecting away from the linear subspace spanned by these coordinates; it is the quadric

$$Q: a_1^2 + a_2^2 + a_3^2 = c^2$$

in \mathbb{P}^3 , which splits over $\mathbb{Q}(i)$, so that $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$.

A point on $X \times X = X(8) \times X(8)$ corresponds to a pair of elliptic curves with full level-8 structure. Since \bar{S} maps to $X(4) \times X(4)$, the points on \bar{S} give rise to a pair of elliptic curves with full level-4 structure; dividing by the diagonal action of G_0 corresponds to only keeping the induced isomorphism between the 8-torsion subgroups. Therefore, a point on \bar{S} corresponds to a pair of elliptic curves E and E' with isomorphisms ϕ and ψ as in the following diagram:

$$\mathbb{Z}/4\mathbb{Z} \times \mu_4 \xrightarrow{\phi} E[4] \subset E[8] \xrightarrow{\psi} E'[8].$$

This holds for all fields containing a square root of -1 . To find out what the correct moduli problem is over \mathbb{Q} , we observe that the sign change $a_3 \mapsto -a_3$ on \bar{S} lifts to the automorphism of $X \times X$ that switches the two factors. This implies that over \mathbb{Q} , \bar{S} is

isomorphic to the quotient of $R_{\mathbb{Q}(i)/\mathbb{Q}}X_{\mathbb{Q}(i)}$, the Weil restriction of scalars down to \mathbb{Q} of X base-changed to $\mathbb{Q}(i)$, by G_0 . The quadric Q is then $R_{\mathbb{Q}(i)/\mathbb{Q}}(X(4)_{\mathbb{Q}(i)})$. A rational point on \bar{S} then corresponds to an elliptic curve E over $\mathbb{Q}(i)$ with a basis P_1, P_2 of $E[4]$ such that there is an isomorphism $\psi: E[8] \rightarrow \bar{E}[8]$ with $\psi(P_1) = \bar{P}_1$ and $\psi(P_2) = -\bar{P}_2$, where the bar denotes the action of the nontrivial automorphism of $\mathbb{Q}(i)$.

Let G_0^+ be the subgroup of G_0 whose elements perform sign changes on an even number of variables. Then X/G_0^+ is the genus 2 curve

$$C: y^2 = 2(x^5 - x).$$

The quotient $Y = (X \times X)/G_0^+$ is a smooth surface that is a double cover of \bar{S} , ramified exactly in the 48 singularities of \bar{S} . On the other hand, $Z = \bar{S}/G_0^+$ has $C \times C$ as a double cover (over $\mathbb{Q}(i)$). Over \mathbb{Q} we find $R_{\mathbb{Q}(i)/\mathbb{Q}}C_{\mathbb{Q}(i)}$ as a double cover of Z . The surface Z in turn is a double cover of the quadric Q , branched over a divisor of type $(6, 6)$, so Z is still of general type.

5 Fibrations in curves of genus 5

We have observed earlier that there are exactly six quadrics of rank 3 that contain \bar{S} . In a similar way, one easily sees that there are exactly eleven quadrics of rank 4 containing \bar{S} . These are diagonal quadrics involving all possible subsets of size 4 of the variables that do not contain one of the sets of three variables corresponding to the rank 3 quadrics. Explicitly, the eleven quadrics are given by

$$\begin{array}{rcl} a_1^2 + a_2^2 + a_3^2 & & - c^2 = 0 \\ a_1^2 - a_2^2 & + b_1^2 - b_2^2 & = 0 \\ & a_3^2 - b_1^2 - b_2^2 & + c^2 = 0 \\ a_2^2 - a_3^2 & + b_2^2 - b_3^2 & = 0 \\ a_1^2 & & - b_2^2 - b_3^2 + c^2 = 0 \\ -a_1^2 & + a_3^2 - b_1^2 & + b_3^2 = 0 \\ a_2^2 & - b_1^2 & - b_3^2 + c^2 = 0 \\ 2a_1^2 & + b_1^2 - b_2^2 - b_3^2 & = 0 \\ 2a_2^2 & - b_1^2 + b_2^2 - b_3^2 & = 0 \\ & 2a_3^2 - b_1^2 - b_2^2 + b_3^2 & = 0 \\ & & b_1^2 + b_2^2 + b_3^2 - 2c^2 = 0 \end{array}$$

They can be associated in the given order to the tetrahedron in (1.3), its six edges, and its four vertices. This partition corresponds to the orbits under the automorphism group G .

