Vector bundles on curves

Lectures by Gerd Faltings held in Bonn 1995

Notes by Michael Stoll

Comments, remarks, questions etc. inserted by me are set in *slanted*.

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Since these notes are quite far from perfection, I very much appreciate any comments, suggestions, corrections or whatever. Send me email to `michael@rhein.iam.uni-bonn.de` or put your message in my mailbox in the staircase of Beringstr. 1.

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Michael Stoll, September 12, 2008

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1 Introduction

Let C be a (smooth projective connected) algebraic curve (over some field k). A *vector bundle* (always on C if not otherwise specified) is a locally free sheaf $\mathcal{E}$.

Goal: Construct a moduli space of vector bundles over C.

There are two invariants associated to $\mathcal{E}$:

- The *rank* $\operatorname{rk}\mathcal{E} = \dim \mathcal{E} \otimes k(x)$ (for any $x \in C$);
- We have the *determinant (bundle)* of $\mathcal{E}$, $\det \mathcal{E} = \Lambda^{\operatorname{rk}\mathcal{E}}\mathcal{E}$ (see below for the construction); it is a line bundle. The *degree* of $\mathcal{E}$ is $\deg \mathcal{E} = \deg \det \mathcal{E}$ (this is the degree of a divisor corresponding to $\det \mathcal{E}$ — e.g. the divisor of a global section (if it exists)).

Construction of $\det \mathcal{E}$

Let $r = \operatorname{rk}\mathcal{E}$ and consider more generally a representation $\rho : \mathbf{GL}(r) \to \mathbf{GL}(r')$ (for $\det \mathcal{E}$ this is $\det : \mathbf{GL}(r) \to \mathbf{GL}(1) = \mathbb{G}_{\mathrm{m}}$). Choose an open cover $C = \bigcup_{i \in I} U_i$ of C such that there are isomorphisms $\alpha_i : \mathcal{E}|_{U_i} \xrightarrow{\cong} \mathcal{O}_{U_i}^r$. On $U_i \cap U_j$, we have

$$\beta_{ij} = \alpha_i \circ \alpha_j^{-1} \in \mathbf{GL}(r, \Gamma(U_i \cap U_j, \mathcal{O}))\,,$$

and on $U_i \cap U_j \cap U_k$, we have $\beta_{ij}\beta_{jk} = \beta_{ik}$. Then $\rho(\beta_{ij}) \in \mathbf{GL}(r', \Gamma(U_i \cap U_j, \mathcal{O}))$ defines a vector bundle $\rho(\mathcal{E})$. As another example, we have the *dual bundle* $\mathcal{E}^{\mathrm{t}}$ given by *the contragredient representation* $\rho(\beta) = (\beta^{\mathrm{t}})^{-1}$.

There exist vector bundles of any degree and rank: $\mathcal{O}_C^{r-1} \oplus \mathcal{L}$ with a line bundle $\mathcal{L}$ has rank r and degree $\deg \mathcal{L}$.

Theorem 1 (Riemann–Roch)

$$\chi(C, \mathcal{E}) = \dim H^0(C, \mathcal{E}) - \dim H^1(C, \mathcal{E}) = (1-g)\operatorname{rk}\mathcal{E} + \deg \mathcal{E}\,,$$

where g is the genus *of C and $\chi(C, \mathcal{E})$ is the* Euler characteristic *of $\mathcal{E}$.*

EXAMPLE: Let k be algebraically closed and look at the case $g = 0$, i.e. $C = \mathbb{P}^1$. There is an exact sequence:

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}^2 \longrightarrow \mathcal{O}(1) \longrightarrow 0\,;$$

$\mathcal{O}(-1)$ is the *universal line* and $\mathcal{O}(1)$ is the *universal quotient*; $\mathcal{O}(1)$ has rank 1 and degree 1 (we have a global section coming from the global section $(1,0)$ of $\mathcal{O}^2$, which has a unique and simple zero at the line $k \cdot (1,0)$).

We let $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$ for $n \in \mathbb{Z}$.

Theorem 2 *On $\mathbb{P}^1$, every vector bundle $\mathcal{E}$ is isomorphic to a direct sum of $\mathcal{O}(n)$'s:*

$$\mathcal{E} \cong \mathcal{O}(m_1) \oplus \mathcal{O}(m_2) \oplus \cdots \oplus \mathcal{O}(m_r)\,.$$

(And this is essentially unique.)

PROOF: Coherent sheaves on $\mathbb{P}^1$ essentially correspond to finitely generated graded modules over $R = k[t_0, t_1]$: To a graded R–module $M = \bigoplus_{n\in\mathbb{Z}} M_n$, we associate the coherent sheaf $\mathcal{F}$ that is defined by

$$\Gamma(D(t_0), \mathcal{F}) = M_{(t_0)}\,, \qquad \Gamma(D(t_1), \mathcal{F}) = M_{(t_1)}\,,$$

where $D(t_j)$ is the open subvariety of $\mathbb{P}^1$ defined by $t_j \neq 0$, and $M_{(t_j)}$ is the *homogeneous localization*, i.e. the elements of degree zero in the usual localization M_{t_j}. We get a trivial $\mathcal{F}$ iff $M_n = 0$ for n sufficiently big (since then $\frac{m}{t_j^d} = \frac{mt_j^h}{t_j^{d+h}} = 0$ for $h \gg 0$), and M_1, M_2 have the same image iff there is a common submodule M_{12} of M_1 and M_2 coinciding with both in sufficiently high degrees.

Given a coherent sheaf $\mathcal{F}$, a suitable M is

$$M = \bigoplus_{n\geq n_0} M_n = \bigoplus_{n\geq n_0} \Gamma(\mathbb{P}^1, \mathcal{F}(n))$$

(where $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}(n)$) *with some (arbitrary)* $n_0 \in \mathbb{Z}$. (M is finitely generated by SERRE's Theorem [H III.5.2?]). For $\mathcal{F} = \mathcal{O}(m)$, we may take $M = R(m)$, defined by $M_n = R_{m+n}$; this is a free R–module, generated by $1 \in M_{-m}$.

Now let $\mathcal{E}$ be a vector bundle. We claim that there are $m_1, \dots, m_r \in \mathbb{Z}$ such that

$$M = \bigoplus_{n\in\mathbb{Z}} \Gamma(\mathbb{P}^1, \mathcal{E}(n)) \cong R(m_1) \oplus \cdots \oplus R(m_r)\,.$$

Im particular, $\Gamma(\mathbb{P}^1, \mathcal{E}(n)) = 0$ for $n \ll 0$. We first show this. By SERRE, $\mathcal{E}^{\,\mathrm{t}}(m)$ is globally generated for $m \gg 0$, hence we have a surjection $\mathcal{O}_{\mathbb{P}^1}^N \twoheadrightarrow \mathcal{E}^{\,\mathrm{t}}(m)$. Then, dually, there is an injection $\mathcal{E}(-m) \hookrightarrow \mathcal{O}_{\mathbb{P}^1}^N$, which by tensoring with $\mathcal{O}(-1)$ gives $\mathcal{E}(-m-1) \hookrightarrow \mathcal{O}_{\mathbb{P}^1}(-1)^N$, and the latter has no non-trivial global sections.

Next, we show that M is a projective R–module. R is regular of dimension 2; thus projective is equivalent to *reflexive*, i.e. $M = \bigcap M_{\mathfrak{p}}$, where $\mathfrak{p}$ runs through the prime ideals of height one in R (see below) [CK]. Consider

$$\begin{array}{ccccc} \mathbb{A}^2 & \supset & \mathbb{A}^2 \setminus \{0\} = U & \xrightarrow{\pi} & \mathbb{P}^1 \\ & & (t_0, t_1) & \mapsto & (t_0 : t_1) \end{array}$$

Then $\pi_* \mathcal{O}_U = \bigoplus_{m\in\mathbb{Z}} \mathcal{O}(m)$:

$$\begin{array}{rcl} \operatorname{Spec} k[t_0, t_0^{-1}] \times \operatorname{Spec} k[\frac{t_1}{t_0}] = \operatorname{Spec} k[t_0, t_0^{-1}, t_1] & \xrightarrow{\pi} & \operatorname{Spec} k[\frac{t_1}{t_0}] = D(t_0) \\ \cup & & \cup \\ \operatorname{Spec} k[t_0, t_0^{-1}, t_1, t_1^{-1}] & \xrightarrow{\pi} & \operatorname{Spec} k[\frac{t_1}{t_0}, \frac{t_0}{t_1}] \\ \cap & & \cap \\ \operatorname{Spec} k[t_1, t_1^{-1}] \times \operatorname{Spec} k[\frac{t_0}{t_1}] = \operatorname{Spec} k[t_0, t_1, t_1^{-1}] & \xrightarrow{\pi} & \operatorname{Spec} k[\frac{t_0}{t_1}] = D(t_1) \end{array}$$

Now,

$$k[t_0, t_0^{-1}] \otimes k[\tfrac{t_1}{t_0}] = \bigoplus_{m\in\mathbb{Z}} k[\tfrac{t_1}{t_0}]\, t_0^m\,;$$

this corresponds to $\bigoplus_{m\in\mathbb{Z}} \mathcal{O}(m)$ (it is glued over $D(t_0) \cap D(t_1)$ with the corresponding expression obtained by exchanging t_0 and t_1). We then have

$$\pi_*(\pi^*\mathcal{E}) = \pi_*\mathcal{O}_U \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{E} \cong \bigoplus_{m\in\mathbb{Z}} \mathcal{E}(m)\,.$$

Taking global sections $\Gamma(\mathbb{P}^1, \text{—})$, we get

$$\Gamma(U, \pi^*\mathcal{E}) = \Gamma(\mathbb{P}^1, \pi_*\pi^*\mathcal{E}) = \bigoplus_{m\in\mathbb{Z}} \Gamma(\mathbb{P}^1, \mathcal{E}(m)) = M\,.$$

Here, $\pi^*\mathcal{E}$ is a vector bundle on $U \subset \mathbb{A}^2 = \operatorname{Spec} R$. Let $\mathfrak{p} \subset R$ be a prime ideal of height 1; it corresponds to a curve in $\mathbb{A}^2$. A $y \in M \otimes \operatorname{Quot}(R)$ is in $M_\mathfrak{p}$ iff the corresponding section is regular on an open subset of [CK] this curve. If $y \in \bigcap_\mathfrak{p} M_\mathfrak{p}$, the singular set can contain only (discrete) points, but then the section must be regular, hence $y \in M$.

Now, since M is finitely generated, there is a surjection $\bigoplus_\nu R(m_\nu) \twoheadrightarrow M$. Because M is projective, it is a direct summand (with the graded structures), hence $M = R(m_1) \oplus \cdots \oplus R(m_r)$ with suitable m_i. To get the *graded* splitting, take a splitting $s : M \to \bigoplus_\nu R(m_\nu)$ as R–modules (not necessarily respecting the grading). Let $pr_m : M \twoheadrightarrow M_m$ be the projection of M onto its degree–m–part and define

$$\tilde{s} : M \to \bigoplus_\nu R(m_\nu) \qquad \text{by} \qquad \tilde{s}(a) = \sum_{m \in \mathbb{Z}} pr_m\Big(s(pr_m(a))\Big) .$$

Then $\tilde{s}$ is a homomorphism of graded R–modules identifying M with a direct summand of $\bigoplus_\nu R(m_\nu)$. [CK]

Since $M/(t_0, t_1)M = k(m_1) \oplus \cdots \oplus k(m_r)$ (where $k(m)$ denotes a one–dimensional graded k–vector space in degree m), this representation of M is unique.

Going back to sheaves finishes the proof. □

On the equivalence of 'projective' and 'reflexive' used in the proof above:

(1) Let A be a regular NOETHERian ring, and let M be a finitely generated A–module. Let $K = \operatorname{Quot}(A)$ be the quotient field of A.

Claim: The following two properties of M are equivalent:
(i) $M \xrightarrow{\cong} M^{**}$ (where $M^* = \operatorname{Hom}_A(M, A)$);
(ii) $M = \bigcap_\mathfrak{p} M_\mathfrak{p} \subset M \otimes_A K$, where the intersection is over all prime ideals $\mathfrak{p}$ of height 1 in A.

If M has these properties, M is called *reflexive*.

PROOF: (i) and (ii) both imply that M is torsion free, i.e., $M \hookrightarrow M \otimes_A K$. Since $M \otimes_A K \xrightarrow{\cong} M \otimes_A K$ canonically, we have inclusions

$$M \subset \bigcap_\mathfrak{p} M_\mathfrak{p} \subset M \otimes_A K \quad \text{and} \quad M \subset M^{**} \subset M \otimes_A K ,$$

hence it is sufficient to show $\bigcap_{\mathfrak{p}} M_{\mathfrak{p}} = M^{**}$.

"$\supset$": Let $\mathfrak{p}$ be a prime ideal of height 1, then $A_{\mathfrak{p}}$ is a discrete valuation ring. Since $M_{\mathfrak{p}}$ is torsion free, we have $M_{\mathfrak{p}} \xrightarrow{\cong} M_{\mathfrak{p}}^{**}$, whence $M^{**} \subset \bigcap_{\mathfrak{p}} M_{\mathfrak{p}}^{**} = \bigcap_{\mathfrak{p}} M_{\mathfrak{p}}$.

"$\subset$": Let $x \in \bigcap_{\mathfrak{p}} M_{\mathfrak{p}}$. Consider $l \in M^*$, i.e., $l : M \to A$. l extends to $l_K : M \otimes_A K \to K$. Since $x \in \bigcap_{\mathfrak{p}} M_{\mathfrak{p}}$, $l_K(x) \in \bigcap_{\mathfrak{p}} A_{\mathfrak{p}} = A$, whence $x \in M^{**}$. $\square$

(2) (Serre) If A is a regular NOETHERian ring of dimension 2, and M is a finitely generated reflexive A–module, then M is projective.

Proof: Without loss of generality, we may assume A to be local. Then we have to show that M is a free A–module, which is equivalent to M being flat over A, which in turn is equivalent to $\mathrm{Tor}_j^A(M,k) = 0$ for all $j > 0$, where $k = A/\mathfrak{m}_A$ is the residue field of A.

Let $K = \mathrm{Quot}\,(A)$. Let $n = \dim_K M \otimes_A K$, then (since M is torsion free), there is an injection $i : M \hookrightarrow A^n$. Let $Y = A^n/M$ and consider the long exact Tor–sequence for

$$0 \longrightarrow M \longrightarrow A^n \longrightarrow Y \longrightarrow 0\,.$$

Since $\mathrm{Tor}_j^A(A,k) = 0$ for $j > 0$, it suffices to show $\mathrm{Tor}_j^A(Y,k) = 0$ for $j > 1$.

Since M is reflexive, all zero divisors of Y belong to some prime ideal $\mathfrak{p}$ of A of height 1; therefore the set of associated prime ideals of Y (i.e., maximal elements in the set of annihilators of elements of Y) consists of finitely many prime ideals of height 1. In particular, $\mathfrak{m}_A$ does not belong to this set. Hence there is some $x \in \mathfrak{m}_A$ such that we have a short exact sequence

$$0 \longrightarrow Y \xrightarrow{\cdot x} Y \longrightarrow Y/xY \longrightarrow 0\,.$$

Looking again at the long exact Tor–sequence, we see that we only need to show $\mathrm{Tor}_j^A(Y/xY,k) = 0$ for $j > 2$. (Note that multiplication by x is zero on $\mathrm{Tor}_j^A(Y,k)$ since it is zero on k.) But this follows from the fact that A is regular of dimension 2. $\square$

[CK]

2 General Observations

Suppose a moduli space $\mathcal{M}$ exists. Let S be some 'parameter space' and consider vector bundles on $C \times S$. The isomorphism classes of these objects should be given by Map $(S, \mathcal{M})$.

$\mathcal{M}$ is smooth

Lift over infinitesimals: Let R be a ring, $I \subset R$ an ideal with $I^2 = 0$, and let $S = \operatorname{Spec} R$, $S_0 = \operatorname{Spec} R/I$. We have to show that any morphism $S_0 \to \mathcal{M}$ extends to S. This means that any vector bundle $\mathcal{E}_0$ on $C \times S_0$ comes from a vector bundle $\mathcal{E}$ on $C \times S$. (It will be important that C has dimension 1.) *(This criterion looks a bit like [H III.10.4 (iii)].)*

Choose an open affine cover $C \times S_0 = \bigcup_i U_{i,0}$ such that we have isomorphisms

$$\beta_{i,0} : \mathcal{E}_0|_{U_{i,0}} \xrightarrow{\cong} \mathcal{O}^r_{U_{i,0}} .$$

(Since $S = S_0$ as spaces, the $U_{i,0}$ also give an open affine cover $C \times S = \bigcup_i U_i$.) As usual, we then have $\beta_{ij,0} \in \Gamma(U_{i,0} \cap U_{j,0}, \mathbf{GL}(r))$ with

$$\beta_{ij,0}\beta_{jk,0} = \beta_{ik,0} . \tag{1}$$

It is always possible to lift $\beta_{ij,0}$ to $\beta_{ij} \in \Gamma(U_i \cap U_j, \mathbf{GL}(r))$, but we have to preserve (1). So take any lifting β_{ij} and write

$$\beta_{ij}\beta_{jk}\beta_{ik}^{-1} = 1 + \gamma_{ijk}$$

with $\gamma_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, I\mathcal{O}^{r \times r}_{C \times S}) = \Gamma(U_i \cap U_j \cap U_k, I \otimes \mathcal{E}\mathrm{nd}\,(\mathcal{E}))$; this identification is given by the trivialization via β_i. (*Question: $\mathcal{E}$ is not yet constructed — can we use $\mathcal{E}\mathrm{nd}\,(\mathcal{E})$ and β_i here?*). Because of $I^2 = 0$, we get on $U_i \cap U_j \cap U_k \cap U_l$ the relation (by calculating $\beta_{ij}\beta_{jk}\beta_{kl}$ in two ways) [CK]

$$\gamma_{ijk} - \gamma_{ijl} + \gamma_{ikl} - \gamma_{jkl} = 0 . \tag{2}$$

We may change our lifts β_{ij} by some $1 + \delta\beta_{ij}$ with $\delta\beta_{ij} \in \Gamma(U_i \cap U_j, I\mathcal{E}\mathrm{nd}\,(\mathcal{E}))$. Then γ_{ijk} changes by $\delta\beta_{ij} - \delta\beta_{ik} + \delta\beta_{jk}$. We have

$$0 = H^2(C \times S, I\mathcal{E}\mathrm{nd}\,(\mathcal{E})) = \{\gamma_{ijk} \mid (2) \text{ holds}\}/\{\delta\beta_{ij} - \delta\beta_{ik} + \delta\beta_{jk}\}$$

($H^2 = 0$ since C is a *curve*—*[H III.2.7] and [H III.3.7] (remember S is affine)* + KÜNNETH–*formula*), hence we can achieve $\gamma_{ijk} = 0$.

We see that $H^2 = 0$ means that $\mathcal{M}$ is smooth. H^1 also has a meaning: its dimension is that of the tangent space of $\mathcal{M}$.

Let $\operatorname{Spec} k = x \in \mathcal{M}$ be a (rational) point. It corresponds to a vector bundle $\mathcal{E}_0$ on $C \times \{x\}$ ($= C$). Try to extend $\operatorname{Spec} k \to \mathcal{M}$ to $S \to \mathcal{M}$ with $S = \operatorname{Spec} k[\varepsilon]$, $\varepsilon^2 = 0$. The following objects correspond to each other:

$$(\operatorname{Spec} S \to \mathcal{M}) \longleftrightarrow \left(\begin{array}{ccc} \mathcal{O}_{\mathcal{M},x} & \xrightarrow{\mathfrak{m}_x^2 \mapsto 0} & k[\varepsilon] \\ & \searrow_{x} & \downarrow {\scriptstyle \varepsilon \mapsto 0} \\ & & k \end{array} \right) \longleftrightarrow \operatorname{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k \cdot \varepsilon)$$

and the dimension of the Hom_k is that of the tangent space of $\mathcal{M}$ in x.

By the discussion above, the possible lifts correspond to changes by $\delta\beta_{ij}$'s with $\delta\beta_{ij} - \delta\beta_{ik} + \delta\beta_{jk} = 0$; different lifts are isomorphic (with an isomorphism that is the identity mod $I = (\varepsilon)$) iff the $\delta\beta_{ij}$'s differ by $\delta\beta_i - \delta\beta_j$. Hence the different lifts are parametrized by

$$H^1(C \times S, I\mathcal{E}\mathrm{nd}\,(\mathcal{E})) = H^1(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}_0))\,,$$

and the dimension of the tangent space is given by $\dim H^1(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}_0))$.

And what is H^0? Well, it can be interpreted as

$$H^0(C, I\mathcal{E}\mathrm{nd}\,(\mathcal{E})) = \ker(\operatorname{Aut}(\mathcal{E}) \to \operatorname{Aut}(\mathcal{E}_0)) = \{1 + \delta\alpha \mid \delta\alpha \in \Gamma(C, I\mathcal{E}\mathrm{nd}\,(\mathcal{E}))\}.$$

(*Shouldn't it be $H^0(C \times S, I\mathcal{E}\mathrm{nd}\,(\mathcal{E}))$?* They are the same, since I kills $I\mathcal{E}\mathrm{nd}\,(\mathcal{E})$, so that $I\mathcal{E}\mathrm{nd}\,(\mathcal{E})$ is an $\mathcal{O}_C = \mathcal{O}_{C\times S}/I\mathcal{O}_{C\times S}$ module. [CK])

By RIEMANN–ROCH (use that $\mathcal{E}\mathrm{nd}\,(\mathcal{E}_0)$ is a vector bundle of rank r^2 and degree 0):

$$\dim H^0(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}_0)) - \dim H^1(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}_0)) = r^2(1-g)\,.$$

A moduli space for all vector bundles does not exist, but there will be one for those $\mathcal{E}$ with $H^0(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E})) = k$; its dimension then is (by our discussion above) $1 + (g-1)r^2$.

May 5, 1995

(Raw notes by Michel Fontaine.)

Why do moduli spaces not exist?

There are difficulties with automorphisms:

Let $\mathcal{E}$ be a vector bundle on $C \times S \xrightarrow{\pi} S$, let $\mathcal{L}$ be a line bundle on S and take $\mathcal{E}' = \mathcal{E} \otimes \pi^*\mathcal{L}$. There is an open cover $S = \bigcup U_i$ such that $\mathcal{L}|_{U_i}$ is trivial for all i. Hence on $C \times U_i$, we have $\mathcal{E} \cong \mathcal{E}'$. So, if $\mathcal{M}$ exists, there are maps $\alpha, \beta : S \longrightarrow \mathcal{M}$ corresponding to $\mathcal{E}$ and $\mathcal{E}'$, resp. Since $\mathcal{E}|_{C\times U_i} \cong \mathcal{E}'|_{C\times U_i}$, we should have $\alpha|_{U_i} = \beta|_{U_i}$, whence $\alpha = \beta$. But it is possible that $\mathcal{E} \not\cong \mathcal{E}'$ (and so α and β should be distinct, a contradiction): Let $\gamma : \mathcal{E} \longrightarrow \mathcal{E}'$ be a homomorphism. Then

$$\begin{aligned}
\gamma &\in \Gamma(C \times S, \mathcal{H}\mathrm{om}\,(\mathcal{E}, \mathcal{E} \otimes \pi^*\mathcal{L})) \\
&= \Gamma(S, \pi_*(\mathcal{H}\mathrm{om}\,(\mathcal{E}, \mathcal{E})) \otimes \mathcal{L}) \\
&= \underbrace{\mathrm{End}_C(\mathcal{E})}_{=\Gamma(S,\pi_*(\mathcal{E}\mathrm{nd}(\mathcal{E})))?} \otimes \Gamma(S, \mathcal{L})\,.
\end{aligned}$$

If, e.g., $\Gamma(S, \mathcal{L}) = 0$, γ cannot be an isomorphism.

Problem: $\mathbb{G}_{\mathrm{m}} \subset \mathrm{Aut}\,(\mathcal{E})$

Need to consider 'local' isomorphism classes, but then the moduli space will not be very nice.

Solutions:

(a) Consider $\mathcal{M}$ as a (moduli) *stack (see later lectures)*;

(b) Restrict the set of vector bundles that are considered, e.g., take only $\mathcal{E}$ such that $\mathrm{End}\,(\mathcal{E}) = k$, or *(semi-)stable bundles.*

For a vector bundle $\mathcal{E}$, define its *slope* to be $\mu(\mathcal{E}) = \frac{\deg \mathcal{E}}{\mathrm{rk} \mathcal{E}}$.

$\mathcal{E}$ is called *semistable* (*stable*) if for every $0 \neq \mathcal{F} \subset \mathcal{E}$ (with $\mathcal{F} \neq \mathcal{E}$) $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ ($\mu(\mathcal{F}) < \mu(\mathcal{E})$).

REMARK: Any $\mathcal{F} \subset \mathcal{E}$ is torsion free and hence a vector bundle, but $\mathcal{E}/\mathcal{F}$ may have torsion. There exists some $\mathcal{F} \subset \mathcal{F}^* \subset \mathcal{E}$ such that $\mathcal{F}^*/\mathcal{F}$ has finite support and such that $\mathcal{E}/\mathcal{F}^*$ is torsion free. We have

$$\operatorname{rk} \mathcal{F}^* = \operatorname{rk} \mathcal{F} \quad \text{and} \quad \deg \mathcal{F}^* = \deg \mathcal{F} + \operatorname{length}(\mathcal{F}^*/\mathcal{F}) .$$

Explanation: For our purposes, a torsion sheaf *is a finite sum of (coherent) skyscraper sheaves, i.e., sheaves of finite support with finite-dimensional stalks. To such a sheaf* $\mathcal{T}$ *corresponds the positive divisor* $D = \sum_{x \in C} \dim \mathcal{T}_x \cdot x$. *If we want the degree to be additive on exact sequences (which it should be:*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_2 \longrightarrow 0 \quad \Longrightarrow \quad \det \mathcal{F} \cong \det \mathcal{F}_1 \otimes \det \mathcal{F}_2 \,),$$

then we are forced to define $\deg \mathcal{T} = \deg D = \sum_x \dim \mathcal{T}_x = \operatorname{length} \mathcal{T}$, *since we have the exact sequence*

$$0 \longrightarrow \mathcal{L}(-D) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{T} \longrightarrow 0 \,,$$

where $\deg \mathcal{L}(-D) = -\deg D$ *and* $\deg \mathcal{O}_C = 0$.

Over an open affine subset $U = \operatorname{Spec} A \subset C$, *a coherent sheaf corresponds to a finitely generated* A*–module* M. *For suitable* A *(e.g. principal ideal domains),* M *decomposes as*

$$0 \longrightarrow M_{\text{tors}} \longrightarrow M \longrightarrow \overline{M} \longrightarrow 0$$

with $\overline{M}$ *a finitely generated free* A*–module. This corresponds to*

$$0 \longrightarrow \mathcal{F}^*/\mathcal{F} \longrightarrow \mathcal{E}/\mathcal{F} \longrightarrow \mathcal{E}/\mathcal{F}^* \longrightarrow 0$$

in the Remark above. $\operatorname{rk} \mathcal{F}$ *means the generic rank of* $\mathcal{F}$.

REMARK: We can also consider quotients $\mathcal{E} \twoheadrightarrow \mathcal{G}$. The condition for (semi-) stability then is $\mu(\mathcal{G}) \left\{ \substack{\geq \\ >} \right\} \mu(\mathcal{E})$.

PROOF: *For every exact sequence* $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0$, *we have* $\deg \mathcal{E} = \deg \mathcal{F} + \deg \mathcal{G}$ *and* $\operatorname{rk} \mathcal{E} = \operatorname{rk} \mathcal{F} + \operatorname{rk} \mathcal{G}$.

"$\Longrightarrow$": *If* $\mu(\mathcal{G}) \left\{ \substack{< \\ \leq} \right\} \mu(\mathcal{E})$, *then* $\deg \mathcal{G} \cdot \operatorname{rk} \mathcal{E} \left\{ \substack{< \\ \leq} \right\} \deg \mathcal{E} \cdot \operatorname{rk} \mathcal{G}$, *hence*

$\deg \mathcal{E} \cdot \operatorname{rk} \mathcal{F} \left\{ \substack{< \\ \leq} \right\} \deg \mathcal{F} \cdot \operatorname{rk} \mathcal{E}$, *i.e.,* $\mu(\mathcal{E}) \left\{ \substack{< \\ \leq} \right\} \mu(\mathcal{F})$, *contradiction.*
"$\Longleftarrow$": Let $\mathcal{F}^*$ as above. Then the same argument in reverse direction shows $\mu(\mathcal{F}^*) \left\{ \substack{< \\ \leq} \right\} \mu(\mathcal{E})$. *Since $\mu(\mathcal{F}) \leq \mu(\mathcal{F}^*)$, the claim follows.* □

REMARK: If $\mathcal{E}_1$ and $\mathcal{E}_2$ are semistable vector bundles with $\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2)$, then $\operatorname{Hom}(\mathcal{E}_1, \mathcal{E}_2) = 0$:

$$\mathcal{E}_1 \twoheadrightarrow \mathcal{F} \hookrightarrow \mathcal{E}_2 \quad \Longrightarrow \quad \mu(\mathcal{E}_1) \leq \mu(\mathcal{F}) \leq \mu(\mathcal{E}_2)\,, \text{ contradiction.}$$

If $\mu(\mathcal{E}_1) = \mu(\mathcal{E}_2)$, we have $\mathcal{F} = \mathcal{F}^*$, and the map is *strict*, i.e. it has constant rank at each point.
If $\mathcal{E}_1$ and $\mathcal{E}_2$ are even stable (and $\mu(\mathcal{E}_1) = \mu(\mathcal{E}_2)$), then every non-trivial map $\mathcal{E}_1 \to \mathcal{E}_2$ is an isomorphism. In particular, $\operatorname{End}(\mathcal{E})$ is a division algebra for stable $\mathcal{E}$, and for k algebraically closed, we have $\operatorname{End}(\mathcal{E}) = k$.

The HARDER–NARASIMHAN Filtration

For a semistable bundle $\mathcal{E}$, this is given by

$$0 \subset \mathcal{E}\,.$$

If $\mathcal{E}$ is not semistable, take $0 \neq \mathcal{F} \subset \mathcal{E}$ such that $\mathcal{F}$ has maximal slope $\mu(\mathcal{F})$ and among the sub-vector bundles with maximal slope has maximal rank $\operatorname{rk} \mathcal{F}$. (This exists since the slope is bounded: $\dim H^0(C, \mathcal{E}) \geq \dim H^0(C, \mathcal{F}) \geq \deg \mathcal{F} - (g-1)\operatorname{rk} \mathcal{F}$ (by RIEMANN–ROCH), hence $\mu(\mathcal{F}) \leq g - 1 + \dim H^0(C, \mathcal{E})$.) We now have

(a) $\mathcal{F}$ is semistable *(by construction)*;

(b) Any $\mathcal{G}/\mathcal{F} \subset \mathcal{E}/\mathcal{F}$ has $\mu(\mathcal{G}/\mathcal{F}) < \mu(\mathcal{F})$ (Otherwise, $\mu(\mathcal{G}) \geq \mu(\mathcal{F})$ and $\operatorname{rk} \mathcal{G} > \operatorname{rk} \mathcal{F}$);

(c) $\mathcal{F} = \mathcal{F}^*$ *(otherwise, $\mu(\mathcal{F}^*) > \mu(\mathcal{F})$).*

We take $\mathcal{E}_1 = \mathcal{F}$ as the first step in the HARDER–NARASIMHAN filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

of $\mathcal{E}$. The construction is continued with $\mathcal{E}/\mathcal{F}$ replacing $\mathcal{E}$. We get a filtration such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable with slope μ_i, where $\mu_1 > \mu_2 > \ldots > \mu_r$.

We re-index the filtration as follows. For $\alpha \in \mathbb{R}$, let

$$HN^\alpha(\mathcal{E}) = \mathcal{E}_{\max\{i|i=0 \text{ or } \mu_i \geq \alpha\}}\,.$$

Then $HN^\alpha \supset HN^\beta$ if $\alpha \leq \beta$.

REMARK: If $f : \mathcal{E} \longrightarrow \mathcal{F}$ is a homomorphism, then $f(HN^\alpha(\mathcal{E})) \subset f(HN^\alpha(\mathcal{F}))$ (for all α). In particular, HN^α does not depend on the choices made in the definition.

PROOF: Let $\mathcal{E}_i = HN^\alpha(\mathcal{E})$ and choose β maximal (i.e. j minimal) such that $f(\mathcal{E}_i) \subset HN^\beta(\mathcal{F}) = \mathcal{F}_j$. Then we have maps

$$\ldots \subset \mathcal{E}_{i-1} \subset \mathcal{E}_i \xrightarrow{f} \mathcal{F}_j \xrightarrow{\pi} \mathcal{F}_j/\mathcal{F}_{j-1} \longrightarrow 0$$

and the composite $\pi \circ f$ is not zero (since j was minimal). Suppose that $HN^\beta(\mathcal{F}) \not\subset HN^\alpha(\mathcal{F})$, then $\alpha > \beta$. Now $\mathcal{F}_j/\mathcal{F}_{j-1}$ is semistable with slope β, and for $1 \leq \nu \leq i$, $\mathcal{E}_\nu/\mathcal{E}_{\nu-1}$ is semistable with slope $\geq \alpha > \beta$, hence $\pi \circ f$ induces the trivial map on $\mathcal{E}_\nu/\mathcal{E}_{\nu-1}$, for $\nu = 0, 1, \ldots, i$. Hence $\pi \circ f = 0$ on $\mathcal{E}_i$, a contradiction. □

Consider a semistable vector bundle $\mathcal{E}$ of slope μ. Then

$$\sum_{\substack{\mathcal{F}\subset\mathcal{E} \\ \mathcal{F} \text{ stable} \\ \mu(\mathcal{F})=\mu}} \mathcal{F} = \bigoplus_{i\in I} \mathcal{F}_i \qquad \text{with } I \text{ finite.}$$

(For $\mathcal{F}_0$ stable with slope μ, $\mathrm{Hom}\,(\mathcal{F}_0, \sum \mathcal{F})$ *tells us how many copies of $\mathcal{F}_0$ occur in the sum.)* Continue with this in the quotient. We see that we get the semistable bundles from the stable ones by direct sums and extensions: If $\mathcal{E}_1$ and $\mathcal{E}_2$ are semistable with the same slope μ and we have an extension

$$0 \longrightarrow \mathcal{E}_1 \xrightarrow{\iota} \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0\,,$$

then $\mathcal{E}$ is also semistable (look at the HN filtration).

Let t be a section of $\mathcal{O}(1)$ (on $\mathbb{P}^1$) which has its (only and simple) zero in ∞. Consider the following bundles on $C \times \mathbb{P}^1$:

$$\tilde{\mathcal{E}} \;=\; \Big(pr_1^*\mathcal{E} \oplus (pr_1^*\mathcal{E}_1 \otimes pr_2^*\mathcal{O}(1))\Big)\Big/pr_1^*\mathcal{E}_1$$

$$\bigl(\text{where } pr_1^*\mathcal{E}_1 \hookrightarrow \ldots \text{ by } (\iota, 1 \otimes pr_2^*(-t))\bigr)$$

$$\begin{aligned} \tilde{\mathcal{E}}_1 &= pr_1^*\mathcal{E}_1 \otimes pr_2^*\mathcal{O}(1) \\ \tilde{\mathcal{E}}_2 &= pr_1^*\mathcal{E}_2 \end{aligned}$$

We get an extension of bundles on $C \times \mathbb{P}^1$:

$$0 \longrightarrow \tilde{\mathcal{E}}_1 \longrightarrow \tilde{\mathcal{E}} \longrightarrow \tilde{\mathcal{E}}_2 \longrightarrow 0\,.$$

For $x \in \mathbb{P}^1$, $x \neq \infty$, we have $\tilde{\mathcal{E}}|_{C\times\{x\}} \cong \mathcal{E}$ (from the inclusion of $\mathcal{E}$ as the first summand), but $\tilde{\mathcal{E}}|_{C\times\{\infty\}} \cong \mathcal{E}_2 \oplus \mathcal{E}_1$. Looking at the map to moduli space corresponding to $\tilde{\mathcal{E}}$, we get

$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & \text{moduli space of semistable bundles} \\ \cup & & \Cup \\ \mathbb{A}^1 & \longrightarrow & [\mathcal{E}] \\ \infty & \longmapsto & [\mathcal{E}_1 \oplus \mathcal{E}_2] \end{array}$$

so $[\mathcal{E}]$ has to be the same class as $[\mathcal{E}_1 \oplus \mathcal{E}_2]$. Hence we can only expect a correspondence

$$(\text{Points in moduli space}) \quad \longleftrightarrow \quad \begin{pmatrix} \text{Semistable bundles which are} \\ \text{direct sums of stable bundles} \end{pmatrix}$$

Result

Consider $\mathcal{E}$ on $C \times S$; for each $s \in S$ look at the *HN* filtration of $\mathcal{E}$ in $C \times \overline{\{s\}}$. This gives a collection of slopes and ranks.

Claim: This is a *constructible function.* (That is, $S = \coprod S_i$ with constructible S_i ('constructible' = finite union of intersections of an open and a closed set) such that the function is constant on each S_i.)

PROOF: Inductive criterion: We have to show that if $T \subset S$ is irreducible, the function is constant on a non-empty open subset $U \subset T$. Hence we may assume $S = T$ and, if necessary, replace S by an open subset such that $S = \operatorname{Spec} R$ with an integral domain R. Let $\eta \in S$ be the generic point. Over $C \times \overline{k(\eta)}$, $\mathcal{E}$ has a *HN* filtration which is defined over a finite extension K

of $k(\eta)$ (in fact, even over $k(\eta)$). Replace $R \subset k(\eta)$ by its normalization R' in K, then $S' = \operatorname{Spec} R' \twoheadrightarrow S$. Assume the assertion holds over S', then the *HN* filtration is constant on $R'_{f'}$ with some $0 \neq f' \in R'$ and hence it is also constant on R_f with $f = \operatorname{Norm}(f')$. Therefore, without loss of generality, we may assume that the *HN* filtration is defined over $k(\eta) = \bigcup_{f \neq 0} R_f$.

We can assume that the $\mathcal{E}_{i,\eta} = HN^\alpha(\mathcal{E}_\eta)$ extend to subbundles $\mathcal{E}_i$ of $\mathcal{E}$ on $C \times S$. Then it is enough to show that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable of constant slope on each fiber over an (non-empty) open subset of S. Hence we have reduced to showing

$$\mathcal{E}_{\overline{\eta}} \text{ semistable} \quad \Longrightarrow \quad \mathcal{E}|_{C \times \overline{\{s\}}} \text{ semistable for } s \in \text{open subset}. \tag{3}$$

Tensor with a suitable line bundle $\mathcal{L}$: $\mathcal{E} \to \mathcal{E} \otimes \mathcal{L}$ such that $\mu(\mathcal{E}) \geq 2g$ (where g is the genus of C).

Remark: Any semistable $\mathcal{F}$ with $\mu(\mathcal{F}) \geq 2g$ has $H^1(C, \mathcal{F}) = 0$ and is generated by global sections.

Proof: Ω^1_C is a line bundle of degree $2g-2$, hence semistable with $\mu = 2g-2$. *Let $\mathcal{V}$ be a semistable vector bundle with $\mu(\mathcal{V}) > 2g-2$. Then* $\operatorname{Hom}(\mathcal{V}, \Omega^1_C) = 0$. *But this is dual to $H^1(C, \mathcal{V})$, hence $H^1(C, \mathcal{V}) = 0$ as well.* Take $x \in C$ and let $\mathcal{F}(-x) = \mathcal{F} \otimes \mathcal{L}(-x)$ (*$\mathcal{L}(-x)$ is a line bundle of degree -1, consisting of regular functions vanishing in x*); then $\mu(\mathcal{F}(-x)) = \mu(\mathcal{F}) - 1 > 2g-2$, hence $H^1(C, \mathcal{F}(-x)) = 0$, and therefore, $H^0(C, \mathcal{F}) \twoheadrightarrow H^0(C, \mathcal{F}/\mathcal{F}(-x)) = \mathcal{F}_x$, i.e. $\mathcal{F}$ is generated by global sections. □

Suppose that $\mathcal{E}|_{C \times \{s\}}$ is not semistable. Then there is some $\mathcal{F} \subset \mathcal{E}|_{C \times \{s\}}$ which is semistable of slope $\alpha > \mu(\mathcal{E}) \geq 2g$. $\mathcal{F}$ is generated by

$$W = \Gamma(C \times \overline{\{s\}}, \mathcal{F}) \subset V = \Gamma(C, \mathcal{E}) \otimes \overline{k(s)}\,.$$

We therefore have to test all subspaces $W \subset V$ if they generate a vector bundle $\mathcal{F}$ with $\mu(\mathcal{F}) > \mu(\mathcal{E})$.

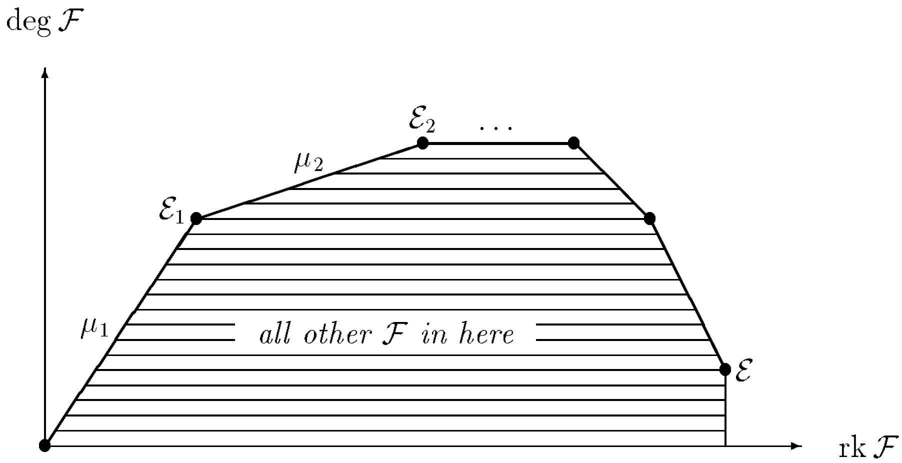
Let $G = \coprod$ Grassmannians be the projective variety parametrizing all possible W's. Over G, we have a universal bundle $\mathcal{W} \subset \Gamma(C, \mathcal{E}) \otimes \mathcal{O}_G$. It generates a subsheaf $\mathcal{F} \subset pr^*\mathcal{E}$ on $C \times G \times S$. Find a decomposition $G \times S = \coprod T_i$ into constructible subsets T_i such that $\mathcal{F}|_{C \times T_i}$ is flat over T_i.

Then the rank and degree are locally constant on T_i, hence (*by taking a finer decomposition*) we may assume them constant, in particular we have constant slope μ_i on T_i.

Consider those T_i with $\mu_i > \mu(\mathcal{E})$: The (by CHEVALLEY) constructible subset $pr_S(T_i)$ of S does not contain the generic point η of S (because $\mathcal{E}_\eta$ is semistable), hence lies in a proper closed subset of S. Now, if we replace S by the open complement of these proper closed subsets, then these T_i must disappear, and the proof is complete. □

Semicontinuity

Proposition 1 *Take some vector bundle $\mathcal{E}$ and look at the pairs* $(\operatorname{rk}\mathcal{F}, \deg\mathcal{F})$ *for all subbundles $\mathcal{F} \subset \mathcal{E}$. Then we get the following picture:*



Under specialization, the domain may at most become bigger.

PROOF: *The picture is clear. To prove the statement about specialization:* Let V be a discrete valuation ring and $S = \operatorname{Spec} V$. Consider $\mathcal{E}$ on $C \times S$. Let $\eta \in S$ be the generic point and $\mathcal{E}_{i,\eta}$ the generic *HN* filtration (defined over η). Extend $\mathcal{E}_{i,\eta}$ to a subsheaf $\mathcal{E}_i \subset \mathcal{E}$ over $C \times S$. Reflexive (?), hence these are vector bundles, and $\mu_i = \mu(\mathcal{E}_i)$ coincides on both fibers. Hence on the special fiber, we have subbundles $\mathcal{E}_i \otimes k(s) \subset \mathcal{E} \otimes k(s)$ mapping to the same points as $\mathcal{E}_{i,\eta}$. Hence the region for $\mathcal{E}_s$ contains that for $\mathcal{E}_\eta$.

If S is arbitrary with $s, t \in S$ such that $s \in \overline{\{t\}}$, there is a discrete valuation ring V and a map $\operatorname{Spec} V \to S$ mapping η to t and the special point of $\operatorname{Spec} V$ to s. (*Is this possible when* $\dim \overline{\{t\}} - \dim \overline{\{s\}} \geq 2$ *? Maybe one has to break this up into several steps.*) Now use the above argument.

By specializing step by step we may assume S to be local and of dimension 1. By pulling $\mathcal{E}$ back to the normalization $\overline{S} \to S$, we may assume $S = \operatorname{Spec} V$, with V a discrete valuation ring. [CK] $\square$

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(Raw notes by BERND STEINERT.)

3 Geometric Invariant Theory

Why do we need this?

Approach to moduli spaces: First, construct moduli space for bundles with additional structure. Then, eliminate additional structure by taking invariants (*see below*).

Let $\mathcal{E}$ be a semistable bundle with $\deg \mathcal{E} = d$, $\operatorname{rk} \mathcal{E} = r$. We may assume $\mu = d/r \geq 2g$, then $\Gamma(C, \mathcal{E})$ generates $\mathcal{E}$ and $H^1(C, \mathcal{E}) = 0$ (*see last lecture*). We have (by RIEMANN–ROCH)

$$\dim \Gamma(C, \mathcal{E}) = d + r(1 - g) = N\,.$$

We want to parametrize $(\mathcal{E}, (e_1, \ldots, e_N))$, where $e_1, \ldots, e_N$ is a basis of the global sections $\Gamma(C, \mathcal{E})$.

Proposition 2 *This gives a* representable functor *(*GROTHENDIECK*):*

PROOF: *(or rather, Explanation)* $\mathcal{E}$ is a quotient of $V \otimes \mathcal{O}$ with a vector space V of dimension $\dim V = N$. There is an algebraic scheme $\mathcal{H}$ (HILBERT scheme) and a universal quotient sheaf $V \otimes \mathcal{O}_{C\times\mathcal{H}} \to \mathcal{F}$, which is flat over $\mathcal{O}_{\mathcal{H}}$.

One has

$$\mathcal{H} = \coprod_{(d,r)} \mathcal{H}_{(d,r)}\,,$$

where on $\mathcal{H}_{(d,r)}$, the degree d and the (generic) rank r are fixed.

(The hard part in the proof is to show that $\mathcal{H}_{(d,r)}$ is of finite type.)

Over a suitable open subset $\mathcal{H}'$ of $\mathcal{H}_{(d,r)}$, we have that

(a) $\mathcal{F}$ is a vector bundle;

(b) $\mathcal{F}$ is semistable on fibres ($\Rightarrow \pi_*\mathcal{F}$ is a vector bundle of rank N, $\pi : C \times \mathcal{H} \to \mathcal{H}$);

(c) $V \otimes \mathcal{O} \to \pi_*\mathcal{F}$ is an isomorphism.

Then $\mathcal{H}'$ is representing the functor described above. □

Now, try to get rid of the additional data.

Two different bases of $\Gamma(C, \mathcal{E})$ are related by $\mathbf{GL}(N)$. So $\mathbf{GL}(N)$ acts on $\mathcal{H}'$. Our moduli space $\mathcal{M}$ should be the quotient $\mathbf{GL}(N)\backslash\mathcal{H}'$. (The scalars act trivially, whence an action of $\mathbf{PGL}(N)$ *(this refers to the first formulation, where bases mod scalars were to be parametrized).*)

Remark: $\mathcal{H}_{(d,r)}$ is projective, hence $\mathcal{H}'$ is quasi-projective.

Proof: (Sketch) Use some Grassmannian.

Take a line bundle $\mathcal{L}$ of degree $\deg \mathcal{L} \gg 0$ such that for any $V \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{E}$ (with $\deg \mathcal{E} = d$, $\operatorname{rk} \mathcal{E} = r$) with kernel $\mathcal{K}$, $\mathcal{K} \otimes \mathcal{L}$ is generated by $H^0(C, \mathcal{K} \otimes \mathcal{L})$, and $H^1(C, \mathcal{K} \otimes \mathcal{L}) = 0$ (bounded family (?)).

Note that

$$H^0(C, \mathcal{K} \otimes \mathcal{L}) \quad \hookrightarrow \quad \underbrace{V \otimes H^0(C, \mathcal{L})}_{\text{has fixed dimension}} \quad \twoheadrightarrow \quad H^0(C, \mathcal{E} \otimes \mathcal{L})$$

defines an embedding of $\mathcal{H}_{(r,d)}$ into some Grassmannian G, and the ample line bundle $\det H^0(C, \mathcal{E} \otimes \mathcal{L})$ on $\mathcal{H}_{(d,r)}$ corresponds to the ample line bundle on G given by the dual of det(subspace), i.e., det(quotient space). □

How to construct quotients?

In $\operatorname{char} k = 0$, we use Invariant Theory.

Let G be a reductive group (e.g. $\mathbf{GL}_N$, $\mathbf{SL}_N$, $\mathbf{PGL}_N$).

Problem: $X \subset \mathbb{P}^N$ quasi-projective, G acts on X via $G \to \mathbf{GL}_N$. What is X/G ?

Try: $R = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}(n)) = \bigoplus_{n \geq 0} R_n$. Is $X/G = \operatorname{Proj} R^G$?

Hilbert: R^G is finitely generated over k.

The key property of G is that for any G–surjection $V \twoheadrightarrow W$ of vector spaces, the induced map $V^G \to W^G$ is again surjective. *(This is equivalent to $H^1(G,V) = 0$ for every vector space V with G–action: It is equivalent to $H^1(G,V) \to H^1(G,W)$ being an injection for every G–injection of vector spaces $V \hookrightarrow W$. Now embed V into an injective kG–module W (cf.* [Brown: Cohomology of Groups]*). Then $H^1(G,W) = 0$, whence $H^1(G,V) = 0$.)* (Since G is reductive, G acts semi-simply on inductive limits of algebraic representations. This implies the 'lifting property' above.) [CK]

(In characteristic $p > 0$ the induced map is no longer surjective, but at least some p–power can always be lifted.)