Projecting away from the space spanned by the remaining variables, each of these gives rise to a rational map from \bar{S} onto a quadric in \mathbb{P}^3 . Since \bar{S} does not meet the first and the last four of these planes, the corresponding rational maps are morphisms. The first of these is the morphism $\bar{S} \rightarrow Q$, where Q is the quadric mentioned in Section 4. On the other hand, \bar{S} does meet the planes corresponding to the remaining six rank 4 quadrics (in eight singular points each), so to obtain a morphism, one has to blow up these points.

Over a field that splits the quadric, we can post-compose with one of the projections to \mathbb{P}^1 . The fibers of the resulting map $\bar{S} \rightarrow \mathbb{P}^1$ are curves of arithmetic genus 5 and degree 8; the generic fiber is a smooth irreducible curve canonically embedded into the \mathbb{P}^4 it spans. In this way, each of the eleven rank 4 quadrics gives rise to two complementary (in the sense that the sum of the classes of their fibers is the hyperplane section) fibrations of \bar{S} in curves of genus 5. The first pair of these, which are defined over $\mathbb{Q}(i)$, are isotrivial and correspond to the fibrations induced from the product $X \times X$ in Section 4. The other fibrations are not isotrivial. (We can check this by considering one of the genus 1 quotient fibrations and verifying that the j -invariant is not constant.) Since the quadrics are invariant under the sign changes of the three variables not occurring in them, the corresponding group $(\mathbb{Z}/2\mathbb{Z})^3$ acts on all the fibers. Quotienting out by subgroups, we obtain surfaces fibered in hyperelliptic (two simultaneous sign changes) or non-hyperelliptic (simultaneous sign change of all three variables) curves of genus 3 or in curves of genus 2 (even number of sign changes). We also find quotients fibered in genus 1 curves (one sign change). Dividing by the full $(\mathbb{Z}/2\mathbb{Z})^3$, we get \bar{S} as a Galois cover of $\mathbb{P}^1 \times \mathbb{P}^1$ (at least geometrically). For the first quadric, the genus 2 fibration is Z in the notation of Section 4; we saw there that this quotient is of general type, which implies that the hyperelliptic genus 3 fibration quotients covering it are of general type as well.

There is a similar construction starting from a rank 3 quadric. Projecting away from the space spanned by the four variables not occurring in the quadric, we map \bar{S} onto a conic. Here we have eight (singular) points of \bar{S} in the base locus, so to get a morphism, we have to blow them up. Post-composing with an isomorphism between the conic and \mathbb{P}^1 (this is always possible already over \mathbb{Q}), we obtain maps $\bar{S} \rightarrow \mathbb{P}^1$ again. The fibers (in \bar{S}) are again generically smooth canonical curves of genus 5, but this time, we get only one fibration from each rank 3 quadric. Here, twice the class of the fiber (on S) is the hyperplane section minus the eight exceptional curves coming from blowing up the singularities in the base locus. The genus 5 curves occurring as fibers are given by diagonal quadrics; we get an action of $(\mathbb{Z}/2\mathbb{Z})^4$ on them. This leads to more quotients fibered into curves of genus 1, 2 or 3.

In total, we obtain $6 + 2 \cdot 11 = 28$ such fibrations.

The (isotrivial) fibrations coming from the first rank 4 quadric above have six bad fibers each (for the fibration given by $[a_1 + ia_2, a_3 + c]$, they are above the points $0, \infty, \pm 1, \pm i$), which each consist of one of the genus 1 curves in \mathcal{G}_2 taken twice. The 12 curves in \mathcal{G}_2 each occur exactly once in this way.

The fibrations coming from the next six quadrics of rank 4 also have six bad fibers each (again over the same points, for $[a_1 + a_2, b_1 + b_2]$, say). Two of them consist of two of the genus 1 curves in \mathcal{G}_3 joined by four of the exceptional curves, the other four bad fibers consist of two conics from \mathcal{G}_1 , each with multiplicity 2 and joined by two exceptional curves. The curves in \mathcal{G}_3 show up exactly once in this way, whereas each conic in \mathcal{G}_1 occurs in three of the fibrations.

The fibrations coming from the last four quadrics of rank 4 have six fibers each splitting into two curves from \mathcal{G}_3 as above and twelve fibers that are hyperelliptic curves of genus 3 with two nodes (at singular points of \bar{S}). These curves form an orbit of size 96 under $\text{Aut}(S)$ and are birational to the curve

$$y^2 = 9x^8 + 20x^6 + 86x^4 + 20x^2 + 9.$$

The fibrations coming from the six quadrics of rank 3 each have four fibers splitting into four conics (from \mathcal{G}_1), four fibers splitting into two genus 1 curves from \mathcal{G}_3 and two fibers splitting into two genus 1 curves from \mathcal{G}_2 .