Let $I_+ = \bigoplus_{n>0} R_n$, this is an ideal of R. $I_+^G\, R$ is a homogeneous ideal, hence $I_+^G\, R = (f_1, \ldots, f_r)$ with homogeneous $f_i \in I_+^G$. Then the f_i generate I_+^G as an R^G–module:

Take a homogeneous $f \in I_+^G$. There are $r_i \in R$ with $f = \sum r_i f_i$, i.e. we have a G–surjection $R^r \twoheadrightarrow I_+^G\, R$. Hence (*by the lifting property above*) $(R^G)^r \twoheadrightarrow (I_+^G\, R)^G = I_+^G$, which means that the r_i can be taken in R^G.

This implies that the f_i generate R^G (as a k–algebra). We now have an embedding

$$\operatorname{Proj} R^G \hookrightarrow \mathbb{P}^M$$

defined via all monomials in f_i of degree $\operatorname{lcm}\{\deg f_i\}$.

There are maps

$$\begin{array}{rcl} \{\text{homog. ideals of } R^G\} & \overset{\longrightarrow}{\longleftarrow} & \{\text{homog. ideals of } R\} \\ I & \mapsto & I\,R \\ I^G = I \cap R^G & \leftarrow\!\shortmid & I \end{array}$$

where $\leftarrow \circ \rightarrow = \mathrm{id}$.

Recall: Proj is given by the homogeneous prime ideals except $I_+ = \bigoplus_{n>0} R_n$. This means that on the Proj–level, there is (*in general*) no map corresponding to $\leftarrow$ above.

Remark: (and **Definition**) One can define a quotient map only on homogeneous prime ideals which do not contain $\bigoplus_{n>0} R_n^G$. The corresponding points x are called *semistable*; they are given by the property that there is some f

in some R_n^G $(n > 0)$ with $f(x) \neq 0$. Sometimes we will also call a point $x^* \in \mathbb{A}^{N+1}$ mapping to $x \in X \subset \mathbb{P}^N$ semistable.

Let X^{ss} be the complement in X of the zero-set of all R_n^G for $n > 0$. This is an open subset of X.

REMARK:

$$x \text{ semistable} \iff 0 \notin \overline{Gx^*}$$

(for some/any $x^* \mapsto x$.)

PROOF: "$\Rightarrow$": *There is a* $f \in R_n^G$ *with* $n > 0$ *and* $f_0 = f(x^*) \neq 0$. *Then* $f - f_0$ *vanishes on* $\overline{Gx^*}$, *but not in* 0.

"$\Leftarrow$": Let $J \subset R$ be the ideal defining $\overline{Gx^*}$, then $J \not\subset I_+ = \bigoplus_{n>0} R_n$ and hence $J \twoheadrightarrow k = R/I_+$, where the G–action on the right hand side is trivial. Hence $J^G \twoheadrightarrow k$, i.e. there is a $f = f_0 + f_1 + \cdots \in J^G$ with $f_0 \neq 0$. But then (since $f(x^*) = 0$) some $f_i(x^*) \neq 0$ for some $i > 0$. □

We get a map

$$\begin{array}{ccc} X^{\mathrm{ss}} & \longrightarrow & \operatorname{Proj} R^G \\ I & \mapsto & I^G \end{array} .$$

What are the fibers of this map?

x and y (or x^* and y^*) lie in the same fiber iff some $f \in R_n^G$ for some $n > 0$ vanishes at both of them *(?—this should read 'iff for all* $n > 0$ *and all* $f \in R_n^G$, f *vanishes on* x *iff* f *vanishes on* y*'.)*. We hope that each fiber contains precisely one closed (in X^{ss}) G–orbit. This is indeed true.

PROOF:

(a) Any fiber contains a closed orbit (take some x with $\dim Gx$ minimal).

(b) We have to show: If Gx and Gy are closed in X^{ss} and x and y have the same image in $\operatorname{Proj} R^G$, then $Gx = Gy$.

Choose some $f \in R_d^G$ with $d > 0$ such that $f(x^*) \neq 0$ and $f(y^*) \neq 0$. Let $S = R_{(f)}$ be the localization and $Y = \operatorname{Spec} S$ the open subset defined by f in $X^* \subset \mathbb{A}^{N+1}$. Y projects to X^{ss}. Gx and Gy define G–invariant ideals $I, J \subset S$, resp. Assume $Gx \neq Gy$, then $Gx \cap Gy = \emptyset$ and hence $I + J = S$. By the lifting property, $I \twoheadrightarrow (S/J)^G \ni 1$ $(\dots)$ *??*

On the other hand, let I_x, $I_y \subset S$ *be the homogeneous ideals corresponding to* x *and* y, *resp. Then* $I_x^G = I_y^G = I'$ *and* $I = I^G \subset I_x^G$, $J \subset I_y^G$. *This* *implies* $I+J \subset I' \subset I_x \neq S$, *a contradiction.*

Better: Let I, J, I_x, J_y be the homogeneous ideals in R corresponding to Gx, Gy, x and y, respectively. Since $Gx \cap Gy = \{0\}$, we have $I+J = I_+ = \bigoplus_{n>0} R_n$. Hence,

$$(I_+)^G = I^G + J^G \subset (I_x)^G + (J_y)^G = (I_x)^G\,,$$

which contradicts the fact that x is semi-stable. (The first equality comes from the lifting property, the last one from the fact that $(I_x)^G = (J_y)^G$ by assumption.) [CK] □

x is called *(properly) stable*, if it is semistable and $\dim G_y = 0$ for all y in an open neighborhood of x. X^{s} = stable points of X.

Example: Let $V = k^{N+1}$, $V = \bigoplus_{m\in\mathbb{Z}} V_m$, operation of $\mathbb{G}_{\mathrm{m}}$ on $\mathbb{P}^N = \mathbb{P}(V)$ is given by $\lambda \cdot v = \lambda^m v$ for $v \in V_m$. Then $x^* = \sum x_m$ is

– semistable $\iff$ $\exists m \geq 0 : x_m \neq 0$ and $\exists m \leq 0 : x_m \neq 0$;

– stable $\iff$ in addition $x \neq x_0$.

To test (semi-)stability in general it turns out to be enough to do it for all $\mathbb{G}_{\mathrm{m}}$ in G.

Theorem 3 *x is semistable for G $\iff$ x is semistable for all $\rho : \mathbb{G}_{\mathrm{m}} \to G$.*

Proof: "⇒": $0 \notin \overline{Gx^*} \Longrightarrow 0 \notin \overline{\rho(\mathbb{G}_{\mathrm{m}})x^*}$.
"⇐": Use valuative criterion: If $0 \in \overline{Gx^*}$ then there is a discrete valuation ring $V \cong k[[t]]$ with quotient field K such that there is a commutative diagram

$$\begin{array}{ccccc} \operatorname{Spec} K & \longrightarrow & G & \ni & g \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} V & \longrightarrow & \mathbb{A}^{N+1} & \ni & g\,x^* \end{array}$$

and such that for $g \in G(K)$, $g\,x^*$ is integral and goes to 0 as $t \to 0$. For $G = \mathbf{GL}_{N+1}$ ($\mathbf{SL}_{N+1}$, $\mathbf{PGL}_{N+1}$):

$$g = g_1 \begin{pmatrix} t^{a_0} & & 0 \\ & \ddots & \\ 0 & & t^{a_N} \end{pmatrix} g_2$$

with $g_1, g_2 \in G(V)$, hence $\begin{pmatrix} t^{a_0} & & \\ & \ddots & \\ & & t^{a_N} \end{pmatrix} g_2(x^*)$ is integral. Write $g_2 = g_2(0) +$ higher order terms and replace x^* by $g_2(0)(x^*)$: We may assume that $g_2 \equiv 1 \bmod t$.

$$g_2(x^*) = \sum_{n \geq 0} x^{*,n} t^n = \sum_{n \geq 0} (x_0^{*,n}, \ldots, x_N^{*,n}) t^n$$

The integrality condition implies $x_i^{*,n} \neq 0 \Rightarrow a_i + n > 0$. For $n = 0$: $x^{*,0} = x^*$ (since $g_2 \equiv 1 \bmod t$), hence for $x_i^* \neq 0$, we have $a_i > 0$. This means that x^* is not semistable for this 1–parameter subgroup of G. □

Theorem 4 *x is stable for G $\iff$ x is stable for all non-trivial $\rho : \mathbb{G}_\mathrm{m} \to G$.*

PROOF: "$\Rightarrow$": Clear.
"$\Leftarrow$": i) $G = \mathbb{G}_\mathrm{m}$: $G \times X^\mathrm{s} \to X^\mathrm{s} \times X^\mathrm{s}$ is a proper map (even finite). Can cover X^s by invariant affines where $x_a \neq 0, x_b \neq 0$ with $a \geq 0, b < 0$ (or $a > 0, b \leq 0$). Invert x_a: $\mathbb{G}_\mathrm{m}$ onaffine space, one coordinate is x_b/x_a (deg < 0), invert this (*??*)
ii) Consider $U \subset X^\mathrm{ss}$ where the stabilizer has dimension 0. U is constructible and open (stbilizer will jump under specialization). Claim: $G \times U \to U \times U$ proper. (…) □

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(Raw notes by BERND STEINERT.)

Recall:

$$\begin{aligned}
x \text{ semistable} &\iff 0 \notin \overline{Gx^*} \\
&\iff \exists d > 0, f \in R_d^G : f(x) \neq 0 \\
&\iff \forall \rho : \mathbb{G}_\mathrm{m} \longrightarrow G \text{ 1–parameter subgroup :} \\
&\qquad \text{the weights of this } \mathbb{G}_\mathrm{m}\text{–action occurring in } x \\
&\qquad \text{are neither all } > 0 \text{ nor all } < 0 \\
x \text{ stable} &\iff G \longrightarrow \mathbb{A}^{N+1}, g \mapsto gx^*, \text{ is proper} \\
&\iff Gx \text{ is closed and } G_x \text{ is finite}
\end{aligned}$$

REMARK:

(a) Assume x semistable, $f \in R_d^G$ (*for some* $d > 0$) with $f(x) \neq 0$. If G_y (the stabilizer of y) is finite for all y with $f(y) \neq 0$, then all orbits in there (*in where ? — maybe in* X_f) have dimension $\dim G$ and are closed. ($\overline{Gx} \setminus Gx = \bigcup$ orbits of smaller dimension.)

(b) Assume Gx closed, G_x finite. *Then (?)* $\{y \mid \dim G_y \geq 1\}$ is closed, G–invariant and does not meet $Gx = \overline{Gx}$. Hence there is some $h \in R^G \setminus \{0\}$ such that h vanishes on $Y = \{y \mid \dim G_y \geq 1\}$ *but not at* x. (Let I_Y and I_{Gx} be the homogeneous ideals corresponding to Y and Gx, resp. Since $Y \cap Gx = \emptyset$, we have $I_+ = I_Y + I_{Gx}$, therefore $(I_+)^G = (I_Y)^G + (I_{Gx})^G$. If we had $(I_Y)^G \subset (I_{Gx})^G$, we would get $(I_+)^G = (I_{Gx})^G = (I_x)^G$, contradicting (semi-)stability of x.) [CK] Since Y is $\mathbb{G}_\mathrm{m}$–invariant, we can take h to be homogeneous. Then change f to fh. *Thus we have the situation of '(a)'.* This implies that the set of stable points is open. [CK]

1–parameter subgroup criterion *for (semi-)stability*:
For all $\rho : \mathbb{G}_\mathrm{m} \longrightarrow G$ non-trivial, positive as well as negative weights occur in x. (*I.e., take a basis* $b_0, \ldots, b_N$ *of* $\mathbb{A}^{N+1}$ *diagonalizing the* $\mathbb{G}_\mathrm{m}$*–action; the*

weight of b_j is w with $\rho(\lambda)b_j = \lambda^w b_j$. w occurs in $x = (x_0 : \ldots : x_N)$ (written w.r.t. the basis b_j) if there is a j such that $x_j \neq 0$ and b_j has weight w.)

Valuative criterion *(for what?)*:
$V \subset K$ discrete (?) valuation ring with quotient field, $g \in G(K)$,

$$gx^* \in \mathbb{A}^N(V) \overset{?}{\Longrightarrow} \underbrace{g \in G(V)}_{\textit{really?}}, g = g_1 \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} g_2 \text{ with } g_1, g_2 \in G(V).$$

Example: V, W vector spaces, $G = \mathbf{SL}(V)$ acting on $\mathrm{Grass}_d(V \otimes W)$. Here, $\mathrm{Grass}_d(V \otimes W)$ classifies quotients $V \otimes W \twoheadrightarrow E$ with $\dim E = d$.

Criterion: E is $\left\{\begin{smallmatrix}\text{semistable}\\ \text{stable}\end{smallmatrix}\right\}$ iff for all non-trivial subspaces $V' \subset V$ and $E' = \mathrm{im}(V' \otimes W \to E)$, we have

$$\frac{\dim E'}{\dim E} \left\{\begin{smallmatrix}\geq\\ >\end{smallmatrix}\right\} \frac{\dim V'}{\dim V}.$$

Proof:
"$\Longrightarrow$": Write $V = V' \oplus V''$ with $\dim V' = a$, $\dim V'' = b$, and let

$$\rho : \mathbb{G}_{\mathrm{m}} \longrightarrow \mathbf{SL}(V), t \mapsto (t^b \,\mathrm{id}_{V'}, t^{-a} \,\mathrm{id}_{V''}).$$

We obtain a projective embedding of $\mathrm{Grass}_d(V \otimes W)$ from $\Lambda^d(V \otimes W) \twoheadrightarrow \Lambda^d E = \det E$. *(Elements of $\Lambda^d(V \otimes W)$ give global sections of* $\det E$, which is an ample line bundle on $\mathrm{Grass}_d(V \otimes W)$ (unique up to powers). *We will look at the weights of E under ρ in this embedding.)*

We choose a basis of $V \otimes W$ consisting of $v \otimes w$ with $v \in$ basis of $V' \cup$ basis of V'' and $w \in$ basis of W. Then

$$v_1 \otimes w_1 \wedge \ldots \wedge v_d \otimes w_d \in \Lambda^d(V \otimes W)$$

has weight $b \cdot \#\{i \mid v_i \in V'\} - a \cdot \#\{i \mid v_i \in V''\}$.

Since E is semistable, a non-negative weight occurs in E. This means that there are v_i, w_i such that $\langle v_i \otimes w_i \rangle \xrightarrow{\cong} E$ and such that $b\alpha \geq a\beta$, where $\alpha = \#\{i \mid v_i \in V'\}$ and $\beta = \#\{i \mid v_i \in V''\}$.

Now, $\langle v_i \otimes w_i \mid v_i \in V' \rangle \hookrightarrow E'$, hence $\dim E' \geq \dim(\mathrm{im}(V' \otimes W)) \geq \alpha$. Since $b\alpha \geq a\beta$, we have

$$\frac{\dim E'}{\dim E} \geq \frac{\alpha}{\alpha + \beta} \geq \frac{a}{a+b} = \frac{\dim V'}{\dim V}.$$

"$\Longleftarrow$": Let ρ be a 1–parameter subgroup and write $V = \bigoplus_{i\in\mathbb{Z}} V_i$ such that $\rho(t)$ acts as t^i on V_i. Then $\sum_{i\in\mathbb{Z}} i \dim V_i = 0$ because $G = \mathbf{SL}(V)$.
Assume the stability criterion holds. Let $\pi : \Lambda^d(V \otimes W) \twoheadrightarrow \Lambda^d E$; then we have to show that $\pi(v) \neq 0$ for some v of weight ≥ 0. *What about weight ≤ 0? I think both should occur.* Take $t \mapsto t^{-1}$. [CK]

Let $F^\mu(V) = \bigoplus_{i\geq\mu} V_i$. Then

$$\begin{aligned}
0 &= \sum_{\mu\in\mathbb{Z}} \mu \dim \mathrm{gr}_F^\mu(V) \\
&= \sum_{\mu\in\mathbb{Z}} \mu(\dim F^\mu(V) - \dim F^{\mu+1}(V)) \\
&= \sum_{\mu\geq -N} \mu \dim F^\mu(V) - \sum_{\mu>-N} (\mu-1) \dim F^\mu(V) \\
&= \sum_{\mu>-N} \dim F^\mu(V) - N \dim \underbrace{F^{-N}(V)}_{=V}
\end{aligned}$$

for N big enough.

We look for $\{v_i \otimes w_i \mid 1 \leq i \leq d\}$ such that $\langle v_i \otimes w_i\rangle \xrightarrow{\cong} E$, and try to maximize the weight. We take a_μ elements $v_i \in F^\mu(V)$. The best strategy obviously is to take $a_\mu = \dim \mathrm{im}(F^\mu(V) \otimes W \to E)$. This is possible: Start with $F^n(V) \otimes W$ (n big), this gives $a_n = 0$. Then go down step by step, choosing $a_{n-1} - a_n$, $a_{n-2} - a_{n-1}$, ..., new basis elements. The weight is then

$$w = \sum_\mu \mu(a_\mu - a_{\mu+1}) = \sum_{\mu>-N} a_\mu - N a_{-N}\,.$$

We know:

$$\frac{a_\mu}{d} \geq \frac{\dim F^\mu(V)}{\dim V}\,.$$

Hence,

$$w \geq d \sum_{\mu>-N} \frac{\dim F^\mu(V)}{\dim V} - Nd = \frac{d}{\dim V}\Big(\sum_{\mu>-N} \dim F^\mu(V) - N \dim V\Big) = 0\,.$$

(For stability, one needs that there are at least two stages in the filtration.)

□

Remark: This *(what?)* never happens when $\dim W = 1$. I.e., for $\dim W = 1$, no E is semistable. [CK]

4 Application to vector bundles on curves

Let C be a smooth projective connected curve with genus $g = g(C)$. For a vector bundle $\mathcal{E}$ on C, let

$$\mu(\mathcal{E}) = \frac{\deg \mathcal{E}}{\operatorname{rk} \mathcal{E}} + 1 - g$$

(i.e. $h^0(C, \mathcal{E}) - h^1(C, \mathcal{E}) = \mu \operatorname{rk} \mathcal{E}$ (Riemann–Roch)). *Note that this is a modification of the slope μ used earlier.*

Let $\mathcal{O}(1)$ be an ample line bundle of degree 1 and define $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$, $\mathcal{E}(n) = \mathcal{E} \otimes \mathcal{O}(n)$. Then $\mu(\mathcal{E}(n)) = \mu(\mathcal{E}) + n$.

$\mathcal{M}(r, \mu)$ will denote the moduli space of (semi-)stable $\mathcal{E}$ with $\operatorname{rk} \mathcal{E} = r$ and $\mu(\mathcal{E}) = \mu$.

Since $\mathcal{E} \mapsto \mathcal{E}(a)$ induces an isomorphism $\mathcal{M}(r, \mu) \to \mathcal{M}(r, \mu + a)$, we may (without loss of generality) assume that $\mu(\mathcal{E}) \gg 0$.

If $\mu \gg 2g$ *($\mu \geq g + 1$ will do)*, then $\mathcal{E}$ is globally generated by $\Gamma(C, \mathcal{E})$ which has dimension $N = \mu\, r$. So,

$$\mathcal{M}(r, \mu) \hookrightarrow * \overset{\mathrm{SL}_N}{\twoheadleftarrow} Q$$

with a 'quotient scheme' Q *classifying quotients $\mathcal{E}$ of $\mathcal{O}^N$ with* Hilbert *polynomial $n \mapsto r(\mu + n) = \dim \Gamma(C, \mathcal{E}(n))$ (n big) (see later lectures).*

We shall choose some $\mu \gg 0$ (by shifting $\mu \mapsto \mu + a$). Once $\mu + a$ is chosen, choose $b \gg a$ such that for any exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}^{r(\mu+a)} \longrightarrow \mathcal{E}(a) \longrightarrow 0$$

(where $\mathcal{E}$ has the correct Hilbert polynomial), $\mathcal{K}(b)$ is generated by global sections, and $H^1(C, \mathcal{K}(b)) = 0$.

This gives an embedding

$$\begin{array}{rcl} Q & \hookrightarrow & \text{Grass} \circlearrowright \mathbf{SL}(r(\mu + a)) \\ \mathcal{E}(a) = \mathcal{O}^{r(\mu+a)}/\mathcal{K} & \mapsto & \big(\Gamma(C, \mathcal{K}(b)) \hookrightarrow \Gamma(C, \mathcal{O}(b))^{r(\mu+a)} \twoheadrightarrow \Gamma(C, \mathcal{E}(a + b))\big) \end{array}$$

Claim: For $a \gg 0$:

$$\big(\mathcal{E}\text{(semi-)stable vector bundle}\big) \iff \begin{pmatrix} \text{corresponding point in Grass} \\ \text{is (semi-)stable} \end{pmatrix}$$

To test stability (in Grass), we have to consider

$$V' \subset V = k^{r(\mu+a)} = \Gamma(C, \mathcal{E}(a))\,.$$

$\Gamma(C, \mathcal{F}(a)) = V' \iff \mathcal{F}(a) \subset \mathcal{E}(a)$ is generated by V'.
Idea: (For $a \gg 0$) $\mathcal{F} \subset \mathcal{E}$ nice sub-bundle with $\operatorname{rk} \mathcal{F} = r'$, $\mu(\mathcal{F}) = \mu'$

$$\begin{aligned} \dim \underbrace{\Gamma(\mathcal{F}(a))}_{=V'} &\sim r'(\mu' + a) \qquad \text{(by RR)} \\ \dim \underbrace{\Gamma(\mathcal{F}(a+b))}_{\supset \operatorname{im}(V \otimes W)} &\sim r'(\mu' + a + b) \end{aligned}$$

Condition:

$$\frac{\mu' + a + b}{\mu' + a} \geq \frac{\mu + a + b}{\mu + a}$$

For $b > 0$: $\mu' \leq \mu$.

Argument: Either $\mu(\mathcal{F}) = \mu(\mathcal{E})$ or $\mu(\mathcal{F}) \leq \mu(\mathcal{E}) - \varepsilon$ with some fixed ε.

As always, C is a nice curve of genus g, and we fix the data $r = \operatorname{rk}\mathcal{E}$ and $\mu = \mu(\mathcal{E}) = \frac{\deg\mathcal{E}}{\operatorname{rk}\mathcal{E}} + 1 - g$ such that $\mu \geq g+1$. Then $H^1(C,\mathcal{E}) = 0$, and by R–R, $N = \dim\Gamma(C,\mathcal{E}) = r\mu$.

(By GROTHENDIECK,) there is a 'quotient scheme' Q *parametrizing (?)* quotients $\mathcal{O}_C^N \twoheadrightarrow \mathcal{E}$ with HILBERT polynomial $n \mapsto r(\mu+n)$. (*I.e., for* $n \gg 0$, $\dim\Gamma(C,\mathcal{E}(n)) = r(\mu+n)$.) Q is of finite type.

Choose $b \in \mathbb{Z}$ so big that for all $\mathcal{K} = \ker(\mathcal{O}_C^N \twoheadrightarrow \mathcal{E})$, $H^1(C,\mathcal{K}(b)) = 0$ and $H^0(C,\mathcal{K}(b))$ generates $\mathcal{K}(b)$. (For b fixed, the condition defines an open subset U_b of Q. We have $\bigcup_b U_b = Q$ since for every $\mathcal{K}$ there is some b satisfying the condition. Because of quasi-compactness (and $U_b \subset U_{b+1}$), there is some b with $U_b = Q$.)

There is a map

$$\begin{array}{ccc} Q & \longrightarrow & \operatorname{Grass}_{r(\mu+b)} \\ (\mathcal{O}_C^N \twoheadrightarrow \mathcal{E}) & \longmapsto & \big(\Gamma(C,\mathcal{O}(b))^N \twoheadrightarrow \Gamma(C,\mathcal{E}(b))\big) \end{array}$$

(Grass parametrizing quotients). If b is chosen as above, $\phi : \Gamma(C,\mathcal{O}(b))^N \to \Gamma(C,\mathcal{E}(b))$ is surjective (since $H^1(C,\mathcal{K}(b)) = 0$ with $\mathcal{K} = \ker(\mathcal{O}_C^N \twoheadrightarrow \mathcal{E})$) (*hence the map is well-defined*), and we recover $\mathcal{K}(b)$ (and hence $\mathcal{K}$) as the sub-bundle of $\mathcal{O}(b)^N$ generated by the kernel of ϕ. Hence the map is an injection. *It is also equivariant with respect to the* $\mathbf{SL}_N$*–action on both sides.*

Let $V = k^N$ and $W \subset V$ a subspace. Consider the following situation:

$$\begin{array}{ccccccccc} & & & & V\otimes\mathcal{O}_C & = & \mathcal{O}_C^N & \twoheadrightarrow & \mathcal{E} \\ & & & & \cup & & & & \cup \\ 0 & \longrightarrow & \mathcal{K}_W & \longrightarrow & W\otimes\mathcal{O}_C & & & \twoheadrightarrow & \mathcal{F} \end{array}$$

W want to choose b in such a way that additionally $H^1(C,\mathcal{K}_W(b)) = 0$ for all these $\mathcal{K}_W$. This is possible: The pairs $(\mathcal{E}, W)$ are parametrized by a quasi-compact scheme S having a constructible stratification $S = \coprod_\alpha S_\alpha$ such that over every S_α, $\mathcal{E}/\mathcal{F}$ is flat (GROTHENDIECK). (*Then use semi-continuity and quasi-compactness arguments as above (?).*)

Theorem 5 MUMFORD *(semi-)stability of (the image of)* $\mathcal{E}$ *(in* Grass *) (for* b *big) is equivalent to (semi-)stability of* $\mathcal{E}$ *as a vector bundle.*

PROOF: *We use the result of the example in the last lecture. In our situation, this means that* $\mathcal{E}$ *is* MUMFORD *semistable iff for all subspaces* $W \subset V$,

$$\frac{\dim \operatorname{im}(W \otimes \Gamma(C, \mathcal{O}(b)) \to \Gamma(C, \mathcal{E}(b)))}{\dim W} \geq \frac{\dim \Gamma(C, \mathcal{E}(b))}{\dim V} = \frac{\mu + b}{\mu}. \qquad (4)$$

"$\Longrightarrow$": Fix a surjection $V \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{E}$. We show first that $V \hookrightarrow \Gamma(C, \mathcal{E})$: Let W be the kernel of $V = \Gamma(C, V \otimes \mathcal{O}_C) \to \Gamma(C, \mathcal{E})$. Then the image of $W \otimes \Gamma(C, \mathcal{O}(b)) \to \Gamma(C, \mathcal{E}(b))$ is zero, so by (4), W must be zero.

Now we have to show that $\mathcal{E}$ *is* a vector bundle (i.e., torsion free), and that it is semistable (in the vector bundle sense) with slope μ. For this, let $\mathcal{T}$ be the torsion of $\mathcal{E}$ and take $\mathcal{F}$ in the *HN* filtration of $\mathcal{E}/\mathcal{T}$ (which is a vector bundle) such that it contains exactly the portion of slope $\geq \mu$. Then $\mathcal{E}/\mathcal{F}$ is the part of slope $< \mu$. *Now,* $\mathcal{E}/\mathcal{T}$ *is semistable of slope* μ *iff* $\mu(\mathcal{E}/\mathcal{T}) \leq \mu$ *and* $\mathcal{E}/\mathcal{T}$ *has no (non-trivial) quotients with slope* $< \mu$. *This is equivalent to* $\mathcal{E}/\mathcal{F} = 0$, *since* $\mathcal{E}/\mathcal{F}$ *is the largest possible quotient of slope* $< \mu$ *and* $\deg(\mathcal{E}/\mathcal{T}) \leq \deg(\mathcal{E})$. *Since equality in this relation holds only for* $\mathcal{T} = 0$, *we also see that* $\mathcal{E}/\mathcal{F} = 0$ *implies that* $\mathcal{E}$ *is a semistable vector bundle of slope* μ. It suffices to show that $\mu' = \mu(\mathcal{E}/\mathcal{F}) \geq \mu$; this implies $\mathcal{E}/\mathcal{F} = 0$. Let $r' = \operatorname{rk} \mathcal{E}/\mathcal{F}$ and $r'' = \operatorname{rk} \mathcal{F}$, $\mu'' = \mu(\mathcal{F})$.

Let $W = \ker(V \to \Gamma(C, \mathcal{E}/\mathcal{F}))$. Since everything in $\mathcal{F}$ has slope $\geq \mu \geq g+1$, $H^1(C, \mathcal{F}) = 0$, and $\mathcal{F}$ is globally generated. Hence

$$0 \longrightarrow \Gamma(C, \mathcal{F}) \longrightarrow \Gamma(C, \mathcal{E}) \longrightarrow \Gamma(C, \mathcal{E}/\mathcal{F}) \longrightarrow 0$$

is exact, and $W = V \cap \Gamma(C, \mathcal{F})$. We get an induced injection

$$\Gamma(C, \mathcal{F})/W \hookrightarrow \Gamma(C, \mathcal{E})/V\,. \qquad (5)$$

The dimension of $\Gamma(C, \mathcal{E})/V$ is $\dim H^1(C, \mathcal{E})$, which injects into $H^1(C, \mathcal{E}/\mathcal{F})$ ($H^1(C, \mathcal{F}) = 0$), and we want to bound the dimension of the latter.

Since $V \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{E}/\mathcal{F}$, there are r' sections which generically generate $\mathcal{E}/\mathcal{F}$, i.e. we have a map $\phi : \mathcal{O}_C^{r'} \to \mathcal{E}/\mathcal{F}$ with finite cokernel. Then

$$H^1(C, \mathcal{O}_C^{r'}) \twoheadrightarrow H^1(C, \operatorname{im} \phi) \twoheadrightarrow H^1(C, \mathcal{E}/\mathcal{F})\,,$$

whence $h^1(C, \mathcal{E}/\mathcal{F}) = \dim H^1(C, \mathcal{E}/\mathcal{F}) \leq r'g$.
By definition of W,

$$W \otimes \Gamma(C, \mathcal{O}(b)) \hookrightarrow \Gamma(C, \mathcal{F}(b)) \subset \Gamma(C, \mathcal{E}(b)),$$

hence the dimension of the image (*let us call it I*) is at most $h^0(C, \mathcal{F}(b)) = r''(\mu'' + b)$. By (4),

$$\dim W \leq \frac{\mu}{\mu + b} \dim I \leq r'' \frac{\mu(\mu'' + b)}{\mu + b},$$

therefore (*using (5)*)

$$r''\mu'' = \dim \Gamma(C, \mathcal{F}) \leq r'' \frac{\mu(\mu'' + b)}{\mu + b} + h^1(C, \mathcal{E}/\mathcal{F}) \leq r'' \frac{\mu(\mu'' + b)}{\mu + b} + r'g. \quad (6)$$

If $h^1(C, \mathcal{E}/\mathcal{F})$ is zero, then from $\mu, b > 0$, we see that $\mu'' \leq \mu$ (or $r'' = 0$, *i.e.*, $\mathcal{F} = 0$). By definition of $\mathcal{F}$, we must have $\mu'' = \mu$, hence $\mu' = \mu$ as well (look at $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}/\mathcal{F} \longrightarrow 0$ and use $\mu = \mu(\mathcal{E})$). By definition of $\mathcal{F}$ again, this means $\mathcal{E}/\mathcal{F} = 0$. [CK]
So assume $H^1(C, \mathcal{E}/\mathcal{F}) \neq 0$. Then in the *HN* filtration of $\mathcal{E}/\mathcal{F}$ there must be a quotient of slope $\leq 2g - 2$ (*should be $g - 1$ — remember we shifted μ by $g - 1$.*) Then, *since the slopes in the HN filtration of $\mathcal{E}/\mathcal{F}$ are all $< \mu$*, we get

$$\deg \mathcal{E}/\mathcal{F} \leq r'(\mu + g - 1) - (\mu + 1 - g),$$

hence

$$\deg \mathcal{F} \geq r''(\mu + g - 1) + (\mu + 1 - g) \quad \Longrightarrow \quad r''\mu'' \geq r''\mu + \mu + 1 - g.$$

Plugging this into (6), we get

$$\mu + 1 - g \leq r''\mu \frac{\mu'' - \mu}{\mu + b} + r'g.$$

Since $\mu' \geq 0$ (everything is globally generated), μ'' can be bounded in terms of μ (*specifically*, $\mu'' \leq \frac{r}{r''}(\mu + g - 1) - (g - 1) \leq r\mu + (r - 1)(g - 1)$.) Hence by choosing μ big ($\mu \mapsto \mu + a$, everything $\otimes \mathcal{O}(a)$) and b even bigger, we get

a contradiction.
"$\Longleftarrow$": We are given a semistable vector bundle $\mathcal{E}$ of slope μ. Let $V = \Gamma(C, \mathcal{E})$, then we have $V \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{E}$. Let $W \subset V$ be a subspace and $\mathcal{F}$ the sub-bundle of $\mathcal{E}$ generated by W. We have to check (4). Let I be the image figuring in the numerator on the right hand side of (4). By our choice of b, $I = \Gamma(C, \mathcal{F}(b))$, and its dimension is the EULER characteristic of $\mathcal{F}(b)$. Let as before $r'' = \operatorname{rk} \mathcal{F}$ and $\mu'' = \mu(\mathcal{F})$. We have to show

$$\dim W \leq r'' \frac{\mu(\mu'' + b)}{\mu + b}.$$

Since $W \subset \Gamma(C, \mathcal{F})$, there is no harm in assuming $W = \Gamma(C, \mathcal{F})$. So we want to show that

$$h^0(C, \mathcal{F}) = \mu'' r'' + h^1(C, \mathcal{F}) \leq r'' \frac{\mu(\mu'' + b)}{\mu + b}.$$

If $h^1(C, \mathcal{F}) = 0$, this follows from $r'' \geq 0$, $\mu, b > 0$ and $\mu'' \leq \mu$. Otherwise, we have some step of slope $\leq g - 1$ in the *HN* filtration of $\mathcal{F}$, hence (as before)

$$\mu'' r'' \leq \mu r'' - (\mu + 1 - g),$$

and

$$\mu'' r'' + h^1(C, \mathcal{F}) \leq \mu r'' - (\mu + 1 - g) + g r'' < \mu r''$$

if μ is sufficiently big. But then, for b big enough, this is also $\leq \mu r''(\mu'' + b)/(\mu + b)$.
One should check that a single b works for all $\mathcal{E}$ in this proof. One should also check that the proof for 'stable' instead of 'semistable' does go through. □

Now we have

$$\begin{array}{ccccc} & & Q & \stackrel{i}{\hookrightarrow} & \text{Grass} \\ & & \cup & & \cup \\ \{\mathcal{E} \text{ semistable}\} & = & Q^{\text{ss}} = i^{-1}(\text{Grass}^{\text{ss}}) & \hookrightarrow & \text{Grass}^{\text{ss}} \\ & & \cup & & \cup \\ \{\mathcal{E} \text{ stable}\} & = & Q^{\text{s}} = i^{-1}(\text{Grass}^{\text{s}}) & \hookrightarrow & \text{Grass}^{\text{s}} \end{array}$$

where i is a closed immersion and we have a $G = \mathbf{SL}_N$–action on both sides. (The correspondence of (semi-)stable points on the left and right hand sides comes from the fact that suitable powers of G–invariants can be lifted.)

We can form the quootient

$$\begin{array}{ccc} Q^{\mathrm{ss}}/G & = & \mathcal{M} \\ \cup & & \cup \\ Q^{\mathrm{s}}/G & = & \mathcal{M}^{\mathrm{s}} \end{array}$$

$\mathcal{M}$ is our moduli space; its points correspond to the closed G–orbits in Q^{ss}. The points in $\mathcal{M}^{\mathrm{s}}$ correspond to the isomorphism classes of stable bundles (*their orbits are closed*).

Claim: Orbit of $\mathcal{E}$ is closed $\iff$ $\mathcal{E} = \bigoplus \mathcal{E}_i$ is a direct sum of stable bundles.

PROOF: (Sketch)

"$\Rightarrow$": If $\mathcal{E}$ is not semisimple (i.e., not of the form on the right hand side), we have a non-trivial extension

$$0 \to \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

with semistable bundles $\mathcal{E}'$, $\mathcal{E}''$ of the same slope. Let

$$V' = \Gamma(C, \mathcal{E}') \subset \Gamma(C, \mathcal{E}) = V = V' \oplus V''$$

and define a 1-parameter subgroup of G by

$$\mathbb{G}_{\mathrm{m}} \ni t \mapsto t^a \cdot 1_{V'} \oplus t^{-b} \cdot 1_{V''}$$

(*where* $a = \dim V''$ *and* $b = \dim V'$). $t \to \infty$ deforms $\mathcal{E}$ into $\mathcal{E}' \oplus \mathcal{E}''$, hence $\mathcal{E}' \oplus \mathcal{E}''$ is in the closure of the orbit of $\mathcal{E}$, hence in the orbit, and the extension was trivial, contradiction.

"$\Leftarrow$": Let $\mathcal{E} = \bigoplus \mathcal{E}_i$. There is a closed orbit within the closure of the orbit of $\mathcal{E}$; let $\mathcal{E}'$ belong to this closed orbit. Then, by "$\Rightarrow$", $\mathcal{E}'$ splits into a direct sum of stable bundles. Now by semicontinuity, the multiplicity of every specific $\mathcal{E}_i$ in $\mathcal{E}'$ is at least that of $\mathcal{E}_i$ in $\mathcal{E}$, hence (since ranks are equal) they must coincide, and $\mathcal{E}'$ is in the orbit of $\mathcal{E}$, whence this is closed. □

We have $\mathcal{M} = Q^{\mathrm{ss}}//\mathbf{SL}(N)$, $\mathcal{M}$ is projective. (The notation $X//G$ indicates that we take the invariants of the homogeneous coordinate ring to construct the quotient.)

We get an ample line bundle on $\mathcal{M}$ from an ample line bundle on Q:

$$\mathcal{O}^N \twoheadrightarrow \mathcal{E} \quad \text{gives a quotient} \quad \Gamma(C, \mathcal{O}(b))^N \twoheadrightarrow \Gamma(C, \mathcal{E}(b))\,,$$

and the determinant $\det \Gamma(C, \mathcal{E}(b))$ defines an ample line bundle on the GRASSMANNian which can be pulled back to Q. *We have to be careful, however, to get a* vector bundle *on the quotient—we have to take the* $\mathbf{SL}(N)$*–action into account. See below.*

Aside: Natural line bundles

Recall our quotient construction $R \to \operatorname{Proj} R^G$. We obtain coherent sheaves on the quotient from modules M^G, where M is a graded R–module with G–action.

A point x in the quotient corresponds to a closed semistable orbit $Y \cong G/G_x$. Then

$$\begin{aligned} M^G \text{ generates } M \text{ at } x &\iff (M|_Y)^G \text{ generates } M|_Y \\ &\iff M(x)^{G_x} \text{ generates } M(x) = M \otimes k(x) \\ &\iff G_x \text{ operates trivally on } M(x) \end{aligned}$$

(The first equivalence comes from the fact that the action of G commutes with taking quotients modulo (*invariant*) ideals.)

In our case, we have $x = [\mathcal{E}]$, and G_x is the subgroup of $\operatorname{Aut} \mathcal{E}$ consisting of elements having determinant 1 on $\Gamma(C, \mathcal{E})$. Hence we have to check whether $\det \Gamma(C, \mathcal{E}(b))$ has trivial $\mathbf{SL}(N)$–action. We have

$$\Gamma(C, \mathcal{O}(b)) \otimes \Gamma(C, \mathcal{E}) \twoheadrightarrow \Gamma(C, \mathcal{E}(b))\,,$$

where $\mathbf{GL}(N)$ acts on $\Gamma(C, \mathcal{E}) = k^N$. Try to find a linear combination

$$\det \Gamma(C, \mathcal{E}(b))^{\alpha} \otimes \det \Gamma(C, \mathcal{E})^{-\beta}$$

such that the scalars operate trivially: We may choose

$$\alpha = \dim \Gamma(C, \mathcal{E}) = r\mu \quad \text{and} \quad \beta = \dim \Gamma(C, \mathcal{E}(b)) = r(\mu + b)\,.$$

In this way we get an ample line bundle

$$\underbrace{\det \Gamma(C, \mathcal{E}(b))^{\otimes r\mu}}_{\text{ample}} \otimes \underbrace{\det \Gamma(C, \mathcal{E})^{\otimes(-r(\mu+b))}}_{\substack{\text{trivial line bundle} \\ \text{with non-trivial action}}}$$

with a $\mathbf{PGL}(N)$–action, which gives an ample line bundle on the quotient (*?*). *If $\mathcal{E}$ is stable, then* $\operatorname{Aut} \mathcal{E}$ = *scalars, whence we have trivial action of G_x and probably get a line bundle on the quotient.*

The Theorem of Narasimhan–Seshadri

We are now in the compley-analytic case: $k = \mathbb{C}$, and C is a compact Riemann surface of genus g. Then the fundamental group is

$$\pi_1(C) = \langle A_1, \ldots, A_g, B_1, \ldots, B_g \mid [A_1, B_1] \cdots [A_g, B_g] = 1 \rangle\,.$$

Let $\tilde{C}$ be the universal cover of C. Given a representation

$$\rho : \pi_1(C) \longrightarrow \mathbf{GL}_r(\mathbb{C})\,,$$

we get a locally constant sheaf E on C as $E = \tilde{C} \times \mathbb{C}^r / \pi_1(C)$ (*diagonal action*). Hence we have a holomorphic (therefore algebraic) vector bundle $\mathcal{E} = E \otimes \mathcal{O}_C$.

The representations ρ are parametrized by an algebraic variety

$$\operatorname{Rep}(\pi_1(C), \mathbf{GL}_r)$$

(take coordinates for $\rho(A_i)$, $\rho(B_i)$; the relation between the A_i and B_i gives equations for the coefficients); we get an analytic flat family of vector bundles (*over* C) on $\operatorname{Rep}(\pi_1(C), \mathbf{GL}_r)$.

A representation ρ is *unitary* if its image $\rho(\pi_1(C))$ is contained in $\mathbf{U}(r)$ (the subgroup of unitary matrices in $\mathbf{GL}_r(\mathbb{C})$). $\mathcal{E}$ then gets a flat hermitian metric.

Claim: ρ unitary $\Longrightarrow \mathcal{E}$ is semistable of degree 0.

PROOF: The degree is essentially the integral over C of the curvature, which is zero here. If $\mathcal{F} \subset \mathcal{E}$ is a sub-bundle, it gets an induced metric that has non-positive curvature. Hence $\deg \mathcal{F} \leq 0 = \deg \mathcal{E}$, and $\mathcal{E}$ is semistable.

To make this reasoning a bit more precise: We have the sub-bundle $\Lambda^{\mathrm{rk}\mathcal{F}}\mathcal{F} \subset \Lambda^{\mathrm{rk}\mathcal{F}}\mathcal{E}$, where $\Lambda^{\mathrm{rk}\mathcal{F}}\mathcal{E}$ is associated to the unitary representation $\Lambda^{\mathrm{rk}\mathcal{F}}\rho$ on $\Lambda^{\mathrm{rk}\mathcal{F}}\mathbb{C}^r$. Hence we may assume that $\mathcal{F}$ is a line bundle *(remember* $\deg \mathcal{F} = \deg \Lambda^{\mathrm{rk}\mathcal{F}}\mathcal{F}$*)*. Take an open cover $C = \bigcup_\alpha U_\alpha$ such that we have local generators f_α of $\mathcal{F}$ on U_α. In a unitary basis of $\mathcal{E}$ (which is local on U_α), f_α is given by holomorphic coordinate functions $(f_{\alpha,1}, \ldots, f_{\alpha,r})$. Consider $\|f_\alpha\|^2 = \sum_{j=1}^{r} |f_{\alpha,j}|^2$. This does not depend on the unitary basis, and $\log \|f_\alpha\|^2$ is either sub- or super-harmonic, i.e. $\partial\overline{\partial} \log \|f_\alpha\|^2 \leq 0$ *(this 2–form should glue over C (!))*. The degree of $\mathcal{F}$ is (essentially) the integral over C of $\partial\overline{\partial} \log \|f_\alpha\|^2$, hence ≤ 0. □

Claim: $\Gamma(C, \mathcal{E}) = (\mathbb{C}^r)^{\pi_1(C)}$ (the $\pi_1(C)$–invariants in $\mathbb{C}^r$).

PROOF: Let $f \in \Gamma(C, \mathcal{E})$. Then $\|f\|^2$ (defined locally with respect to some unitary basis) has a maximum somewhere on C (since C is compact). Locally, $\|f\|^2 = \sum |f_j|^2$ with holomorphic f_j, which then have to be constant *(?)* ($\log \|f\|^2$ is super-harmonic) [CK], *so f is a constant section, which has to come from E and then from an invariant in $\mathbb{C}^r$.* □

If we have two representations *with spaces* E_1 and E_2 and associated vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$, then the vector bundle corresponding to the representation on $\mathrm{Hom}(E_1, E_2)$ is $\mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2)$, and

$$\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2) = \Gamma(C, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2)) = \mathrm{Hom}_{\pi_1(C)}(E_1, E_2)\,.$$

Hence the functor $E \mapsto \mathcal{E}$ from unitary representations of $\pi_1(C)$ to vector bundles over C is fully faithful.

Theorem 6 *The functor we have constructed is an equivalence of categories:*

$$\begin{pmatrix}\text{unitary representations}\\ \text{of } \pi_1(C)\end{pmatrix} \longrightarrow \begin{pmatrix}\text{semisimple, semistable}\\ \text{vector bundles of degree } 0\end{pmatrix}$$
$$E \longmapsto \mathcal{E}$$

Proof: We have already shown that the image consists of semistable vector bundles of degree zero and that the functor is fully faithful. We have to show that $\mathcal{E}$ in the image is semisimple (i.e., direct sum of stable bundles). Since any unitary representation is semisimple (use orthogonal decomposition), we can assume E to be irreducible. We must then show $\mathcal{E}$ is stable. We induct on the rank of $\mathcal{E}$. If the rank is zero, nothing has to be shown. So, assume $\operatorname{rk}\mathcal{E} > 0$ and $\mathcal{E}$ not stable. Then there is a proper sub-budle $\mathcal{F} \subset \mathcal{E}$ of degree 0, which we can assume to be simple (i.e., stable) (take one of smallest rank). By our induction hypothesis, $\mathcal{F}$ comes from a representation F (*I don't see that*) which, by full faithfulness, has to inject into E, which therefore cannot be irreducible, contradiction. *(Maybe one can argue as follows: If $\mathcal{F} \subset \mathcal{E}$ has degree zero, its induced metric must be flat. Then $\mathcal{F} = \mathcal{O} \otimes \overline{F}$, where $\overline{F} \subset \overline{E}$ is a subsheaf of the sheaf of constant sections of $\mathcal{E}$. This should then pull back to a subrepresentation $F \subset E$.)*

Remark: We see that irreducible representations correspond to stable bundles.

It remains to show that we get all stable vector bundles of degree zero (up to isomorphism) from irreducible unitary representations.

Claim: We get a continuous *(or even real analytic?)* map

$$\operatorname{Rep}(\pi_1(C), \mathbf{U}(r)) \longrightarrow \mathcal{M}(r,0)_{\mathbb{C}}$$

($\mathcal{M}(r,0)_{\mathbb{C}}$ is the space for vector bundles of rank r and degree 0, as a complex analytic space).

Remark: This would be clear if we knew already that $\mathcal{M}(r,0)$ *is* a moduli space in some sense: *We have the universal bundle on $C \times \operatorname{Rep}(\pi_1(C), \mathbf{U}(r))$ whose fiber above $C \times (\rho, E)$ is $\mathcal{E}$ as constructed above.*

There is an open subset $\operatorname{Rep}(\pi_1(C), \mathbf{U}(r)) \subset U \subset \operatorname{Rep}(\pi_1(C), \mathbf{GL}_r)$ such that for E and $\mathcal{E}$ parametrized by U, $H^1(C, \mathcal{E}(a)) = 0$. (This is because of semi-continuity of $\dim H^1(C, \mathcal{E}(a))$: $H^1(C, \mathcal{E}(a)) \neq 0$ is a closed condition which is never true on $\operatorname{Rep}(\pi_1(C), \mathbf{U}(r))$.) Furthermore, we want that the kernel of $\Gamma(C, \mathcal{E}(a)) \otimes \mathcal{O}(b) \to \mathcal{E}(a+b)$ is globally generated. *(Similar argument.)*

Choose an open cover $U = \bigcup U_\alpha$ and over U_α a basis of $\Gamma(C, \mathcal{E}(a)) \cong \mathbb{C}^N$. This defines an analytic map $U_\alpha \longrightarrow Q_\mathbb{C}$. ($Q_\mathbb{C}$ is the same as the algebraically defined Q, since all quotients are algebraic.) We may assume that the image is contained in Q^{ss} (the complement is analytic, hence closed—*or recall that* X^{ss} *is open in* X *quite generally*). Then these maps glue to give a well-defined analytic map

$$U = \bigcup_\alpha U_\alpha \longrightarrow Q^{\mathrm{ss}}_\mathbb{C} \longrightarrow \mathcal{M}(r,0)_\mathbb{C}$$

Now, $\mathrm{Rep}\,(\pi_1(C), \mathbf{U}(r))$ is compact (can be identified with a closed subset of $\mathbf{U}(r)^{2g}$, and $\mathbf{U}(r)$ is compact), hence by restricting, we get a real-analytic (hence continuous), *closed* and proper map

$$\mathrm{Rep}\,(\pi_1(C), \mathbf{U}(r)) \longrightarrow \mathcal{M}(r,0)_\mathbb{C}\,.$$

The pre-image of the stable bundles consists exactly of the irreducible representations, hence we get a proper map

$$\phi : \mathrm{IrrRep}\,(\pi_1(C), \mathbf{U}(r)) \longrightarrow \mathcal{M}(r,0)^{\mathrm{s}}_\mathbb{C}$$

with closed image.

The image is also open: Both sides are manifolds. We show the map induces a surjection on tangent spaces *(then the map has to be open by the implicit function Thm.). But first show that* IrrRep *is a manifold:*

Claim: $\mathbf{GL}_r^{2g} \longrightarrow \mathbf{SL}_r, (A_1, \ldots, A_g, B_1, \ldots, B_g) \mapsto [A_1, B_1] \cdots [A_g, B_g]$, is smooth at points coming from irreducible representations of $\pi_1(C)$.

It is perfectly clear that this map is smooth everywhere. Does Faltings *think of representations modulo conjugation? He said something of cheating around here. Anyway, I don't quite understand the following lines:*

Tangent map is approx. $\sum[\delta A_i, B_i] - \sum[A_i, \delta B_i]$.
Adjoint map $\leftrightarrow$ components of commutators with A_i and $B_i \Rightarrow$ (only) scalars because of irreducibility.