We list equations for the fibers of some representatives of the four different types of fibrations described above.

- The rank 3 quadric $a_1^2 + b_1^2 = c^2$ with $t = (a_1 + c)/b_1$:

$$(t^2 + 1)a_1 - (t^2 - 1)c = 0$$

$$(t^2 + 1)b_1 - 2tc = 0$$

$$(t^2 + 1)^2 a_2^2 - (t^2 + 1)^2 b_3^2 + (t^2 - 1)^2 c^2 = 0$$

$$a_3^2 + b_3^2 - c^2 = 0$$

$$(t^2 + 1)^2 b_2^2 + (t^2 + 1)^2 b_3^2 - 2(t^4 + 1)c^2 = 0$$

- One of the two fibrations associated to $a_1^2 + a_2^2 + a_3^2 = c^2$ with $t = (c + a_1)/(a_2 + ia_3) = (a_2 - ia_3)/(c - a_1)$:

$$(t^2 + 1)a_1 - 2ita_3 - (t^2 - 1)c = 0$$

$$(t^2 + 1)a_2 + i(t^2 - 1)a_3 - 2tc = 0$$

$$a_3^2 + b_3^2 - c^2 = 0$$

$$(t^2 + 1)^2 b_1^2 + 4t^2 b_3^2 + 4it(t^2 - 1)a_3 c - 8t^2 c^2 = 0$$

$$(t^2 + 1)^2 b_2^2 + (t^2 - 1)^2 b_3^2 - 4it(t^2 - 1)a_3 c - 2(t^2 - 1)^2 c^2 = 0$$

- One of the two fibrations associated to $a_1^2 - a_2^2 + b_1^2 - b_2^2 = 0$ with $t = (a_1 + a_2)/(b_2 + b_1) = (b_2 - b_1)/(a_1 - a_2)$:

$$2ta_1 - (t^2 - 1)b_1 - (t^2 + 1)b_2 = 0$$

$$2ta_2 - (t^2 + 1)b_1 - (t^2 - 1)b_2 = 0$$

$$2(t^4 - 1)b_1 b_2 - (t^2 - 1)^2 b_3^2 + 2(t^4 + 1)c^2 = 0$$

$$b_1^2 + b_2^2 + b_3^2 - 2c^2 = 0$$

$$a_3^2 + b_3^2 - c^2 = 0$$

- Finally, one of the two fibrations associated to $b_1^2 + b_2^2 + b_3^2 = 2c^2$ with $t = (b_1 + ib_2)/(\sqrt{2}c + b_3) = (\sqrt{2}c - b_3)/(b_1 - ib_2)$:

$$2tb_1 - (t^2 - 1)b_3 - \sqrt{2}(t^2 + 1)c = 0$$

$$2tb_2 + i(t^2 + 1)b_3 + i\sqrt{2}(t^2 - 1)c = 0$$

$$4t^2 a_1^2 + (t^2 - 1)^2 b_3^2 + 2\sqrt{2}(t^4 - 1)b_3 c + 2(t^4 + 1)c^2 = 0$$

$$4t^2 a_2^2 - (t^2 + 1)^2 b_3^2 - 2\sqrt{2}(t^4 - 1)b_3 c - 2(t^4 + 1)c^2 = 0$$

$$a_3^2 + b_3^2 - c^2 = 0$$

The genus 2 quotient of the third fibration (which is defined over \mathbb{Q}) is birational to

$$y^2 = -2(t^4 - 1)((t^8 - 1)x^6 + 4(3t^8 + 2t^4 + 3)x^5 + 30(t^8 - 1)x^4 + 8(5t^8 - 2t^4 + 5)x^3 + 30(t^8 - 1)x^2 + 4(3t^8 + 2t^4 + 3)x + (t^8 - 1))$$

(whose right hand side splits completely over $\mathbb{Q}(i)$).

6 K3 quotients

Projecting away from one of the coordinate points in \mathbb{P}^6 , which is equivalent to taking the quotient under the involution that changes the sign of the corresponding coordinate, we obtain various K3 surfaces given as intersections of three quadrics in \mathbb{P}^5 that are doubly covered by \bar{S} . We will denote them by $\bar{K}_{a_1}, \dots, \bar{K}_c$ and their minimal desingularizations by K_{a_1}, \dots, K_c , indexed by the coordinate we ‘forget’. The action of $\text{Aut}(\bar{S})$ implies that the three K_{b_j} are isomorphic over \mathbb{Q} and that the three K_{a_j} are isomorphic over \mathbb{Q} and isomorphic to K_c over $\mathbb{Q}(i)$.