The *real* dimension *of* $\mathrm{Rep}\,(\pi_1(C), \mathbf{U}(r))$ is $2gr^2 - (r^2 - 1)$.

As to the dimension of $\mathcal{M}^{\mathrm{s}}$, we have for a stable vector bundle $\mathcal{E}$ that $\mathrm{Aut}\,(\mathcal{E}) = \mathbb{G}_{\mathrm{m}}$ (so that we have only 'trivial' action, which implies $\mathcal{M}^{\mathrm{s}}$ is smooth at $\mathcal{E}$). The tangent space is given by $H^1(C, \mathcal{E}\text{nd}\,(\mathcal{E}))$. We know the EULER characteristic is $r^2(1-g)$. Since $H^0(C, \mathcal{E}\text{nd}\,(\mathcal{E})) = \text{End}\,(\mathcal{E}) = \mathbb{C}$, we get for the *(complex)* dimension of $\mathcal{M}^{\mathrm{s}}$:

$$\dim \mathcal{M}(r,0)^{\mathrm{s}}_{\mathbb{C}} = \dim H^1(C, \mathcal{E}\text{nd}\,(\mathcal{E})) = r^2(g-1)+1\,.$$

Now look at the tangent map of ϕ:

$$H^1(\pi_1(C), \mathfrak{su}(E)) \xrightarrow{\cong} H^1(\underbrace{\pi_1(C)}_{\textit{was } C}, \text{End}_{\mathbb{R}}(E)) \longrightarrow H^1(C, \mathcal{E}\text{nd}\,(\mathcal{E}))$$

It should be $\mathfrak{u}(E)$ *instead of* $\mathfrak{su}(E)$*. In general, if* Γ *is a (discrete) group and* G *is a* LIE *group with* LIE *algebra* $\mathfrak{g}$*, then* $H^1(\Gamma, \mathfrak{g})$ *is the tangent space of* $\text{Rep}\,(\Gamma, G)$ *at* ρ*, where the action of* Γ *on* $\mathfrak{g}$ *is given by* $Ad \circ \rho$*. (Deforming* $\rho(\gamma) \mapsto \rho(\gamma)(1+\delta\rho(\gamma))$ *gives a 1–cocycle* $\gamma \mapsto \delta\rho(\gamma) \in \mathfrak{g}$*; coboundaries correspond to conjugation by elements of* G*.) NB: This holds for representations* **modulo conjugation**.

By HODGE Theory,

$$H^1(\underbrace{\pi_1(C)}_{\textit{was } C}, \mathfrak{su}(E)\otimes\mathbb{C}) = \Gamma(C, \mathcal{E}\text{nd}\,(\mathcal{E})\otimes\Omega^1) \oplus \underbrace{\overline{\Gamma(C, \mathcal{E}\text{nd}\,(\mathcal{E})\otimes\Omega^1)}}_{\cong H^1(C, \mathcal{E}\text{nd}\,(\mathcal{E}))}\,.$$

The real points cannot go into the first summand; *they have to map diagonally since they are invariant under complex conjugation.* Hence they surject onto $H^1(C, \mathcal{E}\text{nd}\,(\mathcal{E}))$, i.e., the tangent map is a surjection, our map ϕ is a submersion, and the image is open.

(One really should look at representations modulo conjugation; then the real dimension of $\text{IrrRep}\,(\pi_1(C), \mathbf{U}(r))$ is $2gr^2 - 2(r^2-1)$, which is the same as the real dimension of $\mathcal{M}(r,0)^{\mathrm{s}}_{\mathbb{C}}$. Hence we have an *isomorphism* on tangent spaces.)

Since the image is open and closed, it must consist of connected components. Because $\mathcal{M}(r,0)^{\mathrm{s}}_{\mathbb{C}}$ is irreducible (hence connected) (we postpone the proof of this fact) and the image is non-empty (there are irreducible unitary representations *of dimension* r (!)), it must be everything.

Hence, every stable vector bundle of degree zero is obtained (up to isomorphism) from an irreducible unitary representation of $\pi_1(C)$. Of course, we then also get every semisimple semistable vector bundle of degree zero from a unitary representation. □

Corollary 1 *If $\mathcal{E}_1$ and $\mathcal{E}_2$ are semistable vector bundles of degree zero on C, then the same holds for $\mathcal{E}_1 \otimes \mathcal{E}_2$.*

Proof: First assume $\mathcal{E}_1$ and $\mathcal{E}_2$ are stable. Then $\mathcal{E}_1$ and $\mathcal{E}_2$ come from (irreducible) unitary representations E_1 and E_2 of $\pi_1(C)$, and $\mathcal{E}_1 \otimes \mathcal{E}_2$ is obtained from $E_1 \otimes E_2$, which is again unitary. Hence, $\mathcal{E}_1 \otimes \mathcal{E}_2$ is semisimple semistable of degree zero.

Now let $\mathcal{E}_1$ and $\mathcal{E}_2$ be arbitrary semistable vector bundles of degree 0. Then there are filtrations $(\mathcal{E}_i^j)$ of $\mathcal{E}_i$ $(i \in \{1,2\})$ with $\mathcal{E}_i^{j+1}/\mathcal{E}_i^j$ stable of degree 0. The proof for $\mathcal{E}_i$ stable now shows that $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$ has a similar filtration $(\mathcal{E}^j)$. If $\mathcal{E}$ were not semistable, there would be some $\mathcal{F} \subset \mathcal{E}$ of positive degree. Then in the filtration $\mathcal{F}^j = \mathcal{F} \cap \mathcal{E}^j$, there would occur a quotient of positive degree. Since $\mathcal{F}^{j+1}/\mathcal{F}^j$ injects into $\mathcal{E}^{j+1}/\mathcal{E}^j$, we have a contradiction. [CK] □

Remark: Faltings says he doesn't know a purely algebraic proof of this.

5 Moduli space as stack

Suppose $\mathcal{M}$ is a moduli space and suppose further that we have a vector bundle $\mathcal{H}$ on $\mathcal{M}$. If we then have a vector bundle family $\mathcal{E}$ on $C \times S$, we get a classifying map $\phi_{\mathcal{E}} : S \longrightarrow \mathcal{M}$, which we can use to pull back $\mathcal{H}$ to a vector bundle $\mathcal{H}(\mathcal{E}) = \phi_{\mathcal{E}}^*(\mathcal{H})$ on S.

If $f : S_1 \longrightarrow S_2$ is a morphism, and $\mathcal{E}$ is a vector bundle on $C \times S_2$, we have $\mathcal{H}((1_C \times f)^*\mathcal{E}) \cong f^*\mathcal{H}(\mathcal{E})$ and all sorts of further nice compatibilities. We also get an action of $\mathrm{Aut}\,(\mathcal{E})$ on $\mathcal{H}(\mathcal{E})$ (*How?*).

So, even if a moduli space does not exist, we can study vector bundles on it by defining a vector bundle $\mathcal{H}$ on $\mathcal{M}$ to be an association $\mathcal{E} \mapsto \mathcal{H}(\mathcal{E})$ with suitable properties.

Example:

(a) $\mathcal{H}(\mathcal{E}) = \mathcal{O}_S$. This should correspond to $\mathcal{O}_{\mathcal{M}}$.

(b) Let $x \in C$. Then we may take $\mathcal{H}(\mathcal{E}) = \mathcal{E}|_{\{x\}\times S}$ *or—if we want a line bundle—* $\mathcal{H}(\mathcal{E}) = \det \mathcal{E}|_{\{x\}\times S}$.

(c) *An important example is the determinant of cohomology*: If $C \times S \xrightarrow{\pi} S$, then locally in S, we can find a complex of sheaves

$$0 \longrightarrow \mathcal{H}^0 \xrightarrow{d} \mathcal{H}^1 \longrightarrow 0$$

such that $\pi_*\mathcal{E} = \ker d$ and $R^1\pi_*\mathcal{E} = \operatorname{coker} d$ (EGA III). For example, take $x \in C$ and $N \gg 0$ such that $R^1\pi_*\mathcal{E}(Nx) = 0$. Then we can take

$$\mathcal{H}^0 = \pi_*(\mathcal{E}(Nx)) \longrightarrow \pi_*(\mathcal{E}(Nx)/\mathcal{E}) = \mathcal{H}^1 .$$

The assertion follows from the long exact cohomolgy sequence for π_*.

This complex is unique up to a quasi-isomorphism which in turn is determined up to homotopy equivalence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{H}^0 & \longrightarrow & \mathcal{H}^1 & \longrightarrow & 0 \\ & & \downarrow & \swarrow & \downarrow & & \\ 0 & \longrightarrow & \tilde{\mathcal{H}}^0 & \longrightarrow & \tilde{\mathcal{H}}^1 & \longrightarrow & 0 \end{array}$$

('quasi-isomorphism' means inducing isomorphism on cohomology). The determinant of cohomology is now

$$\det H^*(C, \mathcal{E}) = \det \mathcal{H}^0 \otimes (\det \mathcal{H}^1)^{\otimes(-1)} ;$$

this is a well-defined line bundle, independent of all choices made.

For a fixed vector bundle $\mathcal{F}$ *on* C, $\det H^*(C, \mathcal{E} \otimes \mathcal{F})$ is an example of a 'vector bundle on moduli space'.

Remark: An exact sequence of vector bundles

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_2 \longrightarrow 0$$

induces an isomorphism

$$\det H^*(C, \mathcal{E} \otimes \mathcal{F}_1) \otimes \det H^*(C, \mathcal{E} \otimes \mathcal{F}_2) \cong \det H^*(C, \mathcal{E} \otimes \mathcal{F}) .$$

Hence it suffices to consider line bundles $\mathcal{L}$ instead of general vector bundles $\mathcal{F}$. For a line bundle $\mathcal{L}$ and a point $x \in C$, consider

$$0 \longrightarrow \mathcal{L}(-x) \longrightarrow \mathcal{L} \longrightarrow k_x \longrightarrow 0\,.$$

This induces an isomorphism

$$\det H^*(C, \mathcal{E} \otimes \mathcal{L}(-x)) \otimes \underbrace{\det \mathcal{E}_x}_{\text{this is of type (b)}} \cong \det H^*(C, \mathcal{E} \otimes \mathcal{L})\,.$$

This means that there is basically only one 'new' line bundle $\det H^*(C, \mathcal{E})$ on $\mathcal{M}$.

If $D = \sum n_i\, x_i = \operatorname{div}(f)$ is a principal divisor on C, then $\mathcal{O}(D)$ is trivial, hence $\det H^*(C, \mathcal{E}) \cong \det H^*(C, \mathcal{E} \otimes \mathcal{O}(D))$, and therefore (using the considerations above), $\bigotimes_i (\det \mathcal{E}(x_i))^{\otimes n_i}$ is trivial. So there is essentially only one line bundle of type (b). *(This should be made clearer. At this point, we get a map from divisors on C to line bundles on $\mathcal{M}$ factoring over the principal divisors. Hence we get a map from (isomorphism classes of) line bundles on C to line bundles on $\mathcal{M}$.* FALTINGS *said something about* JACOBI*ans and algebraic equivalence.)*

EXAMPLE: Consider our ample line bundle

$$\det \Gamma(C, \mathcal{E}(a+b))^{\otimes \alpha} \otimes \det \Gamma(C, \mathcal{E}(a))^{\otimes(-\beta)}\,.$$

Since $H^1 = 0$, we may replace Γ by H^*. Then we get something of the form

$$\begin{aligned} &\det H^*(C, \mathcal{E}(a+b))^{\otimes \alpha} \otimes \det H^*(C, \mathcal{E}(a))^{\otimes(-\beta)} \\ &\quad \cong \ (\text{something of type (b)}) \otimes (\det H^*(C, \mathcal{E}))^{\otimes(\alpha-\beta)}\,, \end{aligned}$$

where $\alpha - \beta < 0$.

This means roughly that we "can expect (some kind of) ampleness" for $(\det H^*(C, \mathcal{E}))^{\otimes(-1)}$.

May 26, 1995

Consider the following *moduli functor*:

$$S \longmapsto \Big(\text{vector bundles on } C \times S, \text{isomorphisms}\Big)$$

The pair on the right hand side are the objects and morphisms of a category, which is a *groupoid* (i.e., all morphisms are isomorphisms). *So we have here* a **groupoid functor**.

The isomorphism classes of vector bundles *(which is what we are interested in)* constitute the π_0 (set of connected components) of the groupoid. The corresponding functor is not representable *("moduli space does not exist")*.

But we may consider it represented by a (so-called) **(moduli) stack** $\mathfrak{M}$*—see below what a stack is.*

(I don't know what the significance of the following remark is at this place:

We can glue vector bundles: If $S = \bigcup U_\alpha$ is an open cover and we are given vector bundles on $C \times U_\alpha$ (for all α) with isomorphisms over $C \times (U_\alpha \cap U_\beta)$ which are compatible on triple intersections, the data glue to give a vector bundle on $C \times S$.)

When constructing a representing object, we have to face two problems:

- a vector bundle can have automorphisms *(other than $\mathbb{G}_\mathrm{m}$): the action of* $\mathbf{SL}(N)$ *(or* $\mathbf{PGL}(N)$*) is not free.* This will be OK—*this seems to be what stacks are made for, after all*;
- the object will not be of finite type, i.e., not quasi-projective (*HN*–filtration can be wild).

To overcome the second problem, we construct 'open substacks' $\mathfrak{M}_n$:

Fix some n. We want to restrict to those $\mathcal{E}$ with $H^1(C, \mathcal{E}(n)) = 0$ (maybe this isn't necessary, but it won't hurt) and such that $H^0(C, \mathcal{E}(n))$ generates $\mathcal{E}(n)$.

Given $\mathcal{E}$ on $C \times S$, let

$$S_n = \{s \in S \mid H^1(C, \mathcal{E}(n)_s) = 0 \quad \text{and} \quad H^0(C, \mathcal{E}(n)_s) \text{ generates } \mathcal{E}(n)_s\}$$

(*where* $\mathcal{E}(n)_s = i^*\mathcal{E}(n)$ *with* $i : C \cong C \times \{s\} \hookrightarrow C \times S$). Then S_n is *open* in S: By semi-continuity (Grauert), "$H^1 = 0$" is an open condition, hence satisfied on an open subset U of S. Then $\pi_*\mathcal{E}(n)|_{C\times U}$ is a vector bundle on U (by some Base Change Thm.) which generates a subsheaf of $\mathcal{E}(n)|_{C\times U}$ (*fiber-wise,* $\pi_*\mathcal{E}(n)$ *corresponds to global sections of* $\mathcal{E}(n)_s$*, hence generates a sub-bundle in each fiber*). The support of the quotient is closed, and π is proper, hence the image of the support in U is closed, and the points where $\mathcal{E}(n)_s$ is generated by global sections constitute an open subset again.

The substack $\mathfrak{M}_n$ *should probably be viewed as consisting of all the images of* S_n *under the classifying maps* $S \to \mathfrak{M}$*. Since* S_n *is always open,* $\mathfrak{M}_n$ *should be kind-of-open in* $\mathfrak{M}$*.*

Clearly, $S = \bigcup_n S_n$ *and so* $\mathfrak{M} = \bigcup_n \mathfrak{M}_n$.

These substacks will be quasi-compact *(whatever this will mean).*

Let $\mathfrak{T}_n$ be the moduli space of vector bundles $\mathcal{E}$ on C such that $H^0(C, \mathcal{E}(n))$ generates $\mathcal{E}(n)$ and $H^1(C, \mathcal{E}(n)) = 0$, together with a basis of $\Gamma(C, \mathcal{E}(n))$. Then $\mathfrak{T}_n$ has an open embedding into the quotient scheme Q classifying all quotients of $\mathcal{O}_C^N$ (where $N = r(\mu + n)$ is the dimension of $\Gamma(C, \mathcal{E}(n))$). (The conditions defining $\mathfrak{T}_n$ are all open: quotient locally free of rank r, $H^1 = 0$, H^0 generates.) So $\mathfrak{T}_n$ is a nice scheme representing the corresponding functor, and we will identify the two. Then we have a $\mathbf{GL}_N$*–torsor*

$$\mathfrak{T}_n \xrightarrow{\pi} \mathfrak{M}_n \,,$$

i.e., a kind of bundle with fiber $\mathbf{GL}_N$ *(but beware of automorphisms of* $\mathcal{E}$ *!). (On* $C \times \mathfrak{M}_n$*, there is the universal vector bundle* $\mathcal{E}$*, and)* $\pi_*\mathcal{E}$ is a locally free sheaf on $\mathfrak{M}_n$ of rank n *(?? I guess it is* $\pi_*\mathcal{E}(n)$ *and has rank* N*).*

The stack $\mathfrak{M}_n$ should therefore be 'the quotient $\mathfrak{T}_n/\mathbf{GL}_N$'. This quotient won't exist as a scheme, since the $\mathbf{GL}_N$–action is not free: the stabilizer of $t \in \mathfrak{T}_n$ corresponding to $(\mathcal{E} + \text{basis})$ is given by $\mathrm{Aut}\,(\mathcal{E})$. For example, we always have $\mathbb{G}_\mathrm{m} \subset \mathrm{Aut}\,(\mathcal{E})$.

The idea now is not to worry about the quotient, but to work with $\mathfrak{T}_n$ and its $\mathbf{GL}_N$–action instead: We represent the stack by '$\mathfrak{T}_n/\mathbf{GL}_N$' and argue as if $\mathbf{GL}_N$ would operate freely.

EXAMPLE: A coherent sheaf on $\mathfrak{T}_n/\mathbf{GL}_N$ should correspond to a coherent sheaf $\mathcal{F}$ on $\mathfrak{T}_n$ 'with $\mathbf{GL}_N$–action': Let $\mathbf{GL}_N \times \mathfrak{T}_n \xrightarrow{\mu} \mathfrak{T}_n$ be the action of $\mathbf{GL}_N$ on $\mathfrak{T}_n$. Then we require that for any two maps

$$\mathbf{GL}_N \xleftarrow{\beta} S \xrightarrow{\alpha} \mathfrak{T}_n ,$$

we have an isomorphism of sheaves on S:

$$\phi_{\alpha,\beta} : \alpha^*\mathcal{F} \xrightarrow{\cong} \alpha'^*\mathcal{F}$$

(where $\alpha' = \mu(\beta \times \alpha)$), which is transitive under multiplication of maps β. *(I take this to mean the following:*

$$\phi_{\mu(\beta_1 \times \alpha),\beta_2} \circ \phi_{\alpha,\beta_1} = \phi_{\alpha,\beta} ,$$

where $\beta(x) = \beta_1(x)\beta_2(x)$ (multiplication in $\mathbf{GL}_N$).)

(Using $\mathbf{GL}_N \xleftarrow{pr_1} \mathbf{GL}_N \times \mathfrak{T}_n \xrightarrow{pr_2} \mathfrak{T}_n$, one can also write down explicit conditions on $\mathcal{F}$.)

These sheaves are the same as 'sheaves on the moduli functor' *(whatever this is)* (restricted to $\mathcal{E}$ with $H^1(C, \mathcal{E}(n)) = 0$ and globally generating $H^0(C, \mathcal{E}(n))$).

Given a family of vector bundles $\mathcal{E}$ on $C \times S$ *(and such that $S = S_n$)*, $\pi_*\mathcal{E}(n)$ is locally free of rank N. Hence we can choose an open cover $S = \bigcup U_\alpha$ such that $\pi_*\mathcal{E}(n)|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}^N$. Fixing this isomorphism, we get a classifying map $\varphi_\alpha : U_\alpha \to \mathfrak{T}_n (\hookrightarrow Q)$. Further, we have transition maps $\tau_{\alpha\beta} : U_{\alpha\beta} = U_\alpha \cap U_\beta \to \mathbf{GL}_N$, given by the matrices changing the one basis into the other: $\varphi_\beta = \mu(\tau_{\alpha\beta} \times \varphi_\alpha)$ on $U_{\alpha\beta}$. *The transition maps are compatible on triple intersections.* An equivariant sheaf $\mathcal{F}$ on $\mathfrak{T}_n$ now induces sheaves $\mathcal{F}_\alpha$ on U_α by pull-back. Looking at $\mathbf{GL}_N \xleftarrow{\tau_{\alpha\beta}} U_\alpha \cap U_\beta \xrightarrow{\varphi_\alpha} \mathfrak{T}_n$, we see that we get isomorphisms $\mathcal{F}_\alpha|_{U_{\alpha\beta}} \xrightarrow{\cong} \mathcal{F}_\beta|_{U_{\alpha\beta}}$ compatible on triple intersections. Hence the $\mathcal{F}_\alpha$ glue to give a sheaf on S. *This shows that we get for each $\mathcal{E}$ on $C \times S$ a sheaf $\mathcal{F}(\mathcal{E})$ on S (with some properties) as discussed in the previous lecture. Is this meant by a 'sheaf on the moduli functor'?*

If we have two isomorphic families $\mathcal{E}$ and $\mathcal{E}'$ *on $C \times S$, we get maps $U_\alpha \to \mathbf{GL}_N$ giving the isomorphism*

$$(\pi_*\mathcal{E}(n)|_{U_\alpha} + \text{basis}) \cong (\pi_*\mathcal{E}'(n)|_{U_\alpha} + \text{basis}) .$$

This in turn yields isomorphisms $\mathcal{F}_\alpha \cong \mathcal{F}'_\alpha$, which are compatible with those induced by the respective transition maps. Hence they glue to give an isomorphism $\mathcal{F}(\mathcal{E}) \cong \mathcal{F}(\mathcal{E}')$. *In particular, we get an action of* $\mathrm{Aut}\,(\mathcal{E})$ *on* $\mathcal{F}(\mathcal{E})$.

It should be easy now to see the existence of the sheaf $\pi_*\mathcal{E}(n)$ on $\mathfrak{M}_n$: It certainly exists on $\mathfrak{T}_n$ (we have the universal sheaf $\mathcal{E}$ on $C \times \mathfrak{T}_n \xrightarrow{\pi} \mathfrak{T}_n$, which has $H^1(C, \mathcal{E}(n)_t) = 0$ and $H^0(C, \mathcal{E}(n)_t)$ generating, so $\pi_*\mathcal{E}(n)$ will be locally free of rank $N = \dim H^0(C, \mathcal{E}(n)_t)$), so we have to check equivariance. Let $\varphi : S \to \mathfrak{T}_n$ and $\beta : S \to \mathbf{GL}_N$ be maps. Then $(1 \times \varphi)^*\mathcal{E}$ and $(1 \times \mu(\beta \times \varphi))^*\mathcal{E}$ are isomorphic sheaves over $C \times S$ (we only change the basis of $H^0(C, —)$), hence

$$\varphi^*\pi_*\mathcal{E} = \pi_*(1 \times \varphi)^*\mathcal{E} \cong \pi_*(1 \times \mu(\beta \times \varphi))^*\mathcal{E} = (\mu(\beta \times \varphi))^*\pi_*\mathcal{E}\,.$$

Transitivity is equally clear.

If we want to consider $\mathfrak{M}$ as a limit of $\mathfrak{M}_n$, we have a problem: Increasing n will also increase $\dim \mathfrak{T}_n$ and N, so we don't get anything like open immersions on the $\mathfrak{T}_n$ level. We must introduce a suitable equivalence relation of (representations of) stacks.

Definition of stacks

Remark: "The notion of stack came up in the sixties. But to swallow schemes was already enough for one generation of mathematicians."

(Faltings)

Definition 1 *An* **algebraic groupoid** *is a pair of schemes (S, R) together with maps*

$$\begin{array}{ccc} R & \xrightarrow{(\mathrm{dom},\,\mathrm{ran})} & S \times S \\ & \nwarrow_{\mathrm{id}} & \cup \ \mathrm{diag} \\ & & S \end{array} \qquad \text{and} \qquad R \times_S R \xrightarrow{\mathrm{comp}} R$$

(where $R \times_S R$ is constructed using $\mathrm{ran} : R \to S$ *for the left and* $\mathrm{dom} : R \to S$ *for the right factor) such that*

$$T \longmapsto \big(\mathrm{Hom}\,(T, S), \mathrm{Hom}\,(T, R)\big)$$

is a groupoid functor on schemes. Hom (T, S) *are to be the objects of the groupoid and* Hom (T, R) *the morphisms. Domain and range of a morphism are given by* (dom, ran)*, the identity morphism is given by* id*, and composition of morphisms is given by* comp.

If (S, R) *is an algebraic groupoid, then 'S/R' is a* **stack**.

EXAMPLE:
(a) Let S be a scheme and G a group acting on S via μ. Then we can take $R = G \times S$ with dom $= pr_2$, ran $= \mu$, $\mathrm{id}(s) = (1, s)$ and $\mathrm{comp}((g, s), (g', gs)) = (g'g, s)$. *The resulting stack corresponds to the quotient S/G.*

(b) $R \subset S \times S$ (closed immersion) is an equivalence relation. The stack S/R represents the quotient of S by this equivalence relation.