We will consider K_c in the following. Its singular model \bar{K}_c is defined by

$$a_1^2 + a_2^2 = b_3^2, \quad a_1^2 + a_3^2 = b_2^2, \quad a_2^2 + a_3^2 = b_1^2.$$

There are 12 nodes on this model (whose preimages are 24 of the 48 nodes on \bar{S} ; the other 24 nodes disappear under the quotient map). The cover is branched over the union of eight conics given by $a_1^2 + a_2^2 + a_3^2 = 0$. This K3 surface, which parametrizes ‘Euler bricks’, has an extra involution (already known to Euler) given by

$$[a_1, a_2, a_3, b_1, b_2, b_3] \longmapsto [a_2 a_3, a_1 a_3, a_1 a_2, a_1 b_1, a_2 b_2, a_3 b_3].$$

This involution of K_c does not descend to \bar{K}_c ; in particular, it does not lift to an automorphism of S .

Let $\pi: S \rightarrow K_c$ be the map induced by dividing S by the action of the sign change σ_c on c . Note that π factors as $S \rightarrow S/\langle \sigma_c \rangle \rightarrow K_c$, where the first map is a double cover ramified exactly on the eight conics whose image on \bar{S} is contained in the hyperplane $c = 0$, and the second map contracts the images of the exceptional curves corresponding to singular points with $c = 0$ on \bar{S} . Let \mathcal{E}_π denote the set of these exceptional curves on S . From the factorization of π above, we easily deduce the following, which will be used in Section 7 below.

Lemma 11. *For C a class in $\text{Pic } S$, we have*

$$\pi^* \pi_* C = C + \sigma_c(C) + \sum_{E \in \mathcal{E}_\pi} (C \cdot E) E.$$

Among the quadrics containing \bar{K}_c , there are exactly three of rank 3 (the three defining ones) and six of rank 4 (of the form $a_j^2 - a_k^2 + b_j^2 - b_k^2 = 0$ or $2(a_j^2 - b_j^2) + b_1^2 + b_2^2 + b_3^2 = 0$). In the same way as for the genus 5 fibrations on S , this defines $3 + 2 \cdot 6 = 15$ elliptic fibrations on K_c .

We find that the images on K_c of the curves in \mathcal{G} generate a subgroup G of rank 20, and therefore of finite index, in $\text{Pic } K_c$. The determinant of the intersection pairing is -32 , so the subgroup is saturated at all primes except possibly 2. We can check that the natural map $\pi^*: \text{Pic } K_c \rightarrow \text{Pic } S$ induces an injection $G/2G \rightarrow \text{Pic } S/2 \text{Pic } S$, which shows that G is also 2-saturated, whence $G = \text{Pic } K_c$. This proves the following.

Lemma 12. *The Picard group of K_c is generated by the images of the curves in \mathcal{G} . In particular, every class in $\text{Pic } K_c$ has even intersection with the class of hyperplane sections.*

So there are no curves of odd degree on \bar{K}_c .

Regarding curves of degrees 2 and 4, we have the following.

Lemma 13. *Let $C \subset \bar{K}_c$ be an integral curve.*

- (1) If C is a plane curve, then C is a conic, and all conics on \bar{K}_c are images of curves in \mathcal{G} ; there are 44 such conics.
- (2) If C spans a \mathbb{P}^3 , then $\deg C = 4$ and C is a fiber of one of the 15 elliptic fibrations mentioned above.
- (3) If $\deg C = 4$ and C is not contained in a \mathbb{P}^3 , then C is a smooth rational normal curve that meets the branch locus transversally in eight points. There are 56 such curves on \bar{K}_c .

Proof.

- (1) If C is a plane curve, then C is (contained in) an intersection of quadrics in the plane it spans. Since there are no curves of odd degree on \bar{K}_c , C must be a conic. By enumerating all elements C of $\text{Pic } K_c$ with $C \cdot H' = 2$ (where H' is the hyperplane class) and $C^2 = -2$ and excluding those with negative intersection with the curves we know, we find that the only conics on \bar{K}_c are those obtained as images of curves in \mathcal{G} . There are three different kinds (each forming one orbit under the subgroup of $\text{Aut}(K_c)$ induced by $\text{Aut}(S)$), namely (i) the eight conics whose union is the branch locus, (ii) twelve conics in the hyperplanes $a_j = 0$, and (iii) 24 conics obtained as quotients of the genus 1 curves in \mathcal{G} that are invariant under the involution that changes the sign of c .
- (2) As before, C is contained in an intersection of quadrics in the \mathbb{P}^3 it spans. Since $\deg C$ is even and conics span planes, we must have $\deg C = 4$ and $p_a(C) = 1$, so $C^2 = 0$. By enumerating all relevant elements of $\text{Pic } K_c$ as before, we find that they all correspond to fibers of one of the 15 fibrations.
- (3) Any curve of degree 4 that spans \mathbb{P}^4 is a rational normal curve. We then have $C^2 = -2$. We can enumerate the relevant elements again and check that their intersection with the branch locus is 8. There are 80 such classes, falling into orbits under $\text{Aut}(K)$ of sizes 8, 24 and 48. The orbit of size 8 comes from the images of the conics in the branch locus under the extra involution. The orbit of size 24 does not come from curves, since its elements have intersection multiplicity 2 with one of the exceptional curves (so the curve would have to have a singularity there). The orbit of size 48 comes from curves in hyperplane sections given by $a_1 + \sqrt{2}ia_2 + a_3 + b_2 = 0$ and their images under sign changes and permutations. \square

If $C \subset \bar{K}_c$ is an integral curve of degree 6 contained in a \mathbb{P}^4 , then the residual intersection with any hyperplane that contains C is a conic. If the conic is not contained in the branch locus, then it meets four of the eight branch conics transversally in one point each. So the residual sextic also has to meet these four branch conics transversally in one point each (note that the branch locus is disjoint from the singularities of \bar{K}_c). (It will also meet the other four branch conics with multiplicity 2.) The pull-back of the sextic to \bar{S} will be irreducible of degree 12 unless the sextic meets the branch locus with even multiplicity in each intersection point. So, for the sextic to lift to a sextic on \bar{S} , it has to pass through two points in which two of the branch conics intersect. This means that the hyperplane has to pass through these points as well. Note that the four branch conics split into two pairs such that the conics in each pair intersect in two distinct points, whereas the conics in distinct pairs do not intersect.

There are two orbits of non-branch conics under the subgroup of $\text{Aut}(\bar{K}_c)$ induced by $\text{Aut}(\bar{S})$. For one of these orbits, the intersection points of the non-branch conic with the four relevant branch conics are distinct from the intersection points of the branch conics. So, in this case, we need to consider hyperplanes containing the non-branch

conics and one intersection point for each pair of branch conics. Of these four hyperplanes, only two contain a sextic curve (the other two contain another conic), and each of these sextics meets the branch locus in some points with multiplicity one, and so lifts to an irreducible curve of degree 12 on \bar{S} .

For the other orbit, the non-branch conic intersects the branch conics at intersection points of the pairs (one intersection point for each pair). In this case, the hyperplane must either pass through the other intersection point of a pair or contain the tangent lines at the intersection point of the two branch conics. Three of the possible four combinations of conditions lead to a unique hyperplane, whose intersection with \bar{K} we can check does not contain an integral sextic curve. The last possibility (which occurs when the hyperplane contains the tangent lines at both intersection points) leads to a pencil of hyperplanes. However, the intersection of \bar{K} with their common \mathbb{P}^3 already contains the initial non-branch conic with multiplicity 2, so none of the hyperplanes can contain a sextic on \bar{K} .

If the conic is a branch conic, then we can check explicitly that no hyperplane containing the \mathbb{P}^2 spanned by the conic is tangent to all four branch conics that do not meet the original one. This again implies that the pull-back to \bar{S} of the residual sextic ramifies.

We obtain the following result.

Lemma 14. *If $C \subset \bar{K}_c$ is an integral sextic curve that is contained in a \mathbb{P}^4 , then C lifts to an irreducible curve of degree 12 on \bar{S} .*

7 Curves of low degree on \bar{S}

We can use our explicit knowledge of the Picard group to determine the set of curves on S or \bar{S} of small degree. We freely identify curves on \bar{S} with their strict transforms on S and with their classes in $\text{Pic } S$.

Theorem 15.

- (1) *All conics on \bar{S} are contained in \mathcal{G} .*
- (2) *The surface \bar{S} does not contain smooth rational curves of degree 4.*
- (3) *All curves of degree 4 and arithmetic genus 1 on \bar{S} are in \mathcal{G} . In particular, all such curves are smooth and hence of geometric genus 1.*

Proof.