Definition 2 *A* **sheaf** *on a stack S/R is a sheaf $\mathcal{F}$ on S, together with an isomorphism between the two pull-backs of $\mathcal{F}$ to R (i.e., $\mathrm{dom}^*\mathcal{F} \cong \mathrm{ran}^*\mathcal{F}$), satisfying the following condition: Every map $\rho : T \to S$ gives rise to a sheaf $\mathcal{F}_\rho = \rho^*\mathcal{F}$ on T. For every map $\mu : T \to R$ (which should be interpreted as giving an isomorphism between the 'objects' $\rho_1 = \mathrm{dom} \circ \mu$ and $\rho_2 = \mathrm{ran} \circ \mu$) we get an isomorphism $\phi_\mu : \mathcal{F}_{\rho_1} \xrightarrow{\cong} \mathcal{F}_{\rho_2}$ such that $\phi_{\mathrm{id} \circ \rho} = id_{\mathcal{F}_\rho}$ and such that for μ_1, μ_2 with $\mathrm{ran} \circ \mu_1 = \mathrm{dom} \circ \mu_2$, we get $\phi_{\mathrm{comp} \circ (\mu_1 \times \mu_2)} = \phi_{\mu_2} \circ \phi_{\mu_1}$.*

To put this into abstract nonsensical language: A sheaf on S/R is a natural transformation from the groupoid functor represented by (S, R) to the groupoid functor $T \mapsto$ (sheaves on$T$, isomorphisms). The corresponding sheaf on S is obtained as the image of $id_S \in$ Hom (S, S) in the sheaves on S.

To get a nice (abelian) category, we will only consider *smooth* stacks.

Definition 3 *A stack S/R is* **smooth** *if S and* dom, ran $: R \to S$ *are smooth.*

Since a stack is meant to represent isomorphism classes of something, which can be obtained in a multitude of different ways (looking at objects with additional structure and modding the additional data out, for example), we need a notion of when two stacks represent 'the same thing'.

Definition 4 *Two (smooth) stacks S_1/R_1 and S_2/R_2 are* **equivalent** *(vulgo 'the same stack') if there is a third stack S_{12}/R_{12} such that we have smooth and surjective maps*

$$S_1 \twoheadleftarrow S_{12} \twoheadrightarrow S_2$$

such that

$$R_1 \times_{S_1 \times S_1} (S_{12} \times S_{12}) \cong R_{12} \cong R_2 \times_{S_2 \times S_2} (S_{12} \times S_{12}) .$$

(This means that we can check 'equivalence' of objects in S_{12} by sending them to S_1 or equally well to S_2. Heuristically this means that we get 'the same' equivalence classes.)

Another way to define this is to say that we take the equivalence relation generated by pairs $(S'/R', S/R)$ such that $S' \twoheadrightarrow S$ and $R' \cong R \times_{S \times S} (S' \times S')$.

As an example, we get 'the same' coherent sheaves on equivalent stacks: Let $S' \stackrel{\alpha}{\twoheadrightarrow} S$ as in the definition above. Then $\mathcal{F} \mapsto \alpha^* \mathcal{F} = \mathcal{F}'$ should induce an equivalence of categories between coherent sheaves on S/R and coherent sheaves on S'/R'. By descent, a sheaf on S corresponds to a sheaf on S' such that the two possible pull-backs to $S' \times_S S'$ are isomorphic plus a transitivity condition. Now we have

$$S' \times_S S' \cong S \times_{S \times S} (S' \times S') \cong \mathrm{id}(S) \times_{S \times S} (S' \times S') \hookrightarrow R \times_{S \times S} (S' \times S') \cong R'$$

($S \to S \times S$ by the diagonal embedding), which implies that a sheaf on S'/R' descends to a sheaf on S/R. The converse is clear. *(Exercise: Write this out in terms of groupoid functors!)*

Back to our moduli problem

Let $m < n$. Then we have two smooth stacks $\mathfrak{M}_m = \mathfrak{T}_m/\mathbf{GL}_M$ and $\mathfrak{M}_n = \mathfrak{T}_n/\mathbf{GL}_N$ which we want to relate.

We can assume that $\mathcal{E} \in \mathfrak{M}_m \Rightarrow \mathcal{E} \in \mathfrak{M}_n$. Then we get an open subset $\mathfrak{T}'_n \subset \mathfrak{T}_n$ defined by '$H^1(C, \mathcal{E}(m)) = 0$ and $H^0(C, \mathcal{E}(m))$ generates $\mathcal{E}(m)$'. $\mathfrak{T}'_n$ is stable under the $\mathbf{GL}_N$–action.

Claim: $\mathfrak{T}_m/\mathbf{GL}_M$ and $\mathfrak{T}'_n/\mathbf{GL}_N$ define equivalent stacks.

(Hence we get some kind of open immersion $\mathfrak{T}_m/\mathbf{GL}_M \cong \mathfrak{T}'_n/\mathbf{GL}_N \subset \mathfrak{T}_n/\mathbf{GL}_N$.)

PROOF: Let $\mathfrak{T} \longrightarrow \mathfrak{T}_m \times \mathfrak{T}'_n$ represent isomorphisms $pr_1^*\mathcal{E}_m \cong pr_2^*\mathcal{E}_n$ *(I take this to mean the following functor:* $S \mapsto$ *isomorphisms* $\mathcal{E}_m \cong \mathcal{E}_n$ *of vector bundles on* $C \times S$ *such that* $S = S_m$ *with respect to* $\mathcal{E}_m$ *and* $S = S_m = S_n$ *with respect to* $\mathcal{E}_n$*, plus bases of* $\pi_*\mathcal{E}_m(m)$ *and of* $\pi_*\mathcal{E}_n(n)$*)* (this *is* representable by the HILBERT scheme formalism). The projections $\mathcal{E}_m \leftarrow\!\shortmid (\mathcal{E}_m \cong \mathcal{E}_n) \mapsto \mathcal{E}_n$ are surjective by definition. They are also smooth: We use the infinitesimal criterion. Let $T = \operatorname{Spec} A$ with a local algebra A, $I \subset A$ an ideal such that $I^2 = 0$, and let $T_0 = \operatorname{Spec} A/I$. We must show that we can lift α to β:

$$\begin{array}{ccc} T_0 & \xrightarrow{\beta_0} & \mathfrak{T} \\ \cap & \overset{\beta}{\nearrow} & \downarrow \\ T & \xrightarrow{\alpha} & \mathfrak{T}_m \end{array}$$

(and similarly for $\mathfrak{T}'_n$). α corresponds to a vector bundle $\mathcal{E}$ on $C \times T$ together with a basis of $\pi_*\mathcal{E}(m)$. Modulo I, we have an isomorphism $\overline{\mathcal{E}} \cong \overline{\mathcal{F}}$, where $\overline{\mathcal{F}}$ is a vector bundle on $C \times T_0$ for which we also have given a basis of $\pi_*\overline{\mathcal{F}}(n)$. We use the isomorphism to identify $\overline{\mathcal{E}}$ and $\overline{\mathcal{F}}$. We lift the resulting identity on $\overline{\mathcal{E}}$ to the identity on $\mathcal{E}$. The basis of $\pi_*\overline{\mathcal{E}}(n)$ can also be lifted because of local freeness. *So we have defined* β *as the classifiying map of* $\mathcal{E} = \mathcal{E}$ *with the two bases.* We still have to check that the pull-backs of $\mathfrak{R}_m = \mathbf{GL}_M \times \mathfrak{T}_m$ and $\mathfrak{R}'_n = \mathbf{GL}_N \times \mathfrak{T}'_n$ are isomorphic. *This says that for two isomorphisms* $\mathcal{E}_m \cong \mathcal{E}_n$ *and* $\mathcal{E}'_m \cong \mathcal{E}'_n$*,* $\mathcal{E}_m$ *and* $\mathcal{E}'_m$ *are isomorphic iff* $\mathcal{E}_n$ *and* $\mathcal{E}'_n$ *are isomorphic. This is of course clear.*

Hence by our definition, the two stacks are equivalent, and the $\mathcal{E}$'s correspond *(does this mean the universal sheaves are 'the same'?).* □

Naturally, one wants to have a stack which is not 'bigger' than necessary, i.e., a stack S/R with minimal possible dimension of S.

So, let $S/R \cong \mathfrak{M}_n$. We will get a lower bound on $\dim S$ from the fact that S will be a *versal deformation*: For $s \in S$, we have a KODAIRA–SPENCER map $\mathcal{T}_{S,s} \longrightarrow H^1(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}_s))$ which is surjective. *(This should be so, since*

$H^1(\ldots)$ *is the 'tangent space at* $\mathcal{E}_s$ *of the moduli space'—see first lecture.* $\mathcal{T}_S$ *is the tangent bundle of* S.*)* Hence any $\mathcal{E}$ gives a lower bound on $\dim S = \dim \mathcal{T}_{S,s}$.

Let $s \in S$ be a k–valued point. Let $\varepsilon^2 = 0$ and consider $k[\varepsilon]$. Then $\mathcal{T}_{S,s}$ is given by all morphisms $\operatorname{Spec} k[\varepsilon] \to S$ extending $\operatorname{Spec} k \xrightarrow{s} S$. Each element of $\mathcal{T}_{S,s}$ defines a vector bundle on $C \times \operatorname{Spec} k[\varepsilon]$, which can be compared to the constant bundle $\mathcal{E}$ (obtained from $k[\varepsilon] \twoheadrightarrow k \hookrightarrow k[\varepsilon]$). The 'difference' lies in $H^1(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}))$: Choose an open affine cover $C = \bigcup U_\alpha$ such that $\mathcal{E}|_{U_\alpha} \cong \mathcal{O}^r_{U_\alpha}$. Write the transition maps into $\mathbf{GL}_r(\mathcal{O}_{U_\alpha \cap U_\beta}[\varepsilon])$ as $g_{\alpha\beta} + \varepsilon \cdot \delta g_{\alpha\beta}$. Then $\delta g_{\alpha\beta}$ defines a cocycle in $H^1(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}))$. *This defines the* Kodaira–Spencer *map.*

This map is surjective: *We first show that surjectivity is invariant under going over to an equivalent stack.* Let $S' \twoheadrightarrow S$ induce an equivalence of stacks. Let $s' \in S'$ map to $s \in S$. Every map $\operatorname{Spec} k[\varepsilon] \to S'$ (extending s') maps to a map $\operatorname{Spec} k[\varepsilon] \to S$ (extending s), and their images in $H^1(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}))$ coincide. On the other hand, every map $\operatorname{Spec} k[\varepsilon] \to S$ (extending s) can be lifted to a map $\operatorname{Spec} k[\varepsilon] \to S'$ (extending s') because of smoothness. Hence $\mathcal{T}_{S',s'}$ and $\mathcal{T}_{S,s}$ have the same image in $H^1(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}))$.

From this we see that it suffices to show surjectivity for $\mathfrak{X}_n$. An element of $H^1(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}))$ corresponds to a deformation of $\mathcal{E}$ to $C \times \operatorname{Spec} k[\varepsilon]$. Since we can lift the given basis of $\pi_*\mathcal{E}(n)$ to a basis over $k[\varepsilon]$, we see that our deformation of $\mathcal{E}$ comes from a deformation in $\mathfrak{X}_n$.

There is a kind of converse:

Proposition 3 *Given* S *smooth,* $\mathcal{E}$ *on* $C \times S$ *such that* $\mathcal{T}_{S,s} \to H^1(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}_s))$ *is surjective for all* $s \in S$ *and such that for all* $s \in S$, $H^1(C, \mathcal{E}(n)_s) = 0$ *and* $H^0(C, \mathcal{E}(n)_s)$ *generates* $\mathcal{E}(n)_s$, *then there is an open subset* $\mathfrak{X}'_n \subset \mathfrak{X}_n$ *such that for* $s \in \mathfrak{X}'_n$, $\mathcal{E}_s$ *is isomorphic to a vector bundle parametrized by* S, *and* $\mathfrak{X}'_n/\mathbf{GL}_N \cong S/R$ *as stacks, where* $R \to S \times S$ *represents* $\operatorname{Isom}(pr_1^*\mathcal{E}, pr_2^*\mathcal{E})$.

Proof: R is smooth: Take as usual a local algebra A with an ideal $I \subset A$

such that $I^2 = 0$ and let $T = \operatorname{Spec} A$, $T_0 = \operatorname{Spec} A/I$. We have to lift:

$$\begin{array}{ccc} T_0 & \longrightarrow & R \\ \cap & \nearrow & \downarrow \\ T & \longrightarrow & S \end{array}$$

The lower map corresponds to a vector bundle $\mathcal{E}$ on $C \times T$. Since $T_0 \longrightarrow R \longrightarrow S \times S$, we have another vector bundle $\overline{\mathcal{E}}'$ and an isomorphism $\overline{\mathcal{E}} \cong \overline{\mathcal{E}}'$ on $C \times T_0$. Since S is smooth, we can lift $\overline{\mathcal{E}}'$ to some $\mathcal{E}'$ on $C \times T$. Then $\mathcal{E}$ and $\mathcal{E}'$ differ by a class in $H^1(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes I)$, and two extensions of $T_0 \to S$ differ by something in $\mathcal{T}_S \otimes I$. Now,

$$\mathcal{T}_S \otimes I \twoheadrightarrow H^1(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E})) \otimes I \twoheadrightarrow H^1(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes I)$$

(the first by versality, the second because H^1 is right exact here). This means that we can adjust $\mathcal{E}'$ to become isomorphic to $\mathcal{E}$, whence the lift $T \to R$.

Now let $\mathfrak{X} \to \mathfrak{X}_n \times S$ classify isomorphisms. Then the projections are smooth (use a similar argument as above), hence the image $\mathfrak{X}'_n$ in $\mathfrak{X}_n$ is open. *Thus we get an equivalence of stacks* $\mathfrak{X}'_n/\mathbf{GL}_N \cong S/R$. □

There is a construction of ARTIN: Given a vector bundle $\mathcal{E}$ on $C \cong C \times \operatorname{Spec} k$, construct a smooth scheme S of dimension $\dim S = \dim H^1(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}))$ *(and a vector bundle on $C \times S$ extending $\mathcal{E}$, probably)*. In this way, one can construct parts of the moduli stack directly.

May 30, 1995

We have our moduli stacks $\mathfrak{M} = \bigcup_n \mathfrak{M}_n$, where $\mathfrak{M}_n = S_n/R_n$ with smooth schemes S_n and groupoid structures R_n on S_n. The inclusions $\mathfrak{M}_n \subset \mathfrak{M}_{n+1}$ can be represented by open immersions $S_n \hookrightarrow S_{n+1}$ *on suitable representatives.* The S_n are *versal deformations*, i.e., the tangent map

$$\mathcal{T}_{S_n} \twoheadrightarrow R^1\pi_*\mathcal{E}\mathrm{nd}\,(\mathcal{E})$$

is surjective.

There is a method of M. Artin *to construct neighborhoods in S_n of a given point $\mathcal{E}$:*

Given $\mathcal{E}/k$, we can construct a versal deformation of $\mathcal{E}$ over $C \times \operatorname{Spec} k[[t_1, \ldots, t_d]]$ such that any other deformation can be obtained by base-change from it (we can choose $d = \dim H^1(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}))$ —in general, it has to be $\geq$).

By the Artin Approximation Theorem, there is a subring $k[t_1, \ldots, t_d] \subset A \subset k[[t_1, \ldots, t_d]]$, étale and of finite type over $k[t_1, \ldots, t_d]$, such that the versal deformation of $\mathcal{E}$ can be defined over A, and we get a versal family over $\operatorname{Spec} A$, *which constitutes a neighborhood of $\mathcal{E}$.*

Choose $S_n = \coprod_{\text{finite}} \operatorname{Spec} A$ (with various A's) (we can take the union to be finite because of quasi-compactness) and R_n = isomorphisms. *This gives a kind of direct construction of $\mathfrak{M}_n$.*

Moduli stack is connectd

As promised:

Theorem 7 *The moduli stack (for fixed rank and degree) is* **connected**, *i.e., each S_n (for $n \gg 0$) can be chosen connected, or alternatively, there is no open and closed non-trivial subset $S'_n \subset S_n$ that is stable under R_n (i.e., $\operatorname{ran}(\operatorname{dom}^{-1}(S'_n)) = S'_n$).*

Proof: The idea is to use the known result for line bundles and reduce the general case to it by filtering vector bundles by line bundles.

Claim: Let $\mathcal{E}$ have rank > 1. Then for $N \gg 0$, $\mathcal{E}(N)$ has a section s with $s(x) \neq 0$ for all $x \in C$.

PROOF: Let $T = \{(s, x) \in \Gamma(C, \mathcal{E}(N)) \times C \mid s(x) = 0\}$; this is a closed subvariety of $\Gamma(C, \mathcal{E}(N)) \times C$. Its fiber over $x \in C$ under pr_2 is $\Gamma(C, \mathcal{E}(N)(-x))$. If N is sufficiently big, then $H^1(C, \mathcal{E}(N)(-x)) = 0$, hence by RR, *applied to* $\mathcal{E}(N)$ *and* $\mathcal{E}(N)(-x)$,

$$\dim(\text{fiber}) = \dim \Gamma(C, \mathcal{E}(N)) - \operatorname{rk} \mathcal{E}\,,$$

and therefore,

$$\dim T \leq \dim \Gamma(C, \mathcal{E}(N)) - \operatorname{rk} \mathcal{E} + 1 < \dim \Gamma(C, \mathcal{E}(N))\,.$$

Hence $pr_1(T)$ is a proper subset of $\Gamma(C, \mathcal{E}(N))$, and an element in the complement is the section claimed to exist. □

We get $\mathcal{E}$ as an extension

$$0 \longrightarrow \mathcal{O}(-N) \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

(with $\deg \mathcal{F} = \deg \mathcal{E} + N$). Since H^1 is right exact on a curve, we have

$$0 = H^1(C, \mathcal{E}(M)) \twoheadrightarrow H^1(C, \mathcal{F}(M))$$

(and analogously for $\mathcal{E}(M)(-x)$ *and* $\mathcal{F}(M)(-x)$*)* for $M \gg 0$. Hence we can repeat the construction to obtain a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

(where $r = \operatorname{rk} \mathcal{E}$) such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{O}(-N_i)$ for $i = 1, \ldots, r-1$ and $\mathcal{E}_r/\mathcal{E}_{r-1} \cong \det \mathcal{E} \otimes \mathcal{O}(\sum_{i=1}^{r-1} N_i)$. Since we can always increase N in the construction above, we may assume $N_1 \gg N_2 \gg \ldots \gg N_{r-1} \gg 0$.

Claim: Such extensions (with fixed r, N_i and $\deg \mathcal{E}$) are classified by an irreducible variety.

PROOF: For line bundles, we get a copy of the JACOBIan (which is irreducible). So suppose we know the assertion for $r-1$ and bundles $\mathcal{F}$. We have to classify extensions

$$0 \longrightarrow \mathcal{O}(-N) \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0\,.$$

We have $\mathrm{Ext}^1(\mathcal{F}, \mathcal{O}(-N)) = H^1(C, \mathcal{H}\mathrm{om}\,(\mathcal{F}, \mathcal{O}(-N)))$. Since

$$H^0(C, \mathcal{H}\mathrm{om}\,(\mathcal{F}, \mathcal{O}(-N))) = \mathrm{Hom}\,(\mathcal{F}, \mathcal{O}(-N)) = 0$$

(induction on filtration of $\mathcal{F}$, use that degrees in there are $\gg -N$), H^1 defines a vector bundle. *(Let F be the variety classifying $\mathcal{F}$'s and $\mathcal{F}$ the universal bundle on $C \times F \xrightarrow{\pi} F$. Then $\mathcal{H} = \mathcal{H}\mathrm{om}\,(\mathcal{F}, \mathcal{O}(-N))$ is a bundle on $C \times F$ with $\pi_*\mathcal{H} = 0$, whence $R^1\pi_*\mathcal{H}$ is locally free (by* RR*).)* Its total space classifies extensions $\mathcal{E}$. □

So we have an irreducible variety T classifying filtrations as above. The conditions for S_n define an open subvariety $T' \subset T$, which 'maps' onto S_n/R_n. Since T' is irreducible, S_n/R_n is connected (or also irreducible, if this notion were defined). (By choosing a connected component of S_n, we see that we may take S_n connected.) □

Remark: This argument also works for semistable bundles *(where we have used the result).*

Theorem 8 *Assume $g > 1$.*

(a) The closed subset of non-semistable $\mathcal{E}$ has codimension ≥ 2 in S_n.

(b) Suppose $\mathrm{char}\, k = 0$. *Then the closed subset of non-stable $\mathcal{E}$ has always codimension ≥ 1 in S_n. Unless $g = 2$ and the rank $r = 2$, the codimension is ≥ 2.*

In particular, the stable bundles are dense, hence they exist for any degree and rank.

Proof: To estimate the dimension, we use versality: $\mathcal{T}_{S_n} \twoheadrightarrow R^1\pi_*\mathcal{E}\mathrm{nd}\,(\mathcal{E})$. But in case of singularities, the tangent space may be 'too big'. To overcome this, we look at a generic point.

Let $S = S_n$ and $T = \{\mathcal{E} \mid \mathcal{E} \text{ not semistable}\}$, and let η be the generic point of an irreducible component of T. Then over $C \times \overline{k(\eta)}$, $\mathcal{E}$ has a non-trivial *HN*–filtration. This *HN*–filtration is already defined over $k(\eta)$: It is defined over a finite extension M of $K = k(\eta)$, which is a purely inseparable extension of a GALOIS extension L of K. Since the *HN*–filtration is invariant under the GALOIS group, it will be defined over K, if it is defined over L. If $\operatorname{char} k = 0$, we are done. Otherwise, let $p = \operatorname{char} k$. Consider a purely inseparable extension $K \subset K' = K(\sqrt[p]{t})$. There exists a derivation ∂ on K' such that $\partial(\sqrt[p]{t}) = 1$ and $(K')^{\partial} = K$ *(i.e.,* $K = \{u \in K' \mid \partial u = 0\}$*).* ∂ acts on $\mathcal{O}_{C \times \operatorname{Spec} K'} = \mathcal{O}_C \otimes_K K'$ via its action on the second factor. We claim that the *HN*–filtration on $\mathcal{E}' = \mathcal{E} \otimes_K K'$ is invariant under ∂ (and hence defined over K). Let $0 \subset \mathcal{F}' \subset \mathcal{E}'$ be the first step in the *HN*–filtration and consider

$$\mathcal{F}' \xrightarrow{1 \otimes \partial} \mathcal{E}'/\mathcal{F}'$$

(how is this map defined? It seems to require $\mathcal{F}' = \mathcal{F} \otimes_K K'$*, which probably is what we want to prove!)* This map is $\mathcal{O}_C \otimes_K K'$–linear (!), and must be 0 because of the different slopes.

We can now replace T by a non-empty open $T' \subset T$ and find a filtration $\mathcal{E}_\alpha$ on $\mathcal{E}|_{C \times T'}$ by vector bundles $\mathcal{E}_\alpha$ which induce the *HN*–filtration on each fiber. Consider the following diagram:

$$\begin{array}{ccc}
\mathcal{T}_{T'} & \longrightarrow & R^1\pi_*(\mathcal{E}\mathrm{nd}_{HN}(\mathcal{E})) \\
\cap & & \downarrow \\
\mathcal{T}_S & \twoheadrightarrow & R^1\pi_*(\mathcal{E}\mathrm{nd}(\mathcal{E})) \\
 & \searrow & \downarrow \\
 & & R^1\pi_*(\mathcal{E}\mathrm{nd}(\mathcal{E})/\mathcal{E}\mathrm{nd}_{HN}(\mathcal{E}))
\end{array}$$

where $\mathcal{E}\mathrm{nd}_{HN}$ denotes endomorphisms respecting the *HN*–filtration. The map on the lower right is surjective because $R^1\pi_*$ is right exact. The map $\mathcal{T}_{T'} \to R^1\pi_*(\mathcal{E}\mathrm{nd}(\mathcal{E})/\mathcal{E}\mathrm{nd}_{HN}(\mathcal{E}))$ is zero, hence at a point $\mathcal{E} \in T'$, we have

$$\operatorname{codim}_{\mathcal{T}_{S,\mathcal{E}}} \mathcal{T}_{T',\mathcal{E}} \geq \dim H^1(C, \mathcal{E}\mathrm{nd}(\mathcal{E})/\mathcal{E}\mathrm{nd}_{HN}(\mathcal{E}))\,.$$

Now, $\mathcal{E}\mathrm{nd}(\mathcal{E})/\mathcal{E}\mathrm{nd}_{HN}(\mathcal{E})$ has a filtration with subquotients $\operatorname{Hom}(gr_\alpha(\mathcal{E}), gr_\beta(\mathcal{E}))$ where $\alpha > \beta$ ($gr_\alpha(\mathcal{E})$ has slope α). Because of the slopes *and semistability*, $H^0(C, \mathcal{E}\mathrm{nd}(\mathcal{E})/\mathcal{E}\mathrm{nd}_{HN}(\mathcal{E})) = 0$. Hence the dimension of H^1 is given by RIEMANN–ROCH:

$$\dim H^1(\mathcal{E}\mathrm{nd}(\mathcal{E})/\mathcal{E}\mathrm{nd}_{HN}(\mathcal{E}))$$

$$
\begin{aligned}
&= (g-1)\mathrm{rk}\,\mathcal{E}\mathrm{nd}\,(\mathcal{E})/\mathcal{E}\mathrm{nd}_{HN}(\mathcal{E}) - \sum_{\alpha<\beta} \deg \mathrm{Hom}\,(gr_\alpha(\mathcal{E}), gr_\beta(\mathcal{E})) \\
&\geq 2\,,
\end{aligned}
$$

since $g \geq 2$, rk ≥ 1 and

$$\deg \mathrm{Hom}\,(gr_\alpha(\mathcal{E}), gr_\beta(\mathcal{E})) = \mathrm{rk}\,(gr_\alpha(\mathcal{E}))\mathrm{rk}\,(gr_\beta(\mathcal{E}))(\beta-\alpha) < 0$$

(with the old definition of slope = deg /rk) *(and at least one of these occurs, since the filtration is non-trivial).* This proves part (a).