- (1.) Any conic C in \bar{S} must be smooth, since \bar{S} does not contain curves of odd degree. We have $C \cdot K_S = 2$ and $C^2 = -4$ by the adjunction formula. We can enumerate all lattice points in the Picard lattice satisfying these two conditions; this results in 2048 elements. If C is a curve, then it has to have nonnegative intersection with all the curves in \mathcal{G} (except possibly itself). Testing this condition leaves only the 32 known conics in \mathcal{G} .
- (2.) If C is a smooth rational curve of degree 4 on \bar{S} , then C spans a \mathbb{P}^4 (otherwise $p_a(C) = 1$, see Theorem 16 below), which we will denote P_4 . The image of C in \bar{K}_c then either is isomorphic to C and hence a rational normal curve of degree 4. But these all lift to hyperelliptic curves of genus 3 on \bar{S} by Lemma 13. Or else the image spans a \mathbb{P}^3 ; this will be the case when P_4 contains the point $[0, 0, 0, 0, 0, 0, 1]$. We can consider K_{α_j} instead of K_c for any $j \in \{1, 2, 3\}$, so the only remaining cases are those when P_4 contains the \mathbb{P}^3 given by $b_1 = b_2 = b_3 = 0$. Projecting away

from this \mathbb{P}^3 , we obtain the morphism $S \rightarrow Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ mentioned in Section 4. Then P_4 must be the pull-back of one of the lines on Q , and $P_4 \cap \bar{S}$ is a fiber of one of the two corresponding fibrations. But no such fiber contains a rational normal curve of degree 4.

- (3.) This statement is proved in the same way as the result on conics. A curve C of degree 4 and arithmetic genus 1 satisfies $C^2 = -4$. We can enumerate the relevant 16 160 classes in $\text{Pic } S$ and check that none of them has nonnegative intersection with all curves in \mathcal{G} . \square

We now show that we know all curves on the surface \bar{S} that are contained in a low-dimensional linear subspace.

Theorem 16. *Let $C \subset \bar{S}$ be an integral curve.*

- (1) *If C is contained in a plane, then C is a conic; in particular, $C \in \mathcal{G}$.*
- (2) *If C spans a \mathbb{P}^3 , then C is one of the genus 1 curves in \mathcal{G} .*
- (3) *If C spans a \mathbb{P}^4 , then C has degree 8 and is a fiber of one of the 28 fibrations defined in Section 5. In particular, C is either a smooth canonical curve of genus 5, or else it is one of the 96 hyperelliptic curves of genus 3 occurring as irreducible singular fibers.*

Proof.

- (1) Let P_2 be the plane spanned by C (C cannot be a line, since there are no curves of odd degree on \bar{S}). Then C is an intersection of quadrics in P_2 , so C is a conic. By Theorem 15, it follows that $C \in \mathcal{G}$.
- (2) Let P_3 be the linear subspace spanned by C . Then C is contained in an intersection of quadrics in P_3 ; in particular, the degree of C is at most 4 (and even). Since C is not a plane curve, the degree must be 4, hence by Theorem 15, $C \in \mathcal{G}$.
- (3) Let P_4 be the linear subspace spanned by C . Again C is contained in an intersection of quadrics, so the degree of C is now 6 or 8. If the degree were 6, then the image of C on \bar{K}_c would be either a cubic (which is impossible) or a sextic contained in a hyperplane. But Lemma 14 shows that curves of the latter type do not come from sextics on \bar{S} . So the degree of C is 8, and C is an intersection of three quadrics. Therefore the four quadrics defining \bar{S} will be linearly dependent when restricted to P_4 , which means that there is a quadric Q containing \bar{S} that contains P_4 . But a quadric in \mathbb{P}^6 that contains a \mathbb{P}^4 has rank at most 4. So Q is either one of the six rank 3 quadrics or one of the eleven rank 4 quadrics vanishing on \bar{S} . In the former case, P_4 is the preimage of a point on the conic obtained by projecting Q , in the latter case, P_4 is the preimage of a line on the quadric in \mathbb{P}^3 obtained by projecting Q . In either case, we see that $C = \bar{S} \cap P_4$ is a fiber of one of the associated fibrations. The bad fibers have been described in Section 5. \square

Theorem 17. *There are no integral curves of degree 6 on \bar{S} .*

Proof. Let $C \subset \bar{S}$ be an integral curve of degree 6. By Theorem 16, we know that C spans a \mathbb{P}^5 or \mathbb{P}^6 ; it follows that $p_a(C) \in \{0, 1\}$, so $C^2 \in \{-8, -6\}$. The image C' of C on \bar{K}_c will still be of degree 6 (there are no curves of odd degree on K_c) and span \mathbb{P}^5 , since by Lemma 14, no curve of degree 6 in a hyperplane section of \bar{K}_c is the image of a sextic on \bar{S} . So $p_a(C') \in \{0, 1\}$ as well, and $(C')^2 \in \{-2, 0\}$. We enumerate all classes C' in $\text{Pic } K_c$ having intersection 6 with the hyperplane section and self-intersection -2 or 0 , such that C' has nonnegative intersection with all curves obtained as images from curves in \mathcal{G} . This results in 1088 such classes with $(C')^2 = 0$ and 1680 classes with

$(C')^2 = -2$, which fall into 14 and 20 orbits, respectively, under the group of linear automorphisms of \bar{K}_c . If such a class C' is the image of an integral curve C on \bar{S} , then by Lemma 11, we must have

$$\pi^*C' = \pi^*\pi_*C = C + \sigma_c(C) + \sum_{E \in \mathcal{E}_\pi} (C \cdot E)E = \tau(C),$$

where $\tau: \text{Pic } S \rightarrow \text{Pic } S$ is the endomorphism defined by the expression above. For a representative C' of each of the relevant orbits, we first check if π^*C' is in the image of τ (otherwise C' is not in the image of π_*). If this is the case, then we find the (finitely many) ‘candidate classes’ $C \in \text{Pic } S$ satisfying (i) $\tau(C) = \pi^*C'$, (ii) $C^2 \in \{-6, -8\}$, (iii) $C \cdot D \geq 0$ for all $D \in \mathcal{G}$. In all cases, it turns out that the set of candidate classes is empty. This shows that there are no integral curves of degree 6 on \bar{S} that span a \mathbb{P}^5 or \mathbb{P}^6 , which proves the claim. \square

Corollary 18. *The set \mathcal{G} consists precisely of the integral curves C on S such that $C \cdot K_S \leq 6$.*

Proof. We know that all curves in \mathcal{G} satisfy the conditions. Observe that for an integral curve C on S , we must have $C \cdot K_S \in \{0, 2, 4, \dots\}$. So if $C \cdot K_S \leq 6$, we have $C \cdot K_S \in \{0, 2, 4, 6\}$. If the intersection number vanishes, then C is an exceptional curve and so $C \in \mathcal{G}$. If $C \cdot K_S = 2$, then the image of C on \bar{S} is a conic, and so $C \in \mathcal{G}$ by statement (1) in Theorem 15. If $C \cdot K_S = 4$, then the image of C on \bar{S} is a curve of degree 4, which has arithmetic genus either zero (then C is a smooth rational curve), one or three. The first two cases are taken care of by statements (2) and (3) in Theorem 15, respectively. In the last case, C is a plane quartic, and this is ruled out by Theorem 16. Finally by Theorem 17, $C \cdot K_S \neq 6$. \square

Even though there are no plane quartics on \bar{S} , we can find some more curves of genus 3. For this, we consider the quotient $S \rightarrow K_c$ we studied in Section 6. The extra involution on K_c does not lift to an automorphism of S , which allows us to produce new curves by projecting a known curve to K_c , applying the involution and pulling the result back to S . For example, taking one of the conics in the branch locus and applying the involution, we obtain a smooth rational curve of degree 4 meeting the branch locus in eight points. Its pull-back to \bar{S} therefore is a hyperelliptic curve of genus 3. It is of degree 8 and contained in a hyperplane section cut out by $b_1 \pm b_2 \pm b_3 = 0$; the residual intersection is another one of these hyperelliptic curves. Since the hyperplane $c = 0$ meets the curve exactly in the eight Weierstrass points, the curve is bicanonically embedded. It is smooth and has self-intersection -4 . The eight curves obtained in this way together with the further 24 similar curves obtained from replacing c by one of the a_j form an orbit under the automorphism group. They are all isomorphic to the ‘octahedral’ curve

$$y^2 = x^8 + 14x^4 + 1.$$

Pulling back the other rational quartics from \bar{K}_c to \bar{S} (and similarly from the \bar{K}_{a_j}), we find 192 further smooth hyperelliptic genus 3 curves of degree 8 and spanning a \mathbb{P}^5 . They sit in hyperplane sections of the form

$$a_1 + \sqrt{2}ia_2 + a_3 + b_2 = 0$$

(modulo the action of $\text{Aut}(S)$) and are isomorphic (over $\mathbb{Q}(i, \sqrt{2})$) to