To prove part (b), we show that the non-stable bundles have the claimed codimension in the semistable ones. Let now T be the set of non-stable semistable bundles, and let again η denote a generic point of a component of T. Let $\mathcal{E}$ on $C\times\eta$ be semistable, but not stable. We get a filtration of $\mathcal{E}$ whose subquotients are direct sums of stable bundles ($\mathcal{E}_1 = \sum \mathcal{F} \subset \mathcal{E}$, where the sum is over stable sub-bundles with the same slope as $\mathcal{E}$, continue with $\mathcal{E}/\mathcal{E}_1$). By analogous arguments for the GALOIS case as used for part (a), we see that this filtration is defined over $k(\eta)$ (since we assume char $k = 0$, we need not worry about inseparable extensions). By the same reasoning as above, we get the estimate $(g-1)\mathrm{rk}\,(\ldots)$ for the codimension. *(Why is in this case $H^0 = 0$? We might have isomorphic stable bundles in different subquotients ... Maybe this is because the filtration is canonical (image of stable bundle is stable?), hence has to be respected by global endomorphisms.)* If the filtration has at least two steps, the rank is ≥ 1, so the comdimension must be ≥ 1 (and can be 1 only if $g = 2$ and the rank is 1, which implies that the filtration has two steps with line bundles as subquotients, i.e., $\mathcal{E}$ has rank 2).

We therefore may assume that the rank is zero, i.e., that $\mathcal{E}\otimes\overline{k(\eta)}$ is a direct sum of stable bundles. This splitting is defined over a separable extension L of $K = k(\eta)$ (this is true even in characteristic p). $\mathrm{Spec}\,L \to \mathrm{Spec}\,K$ is an étale cover which extends to

$$T'' \xrightarrow{\text{étale}} T' \xrightarrow{\text{open}} T\,,$$

hence $\mathcal{T}_{T'',\mathcal{E}} \hookrightarrow \mathcal{T}_{T',\mathcal{E}} \hookrightarrow \mathcal{T}_{S,\mathcal{E}}$, and we can estimate the codimension by the dimension of the cokernel of the map from $\mathcal{T}_{T''}$ *to* $R^1\pi_*(\mathcal{E}\mathrm{nd}\,(\mathcal{E})/\mathcal{E}\mathrm{nd}_{\oplus}(\mathcal{E}))$, *where* $\mathcal{E}\mathrm{nd}_{\oplus}$ *means endomorphisms respecting the direct sum.* Then our old argument finishes the proof. *(Really? It seems there are difficulties with vector bundles of the form* $\mathcal{E}^n$ *(n–fold direct sum of a stable bundle* $\mathcal{E}$*): how can the global endomorphisms be restricted to make the argument work?)* □

EXAMPLE: For $g = 2$ and $\mathrm{rk} = 2$, codimension 1 does indeed occur: Consider non-trivial extensions with all degrees zero

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}_2 \longrightarrow 0$$

with line bundles $\mathcal{L}_1$, $\mathcal{L}_2$. Then $H^0(C, \mathcal{E}\mathrm{nd}(\mathcal{E})) = k$. S_n can be chosen to have dimension $1 + 4(g-1) = 5$. $\mathcal{L}_1$ and $\mathcal{L}_2$ depend on 2 prameters each, which gives a subspace of deimension 4, hence codimension 1. ($\dim H^1(C, \mathcal{H}\mathrm{om}(\mathcal{L}_2, \mathcal{L}_1)) = 1$, since $H^0 = 0$ generically, i.e., the non-trivial extension is unique.)

The GIT-quotient again

We try to construct again the GIT-quotient *(GIT = Geometric Invariant Theory)*: Let $S_n^{\mathrm{ss}} \subset S_n$ the open subset of semistable bundles and R_n^{ss} the induced groupoid structure. We want a projective quotient.

The determinant of cohomology defines a line bundle $\mathcal{L}$ on S_n and on S_n^{ss}. *Suppose* $\mathcal{L}$ is generated by its R_n^{ss}–invariant global sections (on S_n^{ss}). Then we get a map $S_n^{\mathrm{ss}} \xrightarrow{f} \mathbb{P}^N$, which is R_n^{ss}–invariant (meaning $f \circ \mathrm{dom} = f \circ \mathrm{ran} : R_n^{\mathrm{ss}} \to \mathbb{P}^N$). Is the image closed? *(whence projective) Suppose so (?).* Then take the 'R_n^{ss}–invariant normalization' $S_n^{\mathrm{ss}} \longrightarrow M \xrightarrow{\text{finite}} \mathbb{P}^N$ and check the fibers and analyze curves in them *(to see if we have the quotient we want?).*

By LANGTON's Ph.D. Thesis (Harvard; MUMFORD), the image *is* closed.

Theorem 9 *Let* V *be a discrete valuation ring with fraction field* K*, and let* $\mathcal{E}_K$ *be a semistable bundle on* $C \times \mathrm{Spec}\, K$*.*

Then $\mathcal{E}$ *extends to a bundle* $\mathcal{E}$ *on* $C \times \mathrm{Spec}\, V$ *with semistable special fiber.*

PROOF: We may assume V to be complete: Given $\hat{\mathcal{E}}$ on $C \times \mathrm{Spec}\, \hat{V}$ ($\hat{V}$ the completion of V), choose a fixed extension $\mathcal{E}_0$ on $C \times \mathrm{Spec}\, V$. Let π be a uniformizer of V. Then for some n, we have

$$\pi^n \hat{\mathcal{E}}_0 \subset \hat{\mathcal{E}} \subset \pi^{-n} \hat{\mathcal{E}}_0 .$$

Since the sub-bundles of $\pi^{-n}\hat{\mathcal{E}}_0/\pi^n\hat{\mathcal{E}}_0$ are the same over V and over $\hat{V}$, $\hat{\mathcal{E}}$ can be defined over V.

Now let $\mathcal{E}$ be any extension of $\mathcal{E}_K$ and $\mathcal{E}_s$ its special fiber. Let $\mathcal{E}_{s,\alpha}$ be the *HN*–filtration of $\mathcal{E}_s$, and let $\mathcal{F}$ be its first step. If $\mathcal{E}_s$ is not semistable, we have $\mu(\mathcal{F}) > \mu(\mathcal{E}_s)$. *Under the assumption that the Thm. is false,* choose $\mathcal{E}$ such that $\mu(\mathcal{F})$ is minimal and among those such that $\mathrm{rk}\,(\mathcal{F})$ is minimal. *$\mathcal{F}$ lives on the special fiber. Extend it by zero to all of $C \times \operatorname{Spec} V$.* Define $\mathcal{E} \subset \mathcal{E}' \subset \pi^{-1}\mathcal{E}$ such that $\pi\mathcal{E}'/\pi\mathcal{E}$ is the image of $\mathcal{F} \subset \mathcal{E}/\pi\mathcal{E}$. *(I have changed this from my notes which didn't quite make sense.)* Then we have

$$0 \longrightarrow (\mathcal{E}/\pi\mathcal{E})/\mathcal{F} \longrightarrow \mathcal{E}'/\pi\mathcal{E}' \xrightarrow{\pi} \mathcal{F} \longrightarrow 0$$

($\mathcal{F}$ and $\mathcal{E}_s/\mathcal{F}$ 'change places'). Let $\mathcal{F}'$ be the first step in the *HN*–filtration of $\mathcal{E}'/\pi\mathcal{E}'$. Then $\mu(\mathcal{F}') \leq \mu(\mathcal{F})$ by the exact sequence above. By minimality of $\mu(\mathcal{F})$, $\mu(\mathcal{F}') = \mu(\mathcal{F})$ *(note $\mathcal{E}_K = \mathcal{E}'_K$).* We get a strict map $\mathcal{F}' \to \mathcal{F}$, whose kernel must be zero (since $\mathrm{rk}\,\mathcal{F}$ is minimal) *(this doesn't seem to be the right justification).* Hence $\mathcal{F}' \hookrightarrow \mathcal{F}$, and then $\mathcal{F}' \cong \mathcal{F}$ because $\mathrm{rk}\,\mathcal{F}$ was minimal *(here this is OK).* (So the sequence splits by $\mathcal{F} \to \mathcal{F}'$.)

We have found a new extension $\pi\mathcal{E}' \subset \mathcal{E}$ such that

$$\begin{array}{ccc} \mathcal{E}'_s & \xrightarrow{\cdot\pi} & \mathcal{E}_s \\ \cup & & \cup \\ \mathcal{F}' & \xrightarrow{\cong} & \mathcal{F} \end{array}$$

Continuing in this way, we obtain a decreasing sequence

$$\mathcal{E} = \mathcal{E}^0 \supset \mathcal{E}^1 \supset \ldots \supset \mathcal{E}^n \supset \pi^n \mathcal{E}$$

such that the first step $\mathcal{F}^n$ in the *HN*–filtration of $\mathcal{E}^n_s$ is the image of $\mathcal{E}^{n+1}_s$ and is $\cong \mathcal{F}^{n+1}$.

On the formal scheme $C\hat{\otimes}V$, consider $\mathcal{E}^\infty = \bigcap_{n\geq 0} \mathcal{E}^n \subset \hat{\mathcal{E}}$:

Let $\operatorname{Spec} A \subset C$ be an open affine subscheme, then over $A\hat{\otimes}_k V$, we have a decreasing sequence of projective modules (E^n) with $E^n \subset \hat{E}$; let E^∞ be their intersection.

Claim: $\operatorname{rk} E^\infty = \operatorname{rk} F$, and we have a surjection $E^\infty \twoheadrightarrow F$. ($F$ corresponding to $\mathcal{F}$.)

Proof: Let $\mathfrak{p} = \pi(A\hat{\otimes}_k V)$ be the prime ideal induced by the maximal ideal of V. Then

$$\operatorname{length}(E/E^n)_{\mathfrak{p}} = n(\operatorname{rk}\mathcal{E}_s - \operatorname{rk}\mathcal{F}) .$$

By commutative algebra, there is a basis $e_1, \ldots, e_a, e_{a+1}, \ldots, e_{a+b}$ of E such that $a = \operatorname{rk}\mathcal{F}$ and E^n is generated by $e_1, \ldots, e_a$ (which generate F) and $\pi^n e_{a+1}, \ldots, \pi^n e_{a+b}$. We can choose this basis to be independent of n (use completeness of V). Then we see that E^∞ is generated by $a_1, \ldots, e_a$, hence equals F. □

This gives us a formal sub-bundle $\hat{\mathcal{G}} \subset \hat{\mathcal{E}}$ with $\hat{\mathcal{G}}_s = \mathcal{F}$. By Grothendieck, $\hat{\mathcal{G}}$ is algebraic, hence comes from a sub-bundle $\mathcal{G} \subset \mathcal{E}$. Now,

$$\mu(\mathcal{G}_K) = \mu(\mathcal{G}_s) = \mu(\mathcal{F}) > \mu(\mathcal{E}_s) = \mu(\mathcal{E}_K) ,$$

contradicting the semistability of $\mathcal{E}_K$. □

Remark: We see that the construction in the proof (when starting with an arbitrary extension $\mathcal{E}$ of $\mathcal{E}_K$) eventually must lead to a semistable bundle.

June 2, 1995

(Raw notes by BERND STEINERT.)

Recall S. LANGSTON's Theorem from the last lecture:

Theorem: *Let V be a discrete valuation ring with fraction field K, let C be defined over V, and let $\mathcal{E}_K$ be a semistable bundle on $C \otimes_V K$.*

Then $\mathcal{E}_K$ extends to a bundle $\mathcal{E}$ on C with semistable special fiber.

The argument fot its proof was as follows:

Take $\mathcal{E}$ 'best possible' and look at the *HN*–filtration

$$\mathcal{E} = \mathcal{E}^0 \supset \mathcal{E}^1 \supset \mathcal{E}^2 \supset \dots$$

The inclusions $\pi\mathcal{E}^n \subset \mathcal{E}^{n+1}$ give rise to isomorphisms $\mathcal{E}^{n+1}/\pi\mathcal{E}^n \cong \mathcal{F}$, where $\mathcal{F}$ is the first step in the *HN*–filtration of $\mathcal{E}^n/\pi\mathcal{E}^n$.

Let $\mathcal{E}^\infty = \bigcap_n \mathcal{E}^n$; this is a formal vector bundle.

Claim: $\mathcal{E}^n = \mathcal{E}^\infty + \pi^n\mathcal{E}$

PROOF: Write $C = \bigcup_R \operatorname{Spec} R$ as a union of open affine subschemes. $\mathcal{E}/\pi\mathcal{E}$ and $\mathcal{F}$ correspond to free modules over $R \otimes_V V/\pi V$:

$$\begin{array}{ccccc} E^n & \longrightarrow & E^0 & \longrightarrow & E^0/\pi E^0 \\ & & & & \cup \\ & & & & F \end{array}$$

Choose a basis $e_1, \dots, e_a, e_{a+1}, \dots, e_{a+b}$ of E^0 such that $e_1, \dots, e_a$ are in E^n and induce a basis of F. Then $e_1, \dots, e_a, \pi^n e_{a+1}, \dots, \pi^n e_{a+b} \in E^n$ form a basis of E^n (look at index).

This can be done indenpendently of n: Chosen $e_1^{(n)}, \dots, e_a^{(n)}$ good for E^n, modify them by something in $\pi^n E$ to get elements good for E^{n+1} (this is possible because $E^n = E^{n+1} + \pi^n E$). Then $e_i = \lim_{n\to\infty} e_i^{(n)}$ exists in $E \widehat{\otimes} V$ and is good for all n. $\square$

We had the following picture:

Let V be a complete *discrete valuation (?)* ring with field of fractions K.

Let $K \subset L$ and W the normalization of V in L. Then

$$\begin{array}{ccccccc} S & \supset & & S^{ss} & \xrightarrow{\alpha} & \mathbb{P}^N \\ & & \nearrow & & & \uparrow \\ & \Big| & & \operatorname{Spec} V & \hookleftarrow & \operatorname{Spec} K \\ & & & & \nearrow & \\ & \operatorname{Spec} W & & \hookleftarrow & \operatorname{Spec} L & \end{array}$$

where the map α is defined via global sections of $(\det H^*)^{-N}$ (this will be shown today).

Using LANGTON's Theorem, we see that $\mathcal{E}_L$ extends to $\mathcal{E}$ on $C \otimes_R W$, hence

$$\begin{array}{ccccll} \operatorname{Spec} L & \longrightarrow & \mathbf{Isom} & \xrightarrow{pr_1} & S & \text{original map} \\ & & & \overset{pr_2}{\searrow} & & \\ & & & & S & \text{map extends to } \operatorname{Spec} V \end{array}$$

Replace lifting by equivalent one $\Longrightarrow$ can extend.

In this way one sees that the image is closed.

Construction of global sections of $(\det H^*)^{\otimes(-M)}$ on S that are invariant under Isom

REMARK: If $\mathcal{E}$ is not semistable, any such section vanishes at $\mathcal{E}$.
PROOF: Assume $\mu(\mathcal{E}) = g-1$ (for general $\mu(\mathcal{E})$, see below) and let $\mathcal{F} \subset \mathcal{E}$ be a sub-bundle violating the semistability property *(i.e.,* $\mu(\mathcal{F}) > \mu(\mathcal{E})$*)*. Consider the following family over $\mathbb{P}^1$ of vector bundles: $\tilde{\mathcal{E}} = \mathcal{E} \oplus \mathcal{F}(1)/\mathcal{F}$:

$$0 \longrightarrow \mathcal{F}(1) \longrightarrow \tilde{\mathcal{E}} \longrightarrow \mathcal{E}/\mathcal{F} \longrightarrow 0$$

What is the determinant of cohomology? Let $C \times \mathbb{P}^1 \xrightarrow{\pi} \mathbb{P}^1$ be the projection. Then

$$\begin{aligned} \det(R\pi_*\tilde{\mathcal{E}}) &= \det(R\pi_*\mathcal{F}) \otimes \mathcal{O}(\underbrace{\chi(H^*(C,\mathcal{F}))}_{>0}) \otimes \det(R\pi_*\mathcal{E}/\mathcal{F}) \\ &= \mathcal{O}(\text{something } > 0)\,. \end{aligned}$$

Hence $\deg(\det^{-1}) < 0$, and all global sections must vanish. $\square$

How to construct sections:

Choose some fixed $\mathcal{F}$ such that $\mu(\mathcal{F})+\mu(\mathcal{E}) = g-1$. $\det H^*(C, \mathcal{E}\otimes\mathcal{F})^{\otimes(-1)}$ has global section ϑ; $\vartheta = 0 \iff H^0(C, \mathcal{E} \otimes \mathcal{F}) \neq 0$.

Observation: If $\operatorname{rk}\mathcal{F}_1 = \operatorname{rk}\mathcal{F}_2$ and $\det\mathcal{F}_1 \cong \det\mathcal{F}_2$, then

$$\det H^*(C, \mathcal{E} \otimes \mathcal{F}_1) \cong \det H^*(C, \mathcal{E} \otimes \mathcal{F}_2)$$

as line bundles on the moduli stack of $\mathcal{E}$'s.

PROOF: This is OK if $\operatorname{rk}\mathcal{F}_i = 1$. Otherwise, use induction: Write

$$0 \longrightarrow \mathcal{O}(-N) \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{G}_i \longrightarrow 0\,;$$

then $\det\mathcal{G}_i \cong (\det\mathcal{F}_i)(N)$ and

$$\det H^*(C, \mathcal{E} \otimes \mathcal{F}_i) \cong \det H^*(C, \mathcal{E} \otimes \mathcal{O}(-N)) \otimes \det H^*(C, \mathcal{E} \otimes \mathcal{G}_i)$$

$\square$

Fix some $\mathcal{F}_0$. Consider $\mathcal{F}$ with $\operatorname{rk}\mathcal{F} = N\operatorname{rk}\mathcal{F}_0$ and $\det\mathcal{F} \cong (\det\mathcal{F}_0)^{\otimes N}$. Then

$$\det H^*(C, \mathcal{E} \otimes \mathcal{F}) \cong \det H^*(C, \mathcal{E} \otimes (\mathcal{F}_0^{\otimes N})) \cong (\det H^*(C, \mathcal{E} \otimes \mathcal{F}_0))^{\otimes N}$$

Theorem 10 *If $\mathcal{E}$ is semistable, and N is sufficiently big, then there exists an $\mathcal{F}$ as above such that $H^*(C, \mathcal{E} \otimes \mathcal{F}) = 0$.*

PROOF: This condition defines an open subset in the moduli stack of $\mathcal{F}$'s.

We may assume that $\mathcal{E}$ is stable: Write

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_d = \mathcal{E}$$

with $\mathcal{E}_i/\mathcal{E}_{i+1}$ stable. If all $(\mathcal{E}_i/\mathcal{E}_{i+1}) \otimes \mathcal{F}$ have no cohomology, then $\mathcal{E} \otimes \mathcal{F}$ has no cohomology as well.

Look at $S =$ local deformation space of $\mathcal{F}$.

(REMARK: Replace $\mathcal{F}$ by $\mathcal{F} \oplus (\mathcal{F}^* \otimes \mathcal{L})$: $H^1(\mathcal{E} \otimes \mathcal{F}^* \otimes \mathcal{L})$ is dual to $H^1(\mathcal{F} \otimes \mathcal{H}\mathrm{om}\,(\mathcal{E} \otimes \mathcal{L}, \omega_C))$, hence we need not worry about $\det\mathcal{F}$.)

S is smooth, $\mathcal{F}$ *extends to a bundle* $\mathcal{F}$ on $C \times S$, and *there is an epimorphism* $\mathcal{T}_S \twoheadrightarrow R^1\pi_*\mathcal{E}\mathrm{nd}\,(\mathcal{F})$.

Study $R^i\pi_*(\mathcal{E} \otimes \mathcal{F})$.

Replace S by an open subset such that $\pi_*(\mathcal{E}\otimes\mathcal{F})$ is a vector bundle there.

Claim: The cup product $\pi_*(\mathcal{E} \otimes \mathcal{F}) \times R^1\pi_*\mathcal{E}\mathrm{nd}\,(\mathcal{F}) \longrightarrow R^1\pi_*(\mathcal{E} \otimes \mathcal{F})$ is identically zero.

PROOF: *The tangent vectors are given by* $\mathrm{Spec}\,(k[\varepsilon]) \to S$.

Let $\mathcal{F}_0$ be the constant deformation. $\mathcal{F}$ differs from $\mathcal{F}_0$ by a cocycle $1+\varepsilon\alpha$, where α has values in $\mathcal{E}\mathrm{nd}\,(\mathcal{F}_0)$.

Let $s_0 \in \Gamma(C, \mathcal{E} \otimes \mathcal{F}_0)$ be a global section, then $s = s_0 + \varepsilon\,\delta s$ extends (for some 1–ČECH–cocycle δs).

We get $\varepsilon\,\alpha(s_0) + \varepsilon\, d(\delta s) = 0$, where d is the ČECH differential. But this is just what we get by the cup product pairing. □

Choose a closed point $s \in S$. There is a map

$$\Gamma(C, \mathcal{E} \otimes \mathcal{F}) \times \mathrm{Hom}_C(\mathcal{E}\mathrm{nd}\,(\mathcal{F}), \omega_C)^* \longrightarrow \mathrm{Hom}_C(\mathcal{E} \otimes \mathcal{F}, \omega_C)^*\,, \tag{7}$$

and this map vanishes. To get the map, look at

$$\Gamma(C, \mathcal{E} \otimes \mathcal{F}) \times \mathrm{Hom}_C(\mathcal{E} \otimes \mathcal{F}, \omega_C) \longrightarrow \Gamma(C, \mathcal{E}\mathrm{nd}\,(\mathcal{F}) \otimes \omega_C)\,,$$

which is given by 'contracting' $\mathcal{E}$.

Let $\mathcal{G} \subset \mathcal{E}$ be the sub-bundle generically generated by the image of $\Gamma(C, \mathcal{E} \otimes \mathcal{F}) \otimes \mathcal{F}^* \to \mathcal{E}$. This is the smallest sub-bundle $\mathcal{G}$ such that all sections lie in $\mathcal{G} \otimes \mathcal{F}$:

$$\Gamma(C, \mathcal{E} \otimes \mathcal{F}) = \Gamma(C, \mathcal{G} \otimes \mathcal{F})\,.$$

The vanishing of the pairing (7) is equivalent to $\mathrm{Hom}_C(\mathcal{E} \otimes \mathcal{F}, \omega_C) = \mathrm{Hom}_C(\mathcal{E}/\mathcal{G} \otimes \mathcal{F}, \omega_C)$.

Stability of $\mathcal{E}$ implies $\mu(\mathcal{G}) \leq \mu(\mathcal{E}) - \varepsilon$ (with some $\varepsilon \geq 1/\mathrm{rk}\,\mathcal{E}$).

Replacing S by an open subset, we may assume that the $\mathcal{G}$'s define a sub-bundle of $\mathcal{E}$ on $C \times S$.

Idea: Estimate the variation of $\mathcal{G}$.

Deformations of sub-bundles $\mathcal{G} \subset \mathcal{E}$ have tangent space $\operatorname{Hom}_C(\mathcal{G}, \mathcal{E}/\mathcal{G})$. This gives a map $\mathcal{T}_S \to \operatorname{Hom}_C(\mathcal{G}, \mathcal{E}/\mathcal{G})$, which can be described in the following way (ε comes from $k[\varepsilon]$ with $\varepsilon^2 = 0$):

Let $\mathcal{G}_0 \subset \mathcal{E}$ be constant, and let g be a local section of $\mathcal{G}_0$. Let $g + \varepsilon\, e$ be a local section of $\mathcal{G}$. *(...?)*

Upper bound for $\operatorname{Hom}_C(\mathcal{G}, \mathcal{E}/\mathcal{G})$:

$(\mathcal{F}^*)^{\operatorname{rk}\mathcal{G}} \to \mathcal{G}$ is generically surjective (enough to estimate $\operatorname{Hom}_C(\mathcal{F}^*, \mathcal{E}/\mathcal{G})$), so we get a lower bound for $\deg \mathcal{G}$ (may assume $\mathcal{F}^*$ semistable). We then get bounds for $h^0(\mathcal{H}\mathrm{om}\,(\mathcal{G}, \mathcal{E}/\mathcal{G})) - h^1(\mathcal{H}\mathrm{om}\,(\mathcal{G}, \mathcal{E}/\mathcal{G}))$. Since $H^1(\mathcal{E}\mathrm{nd}\,(\mathcal{E})) \twoheadrightarrow H^1(\mathcal{H}\mathrm{om}\,(\mathcal{G}, \mathcal{E}/\mathcal{G}))$ (H^1 is right exact), we get a fixed bound for $\dim \operatorname{Hom}_C(\mathcal{G}, \mathcal{E}/\mathcal{G})$.