$$y^2 = x^8 + 4x^6 + 32x^5 + 86x^4 + 64x^3 + 36x^2 + 32x + 17.$$

We find some further interesting curves by considering images of graphs of automorphisms of X under the map $X \times X \rightarrow \bar{S}$. The centralizer of G_0 in $\text{Aut}(X)$ has order 32. There are three times 32 further elements that centralize one of the subgroups of G_0 given by sign changes of the type $(-, +, +)$ or $(+, -, -)$. The remaining 64 elements of $\text{Aut}(X)$ only commute with the sign change $(-, -, -)$. This implies that among the images in \bar{S} of these 192 curves, we find the 32 conics, three times 16 genus 1 curves (these are the genus 1 curves in \mathcal{G} that are not contained in one of the hyperplanes $b_j = 0$) and finally 16 curves isomorphic to X divided by the central element, which results in the Fermat quartic. Its images on \bar{S} are bicanonically embedded; in particular, they span a hyperplane, which is of the form $a_1 \pm a_2 \pm a_3 \pm ic = 0$ and contains two of these 16 curves. Since they have negative self-intersection, both the hyperelliptic and the non-hyperelliptic genus 3 curves are not contained in the cone in $\text{Pic } S$ spanned by the curves in \mathcal{G} .

The partial results above suggest the following question.

Question 19. *Are all the curves of (geometric) genus at most 1 on S contained in \mathcal{G} ?*

We can say the following. We write E for the exceptional divisor on S .

Lemma 20.

- (1) *Any rational curve C on \bar{S} must satisfy $C \cdot E \geq 6$.*
- (2) *Any curve C of geometric genus 1 on \bar{S} must satisfy $C \cdot E \geq 2$.*

Proof. Recall that Y as defined near the end of Section 4 is a smooth double cover of \bar{S} branched exactly above the singular points of \bar{S} . Note that Y maps to a genus 2 curve C_2 with fibers that are isomorphic to the curve X of genus 5. So Y does not contain curves of genus < 2 . Let C be a curve of geometric genus g on \bar{S} and let $\tilde{C} \rightarrow C$ be its desingularization. The pull-back $\tilde{C} \times_C Y$ must then have geometric genus at least 2, it must therefore be ramified in at least six points when $g = 0$ and in at least two points when $g = 1$. This translates into the statements on intersection numbers with E . \square

In fact, the conics pass through six singularities each, so they lift to 32 sections of the fibration of Y , whereas the genus 1 curves in \mathcal{G} lift to curves of genus 5 or 9 on Y . Since Y maps to $C_2 \times C_2$, any curve of genus 2 in Y must be isomorphic to C_2 (and map to the graph of an automorphism of C_2). If $C' \subset Y$ is a curve isomorphic to C_2 , then its preimage in $X \times X$ is a four-fold étale cover of C' all of whose components must have genus at least 5. This implies that the preimage is connected and isomorphic to X , so it is the graph of an automorphism σ of X (since it cannot be a fiber of one of the projections). For σ to arise in this way, the group G_0^+ must stabilize its graph, which means that σ is in the centralizer of G_0 in $\text{Aut}(X)$. This centralizer has order 32, so all of these curves are accounted for by the lifts of the 32 conics on \bar{S} . This implies that all other curves must lift to curves of genus at least 3 on Y , and we obtain the following improvement of Lemma 20.

Lemma 21.

- (1) *A rational curve C on \bar{S} that is not a conic must satisfy $C \cdot E \geq 8$.*
- (2) *A curve C of geometric genus 1 on \bar{S} must satisfy $C \cdot E \geq 4$.*

The first statement recovers a result from [GFU20].

Note that surfaces of general type are conjectured to have only finitely many curves of genus at most 1. In general, this is only known in very few cases, for instance, when Bogomolov’s inequality $(K_S)^2 > c_2(S)$ holds (see [Bog77]); for surfaces contained in an abelian variety (see [Fal94]); for very general surfaces of large degree in projective space (see work of Demailly, Siu, Diverio-Merker-Rousseau, Bérczi on the Green-Griffiths Conjecture). None of these cases covers the surface S : Bogomolov’s inequality fails for S since $(K_S)^2 = 16 < 80 = c_2(S)$; the surface S is simply connected and hence is not contained in an abelian variety; the surface S is not a surface in \mathbb{P}^3 , since the only such surfaces of general type with primitive canonical divisor are the surfaces of degree five, and a plane section of a quintic is a curve of odd degree.

According to the Bombieri-Lang conjecture, the rational points on a variety of general type are not Zariski-dense. This means that all but finitely many rational points on a surface of general type lie on a finite set of curves of genus zero or one on the surface. If the question above has a positive answer, then the only such curves defined over \mathbb{Q} on \bar{S} are the conics corresponding to degenerate cuboids. So the Bombieri-Lang conjecture would then imply that there are only finitely many distinct rational boxes (up to scaling).

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