Replacing S by an open subset, we get a sub-bundle $\mathcal{T}_S' \subset \mathcal{T}_S$ with $\operatorname{rk}(\mathcal{T}_S/\mathcal{T}_S') \le$ const. such that

$$\mathcal{T}_S' \subset \ker\Big(\mathcal{T}_S \longrightarrow \pi_*\mathcal{H}\mathrm{om}\,(\mathcal{G}, \mathcal{E}/\mathcal{G})\Big)\,.$$

The pairing

$$\begin{array}{ccccc} \pi_*(\mathcal{G}\otimes\mathcal{F}) & \times & \mathcal{T}_S' & \longrightarrow & R^1\pi_*(\mathcal{G}\otimes\mathcal{F}) \\ & & \downarrow & & \\ & & R^1\pi_*\mathcal{E}\mathrm{nd}\,(\mathcal{F}) & & \end{array}$$

vanishes. Consider

$$\Gamma(C, \mathcal{G}\otimes\mathcal{F}) \times \operatorname{Hom}(\mathcal{G}\otimes\mathcal{F}, \omega) \longrightarrow \Gamma(C, \mathcal{E}\mathrm{nd}\,(\mathcal{F})\otimes\omega)\,.$$

The image is contained in a subspace of dimension $\le c$.

Key observation:

$$\begin{array}{rcl} h^0(\mathcal{G}\otimes\mathcal{F}) - h^1(\mathcal{G}\otimes\mathcal{F}) & = & \mu(\mathcal{G}\otimes\mathcal{F})\operatorname{rk}\mathcal{G}\operatorname{rk}\mathcal{F} \\ & \le & -\varepsilon\operatorname{rk}\mathcal{G}\operatorname{rk}\mathcal{F} \\ & \to & -\infty \qquad \text{for } \operatorname{rk}\mathcal{F}\to\infty \end{array}$$

There exist $r \le \operatorname{rk}\mathcal{G}$ sections $g_1, \dots, g_r$ of $\mathcal{G}$ such that $(\mathcal{F}^*)^r \xrightarrow{(g_1,\dots,g_r)} \mathcal{G}$ is generically surjective. This gives an injection

$$\operatorname{Hom}(\mathcal{G}\otimes\mathcal{F}, \omega) \hookrightarrow \Gamma(C, \mathcal{E}\mathrm{nd}\,(\mathcal{F})\otimes\omega)^r\,,$$

and the image is contained in (above subspace)^r, hence $h^1(\mathcal{G} \otimes \mathcal{F}) \leq rc$, a contradiction. $\square$

(Raw notes by BERND STEINERT.)

In the last lecture, we have seen that

$$\mathcal{E} \text{ semistable } \iff \exists \mathcal{F} : H^*(C, \mathcal{E} \otimes \mathcal{F}) = 0 .$$

In particular, $\mu(\mathcal{E}) + \mu(\mathcal{F}) = g - 1$. (One can prescribe $\operatorname{rk} \mathcal{F} \gg 0$ and $\det \mathcal{F}$ *to some extent.*)

Corollary 2 *ϑ–functions generate* $\det H^*(C, \mathcal{E} \otimes \mathcal{F})^{\otimes(-N)}$ *over the semistable locus.*

Today, we will show the following:

Assume given a complete curve B and a vector bundle $\mathcal{E}$ on $B \times C$ such that $\deg \det R^* pr_{1,*}(\mathcal{E} \otimes \mathcal{F}) = 0$ and such that $\mathcal{E}|_{b \times C}$ is semistable for at least one $b \in B$.

Claim: All restrictions $\mathcal{E}_b = \mathcal{E}|_{b \times C}$ are semistable and have the same JORDAN–HÖLDER series.

REMARK 1: All fibers $\mathcal{E}_b$ are semistable.

PROOF: Find $\mathcal{F}'$ with $\operatorname{rk} \mathcal{F}' = M \operatorname{rk} \mathcal{F}$ and $\det \mathcal{F}' = (\det \mathcal{F})^{\otimes M}$ such that $H^*(C, \mathcal{E}_{b_0} \otimes \mathcal{F}') = 0$ (where $\mathcal{E}_{b_0}$ is semistable). Then

$$\det(R^* pr_{1,*}(\mathcal{E} \otimes \mathcal{F}')) = \Big(\det(R^* pr_{1,*}(\mathcal{E} \otimes \mathcal{F}))\Big)^{\otimes M}$$

has degree zero.

Let $\vartheta \in \Gamma(B, (\det(R^* pr_{1,*}(\mathcal{E} \otimes \mathcal{F}')))^{\otimes(-1)})$ with $\vartheta(b_0) \neq 0$. Then $\vartheta(b) \neq 0$ for all b, hence $\mathcal{E}_b \otimes \mathcal{F}$ has trivial cohomology, and $\mathcal{E}_b$ is semistable for all $b \in B$. $\square$

REMARK 2: Replace $\mathcal{E}$ by $\mathcal{E} \otimes \mathcal{F}'$ (which is semistable and has trivial cohomology).

REMARK 3: We may assume $B = \mathbb{P}^1$.

PROOF: Choose a finite map $B \xrightarrow{\pi} \mathbb{P}^1$ and consider $\pi_* \mathcal{E}$ on $\mathbb{P}^1 \times C$. Let $x \in \mathbb{P}^1$. Then

$$(\pi_* \mathcal{E})|_{\{x\} \times C} \;\cong\; \pi_*\Big((\mathcal{O}_B / \mathfrak{m}_x \mathcal{O}_B) \otimes \mathcal{E}\Big)$$

$$
\begin{aligned}
&\cong \pi_*(\mathcal{O}_B/\mathfrak{m}_x\mathcal{O}_B)\otimes\mathcal{E}\\
&\approx \bigoplus_{b\in\pi^{-1}(x)} \mathcal{E}_b
\end{aligned}
$$

where "$\approx$" means "JORDAN–HÖLDER equivalent" and the b are taken with their multiplicities in the last line. Since we are free to vary π as we like, the claim for $\mathbb{P}^1$ implies that for B.

Note that all fibers of $\pi_*\mathcal{E}$ have trivial cohomology, hence the ϑ–function vanishes nowhere, and our assumptions are valid for $\mathbb{P}^1$ and $\pi_*\mathcal{E}$. □

Claim: All restrictions $\mathcal{E}|_{B\times\{c\}}$ are isomorphic.

PROOF: Choose finite subsets $S, S' \subset C$ such that $\mathcal{O}(S)\cong\mathcal{O}(S')$. Look at

$$
\begin{aligned}
R^1 pr_{1,*}(\mathcal{E}\otimes pr_2^*\mathcal{O}(S)) &= 0\\
pr_{1,*}(\mathcal{E}\otimes pr_2^*\mathcal{O}(S)) &\cong \bigoplus_{s\in S}\mathcal{E}|_{B\times\{s\}}
\end{aligned}
$$

This follows essentially from the exact sequence

$$
0\longrightarrow\mathcal{O}\longrightarrow\mathcal{O}(S)\longrightarrow\bigoplus_{s\in S}k\longrightarrow 0\,.
$$

Also:

$$
pr_{1,*}(\mathcal{E}\otimes pr_2^*\mathcal{O}(S))\cong\bigoplus_{s\in S}\mathcal{E}|_{B\times\{s\}}\cong\bigoplus_{s'\in S'}\mathcal{E}|_{B\times\{s'\}}
$$

There is much freedom to choose S and S'. Check multiplicity of indecomposables (which is semi-continuous). $\Longrightarrow$ Claim. □

Corollary 3 *There exists an open cover $C=\bigcup_\alpha U_\alpha$ such that $\mathcal{E}|_{B\times U_\alpha} = pr_1^*(\mathcal{E}|_{B\times\{c_0\}})$.*

PROOF: Use semi-continuity: Consider $R^i pr_{2,*}(\mathcal{H}om\,(pr_1^*\mathcal{E}|_{B\times\{c_0\}},\mathcal{E}))$. The restrictions to each fiber $B\times\{c\}$ are isomorphic. By GRAUERT's semi-continuity, these are vector bundles and commute with base-change.

Apply to $i=0$ and lift isomorphisms on fibers. □

Now look at the HN–filtration: $HN_r(\mathcal{E}_0)$ (slopes $\geq r$). They glue together to form global sub-bundles $HN_r(\mathcal{E})\subset\mathcal{E}$ on $B\times C$.

Also: $HN_r(\mathcal{E})/HN_{r+1}(\mathcal{E}) \cong pr_1^*\mathcal{O}(r) \otimes pr_2^*\mathcal{G}_r$ for vector bundles $\mathcal{G}_r$ on C. (This is as canonical as things can be here.)

We know: $HN_r(\mathcal{E})|_{\{b\}\times C}$ has slope $\leq g-1$, so EULER characteristic ≤ 0.

Claim: We have equality.

PROOF:

$$\begin{aligned}
\det R^* pr_{1,*}\mathcal{E} &= \bigotimes_r \det R^* pr_{1,*}(\underbrace{\mathcal{O}(r) \otimes \mathcal{G}_r}_{\text{exterior tensor prod.}}) \\
&= \mathcal{O}\Big(\sum_r r\,\chi(H^*(C,\mathcal{G}_r))\Big) \\
&= \mathcal{O}\Big(\sum_{r=r_0\ll 0}^{\infty} r(\chi(HN_r) - \chi(HN_{r+1}))\Big) \\
&= \mathcal{O}\Big(\sum_r \chi(HN_r) - r_0\chi(\underbrace{HN_{r_0}}_{=\mathcal{E}})\Big) \\
&= \mathcal{O}\Big(\sum_r \chi(HN_r)\Big)
\end{aligned}$$

Hence $\sum_r \chi(HN_r) \geq 0$ *(because* $\det R^* pr_{1,*}\mathcal{E}$ *has global sections?)*, and since all $\chi(HN_r) \leq 0$, they must be $= 0$. Thus, all HN_r are semistable sub-bundles of the same slope as $\mathcal{E}$, i.e., $g-1$. □

This implies that $\mathcal{G}_r$ is semistable, and $\mathcal{E}_b \approx \bigoplus_r \mathcal{G}_r$.

[We now only have to give the argument for REMARK 2.]

We have shown now: $\mathcal{E} \otimes \mathcal{F}$ is JH–constant (if $H^*(C, \mathcal{E}_b \otimes \mathcal{F}) = 0$).

Idea: Let $\mathcal{G}$ be stable. We want to check that $\mathcal{G}$ occurs with the same multiplicity in each $\mathcal{E}_b = \mathcal{E}|_{\{b\}\times C}$.

We get the JH–series for $\mathcal{E}_b \otimes \mathcal{F}$ by taking the JH–series for $\mathcal{E}$ (i.e., $\mathcal{E} \approx \sum m_\alpha \mathcal{G}_\alpha$ with $\mathcal{G}_\alpha$ stable) plus taking the JH–series for each $\mathcal{G}_\alpha \otimes \mathcal{F}$.

Let $m_\eta(\mathcal{G}_\alpha)$ be the generic multiplicity, i.e., the multiplicity in the generic point η of B ($\mathcal{E}|_{\overline{k(\eta)}\times C}$), and let $m_b(\mathcal{G}_\alpha)$ be the multiplicity in $b \in B$.

Claim (?): $m_b \geq m_\eta$.

PROOF: Consider

$$\begin{array}{ccc}
\operatorname{Spec} V & \longrightarrow & B \\
\eta & \longmapsto & \eta_B \\
s & \longmapsto & b
\end{array}$$

Pull back on $\operatorname{Spec} V \times C$.

["This is really difficult. Why is this so difficult? Maybe I should give up ..."]

Take JH–series *(($\mathcal{E}_\mu$) ?)* on generic fiber $\eta \times C$: $\mathcal{E}_\eta \supset \mathcal{E}_\mu$; extend to $\mathcal{E}_\mu \subset \mathcal{E}$ such that $\mathcal{E}/\mathcal{E}_\mu$ is torsion free.

The $\mathcal{E}_\mu$ are bundles. $\mathcal{E}_\mu|_{C\times\{s\}}$ have the same slope as $\mathcal{E}$.

Look at $\mathcal{E}_\mu|_{C\times\{s\}} \hookrightarrow \mathcal{E}|_{C\times\{s\}}$. Since $\mathcal{E}$ is semistable, this map is strict, and $\mathcal{E}/\mathcal{E}_\mu$ is a bundle.

In some stage this quotient is my JH–constituent.

$\Longrightarrow \mathcal{E}_\mu/\mathcal{E}_{\mu+1}$: If generic fiber $\cong \mathcal{G}_\alpha$, so is special fiber.

So multiplicities can only jump up.

And we have a matrix eq. *??*

Always: $m_\eta \leq m_b$. Adding these up, we get equality everywhere. □

6 New construction of moduli space

As stack: $\mathfrak{M}^{\mathrm{ss}} = S/R$, where S was a versal deformation and R a groupoid. Let

$$\begin{array}{ccc} S & \longrightarrow & \mathbb{P}^N \\ (\det H^*)^{\otimes(-\nu)} & \hookleftarrow & \mathcal{O}(1) \end{array}$$

be the map defined by $\vartheta(\mathcal{E} \otimes \mathcal{F})$.

Factor this as $S \longrightarrow M \xrightarrow{\text{finite}} \mathbb{P}^N$, where M is the normalization (in particular, M is normal): M is defined by the normalization of $k[\vartheta]$ in

$$\bigoplus_{n=0}^{\infty} \Gamma\Big(\mathfrak{M}^{\mathrm{ss}}, (\det H^*)^{\otimes(-n\nu)}\Big).$$

We have the *scalar automorphisms* $\mathbb{G}_{\mathrm{m}} \times S \subset R$; $R/\mathbb{G}_{\mathrm{m}}$ still acts on S.

The stable locus $S^{\mathrm{s}} \subset S$ is open and R–invariant.

$R^{\mathrm{s}}/\mathbb{G}_{\mathrm{m}} \hookrightarrow S^{\mathrm{s}} \times S^{\mathrm{s}}$ eqivalence relation $\Longrightarrow \mathfrak{M}^{\mathrm{s}} = S^{\mathrm{s}}/R^{\mathrm{s}}$ algebraic space.

Claim: $\mathfrak{M}^{\mathrm{s}} \longrightarrow M$ is an open immersion.

PROOF: By ZARISKI's Main Theorem, 'quasi-finite = finite $\circ$ open immersion'. (Here, finite does not matter, because we have taken M to be the normalization.)

Hence we have to show that the map has finite (i.e., 0–dimensional) fibers.

If not, there is a (not necessarily complete) curve B^0 contained in some fiber, whence a family $\mathcal{E}$ of stable vector bundles on $B^0 \times C$. By LANGSTON's Theorem, this extends to a semistable family over B, the completion of B^0. Ratios ϑ_1/ϑ_2 are constant on it, hence det H^* is trivial, and $\mathcal{E}$ is JH–constant. Then all fibers are isomorphic (and stable), hence $B \to *$ maps to a point in $\mathfrak{M}^{\mathrm{s}}$.

Hence

$$\mathfrak{M}^{\mathrm{s}} \overset{\text{open imm.}}{\longrightarrow} M' \overset{\text{finite}}{\longrightarrow} M \,,$$

then $M' = M$, and we are done. □

Comparison with what we get from GIT

Finally:

$$\bigoplus_{n \geq 0} \Gamma(\mathfrak{M}, (\det H^*(C, \mathcal{E} \otimes \mathcal{F}))^{\otimes(-n)}) = \bigoplus_{n \geq 0} A_n = A$$

is a graded ring.

One has:

$$\begin{array}{lll} A & \supset & k[\theta]^{\mathrm{norm.}} \\ A & \supset & \Gamma(Q')^G \quad \text{(subring from GIT)} \end{array}$$

with $\mathfrak{M}^{\mathrm{ss}} = Q'/G$, $G = \mathbf{PGL}_N$.

Claim: All are equal.

(subring from GIT) = sections on moduli space from GIT.

$$\begin{array}{ccc} \mathfrak{M} & \longrightarrow & \text{moduli space} \\ \cup & & \cup \ \text{open} \\ \mathfrak{M}^{\mathrm{s}} & \overset{\cong}{\longleftrightarrow} & M^{\mathrm{s}} \end{array}$$

(...)

Next time: HIGGS bundles.

So far:

$\mathbf{GL}_r$	vector bundles of rank r
$\cup$	
$\mathbf{SL}_r$	vector bundles with trivial det or det a fixed line bundle

There are other reductive groups:

Sp: vector bundles $\mathcal{E}$ with an alternating bilinear map $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{O}$ (identifying $\mathcal{E}$ and $\mathcal{E}^*$)

$\mathbf{O}_r$ or $\mathbf{SO}_r$: vector bundles $\mathcal{E}$ with a symmetric bilinear map $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{O}$ (identifying $\mathcal{E}$ and $\mathcal{E}^*$)

In these cases there is also a notion of (semi-)stability: "parabolic subgroups" or subalgebras. Let

$$\mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}$$

be isotropic sub-bundles. Semistable means the parabolic subalgebra has degree ≤ 0.

(...)

This notion coincides with the old notion.

This is not true for stability (one only excludes isotropic sub-bundles).

June 9, 1995

(Raw notes by BERND STEINERT.)

7 HIGGS bundles

Let C be a curve as before.

Definition 5 *A* HIGGS **bundle** *is a pair* $(\mathcal{E}, \vartheta)$ *such that* $\mathcal{E}$ *is a vector bundle on* C *and* $\vartheta \in \Gamma(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \omega_C)$.

Why do we care about these?
Note: $\Gamma(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \omega_C) =$ dual of $H^1(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}))$
Thus:

$$\text{HIGGS bundles} \approx \begin{pmatrix} \text{cotangent bundle to moduli} \\ \text{space of vector bundles} \end{pmatrix}$$

(Of course there are several difficulties).

Formal computation of the tangent space

We have to look at isomorphism classes over $k[\varepsilon]$ with $\varepsilon^2 = 0$. Let $(\mathcal{E}_0, \vartheta_0)$ be a HIGGS bundle over k. We have to parametrize its extensions to $k[\varepsilon]$.

Write $C = \bigcup_\alpha U_\alpha$ *with open affines* U_α and let $\mathcal{E} \cong \mathcal{E}_0[\varepsilon]$ $(= id \bmod \varepsilon)$ and $\vartheta = \vartheta_0 + \varepsilon\,\delta\vartheta_\alpha$ on U_α.

Gluing $\mathcal{E}$ gives rise to a 1–cocycle $1+\varepsilon\, g_{\alpha\beta}$ with $g_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{E}\mathrm{nd}\,(\mathcal{E}_0))$, and we have $\delta\vartheta_\alpha - \delta\vartheta_\beta = -[g_{\alpha\beta}, \vartheta_0]$ (commutator) modulo trivial deformations, i.e., $g_{\alpha\beta} = g_\alpha - g_\beta$ with $\delta\vartheta_\alpha = -[g_\alpha, \vartheta_0]$.

The quotient gives the tangent space. It can be written as the hypercohomology of the complex

$$\begin{array}{ccc} \mathcal{E}\mathrm{nd}\,(\mathcal{E}) & \xrightarrow{Ad(\vartheta_0)} & \mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \omega_C \\ \deg 0 & \scriptstyle\text{comutator} & \deg 1 \end{array}$$

One can define $H^1(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}) \xrightarrow{Ad(\vartheta_0)} \mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \omega_C)$. To calculate it, one has to look at the Čech complex in degree 1:

$$\begin{array}{ccccccc} & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow \\ Ad(\vartheta_0) & \downarrow & & \downarrow & & \downarrow & \\ & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow \\ & & \longleftarrow & \text{Čech complex} & \longrightarrow & & \end{array}$$

(Differently from vector bundles, there can be an $H^2 \neq 0$ here.)

There is an exact sequence of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{E}\mathrm{nd}\,(\mathcal{E}) & \xrightarrow{=} & \mathcal{E}\mathrm{nd}\,(\mathcal{E}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \omega_C & \xrightarrow{=} & \mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \omega_C & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

It gives rise to a long exact sequence

$$\begin{aligned} \ldots \quad & \longrightarrow H^{i-1}(\mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \omega_C) \longrightarrow H^i(\mathcal{E}\mathrm{nd}\,(\mathcal{E}) \to \mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \omega_C) \\ & \longrightarrow H^i(\mathcal{E}\mathrm{nd}\,(\mathcal{E})) \xrightarrow{Ad(\vartheta_0)} H^i(\mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \omega_C) \longrightarrow \ldots \end{aligned}$$

$\mathcal{K}^* = H^1(\mathcal{E}\mathrm{nd}\,(\mathcal{E}) \to \mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \omega_C)$ has an inner product that is non-degenerate.

$$\mathrm{Hom}\,(\mathcal{K}^*, \omega_C) = \mathcal{K}^*[1] \qquad \text{(i.e., degrees -1 and 0)}$$

Taking Serre duality:

$$H^1(\mathcal{K}^*) \qquad \text{dual to} \qquad H^0(\mathcal{H}\mathrm{om}\,(\mathcal{K}^*, \omega_C)) = H^1(\mathcal{K}^*)$$

Check directly: This product is anti-symmetric.

It is also the same as the canonical symplectic form on the cotangent bundle (at least if restricted to stable bundles, as usual).

Invariants

$$\det(T-\vartheta) = T^r - a_1 T^{r-1} + \cdots + (-1)^r a_r$$

with $a_i \in \Gamma(C, \omega_C^{\otimes i})$. In particular, $a_1 = \operatorname{tr}\vartheta$ and $a_r = \det\vartheta$.

The *characteristic variety* is given by

$$\mathrm{Char} = \operatorname{Spec} S\Big(\bigoplus_{i=1}^{r} \Gamma(C, \omega_C^{\otimes i})^*\Big)$$

(S: symmetric algebra ?) A Higgs bundle over $C \times S$ gives a map $S \longrightarrow \mathrm{Char}$. Look at its fibers.

Conversely: Given $a_i \in \Gamma(C, \omega_C^{\otimes i})$, we construct a finite flat $D \longrightarrow C$:

$$D = \operatorname{Spec}\left(S^*_{\mathcal{O}_C}(\omega^{\otimes(-1)}) \Big/ \omega^{\otimes(-r)}(1, -a_1, \ldots, \pm a_r)\right),$$

i.e. in local coordinates: If $C = \bigcup U_\alpha$ and λ_α is a generator of ω_C on U_α, write $a_i = a_{i,\alpha}\lambda_\alpha$. Then $D = \bigcup V_\alpha$, where

$$\mathcal{O}_D(V_\alpha) = \mathcal{O}_C(U_\alpha)[T_\alpha] \Big/ (T_\alpha^r - a_{1,\alpha}T_\alpha^{r-1} + \cdots \pm a_{r,\alpha}).$$

The overlap rule on $U_\alpha \cap U_\beta$ is $T_\alpha\lambda_\alpha = T_\beta\lambda_\beta$.

D is a curve, but in general it need not be nonsingular. It is a complete intersection.

$\mathcal{E}$ is an $\mathcal{O}_D$–module by $T_\alpha\lambda_\alpha \mapsto \vartheta$. It is torsion free, but not necessarily locally free. I.e.,

$$(\text{Higgs bundle}) \quad \longrightarrow \quad \begin{pmatrix} \text{coherent torsion free sheaf on } C \\ \text{(with some conditions)} \end{pmatrix}$$

Definition 6 *A* Higgs *bundle $(\mathcal{E}, \vartheta)$ is called* **stable** *if for any ϑ–stable sub-bundle $0 \subsetneqq \mathcal{F} \subsetneqq \mathcal{E}$, we have $\mu(\mathcal{F}) < \mu(\mathcal{E})$.*

By GIT (or other methods), there exists a coarse moduli space $\mathrm{Higgs}^{\mathrm{ss}}$ which is projective over Char. It has an open subset $\mathrm{Higgs}^{\mathrm{s}}$ which is a manifold. (The proofs should be similar to the ones before.)

Why is it a manifold?

Look at $H^2(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}) \longrightarrow \mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \omega_C)$; this is dual to $H^0(\mathcal{E}\mathrm{nd}\,(\mathcal{E}) \longrightarrow \mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \omega_C) = k$.

REMARK: If $(\mathcal{E}, \vartheta)$ is a HIGGS bundle, then $(\det \mathcal{E}, \mathrm{tr}\,\vartheta)$ is also a HIGGS bundle.

The obstruction for $\mathcal{E}$ is the same as the obstruction for $\det \mathcal{E}$. *Hence we are reduced to rank 1.* In this case,

$$\mathrm{Higgs} = \mathrm{Pic}(C) \times \Gamma(C, \omega_C)\,,$$

and this is smooth.

We have $\mathrm{Higgs}^{\mathrm{s}} \supset$ (cotangent bundle to $\mathfrak{M}^{\mathrm{s}}$) as a dense open subset (by estimating dimensions); the complement has codimension at least 2.

A Theorem of LAUMON

We now suppose $\mathrm{char}\, k = 0$.

Look at the map $\mathrm{Higgs}^{\mathrm{s}} \longrightarrow \mathrm{Char}$ and recall that we have a non-degenerate inner product on the tangent bundle of $\mathrm{Higgs}^{\mathrm{s}}$.

Theorem 11 (LAUMON**)** *If S is smooth and is mapped into a fiber of the map above, then the image of the tangent bundle $\mathcal{T}_S$ is isotropic.*

Corollary 4 $\dim(\text{fiber}) \leq \frac{1}{2}\dim(\mathrm{Higgs}\,)$.

(This is not really a theorem about stability.)

PROOF: (Thm.) Let $(\mathcal{E}, \vartheta)$ be the familiy of HIGGS bundles on $C \times S$ corresponding to $S \to \mathrm{Higgs}$. By assumption, $a_i(\vartheta) = a_i$ is constant on S. The tangent bundle is mapped

$$\mathcal{T}_S \longrightarrow R^1 pr_{2,*}(\mathcal{E}\mathrm{nd}\,(\mathcal{E}) \longrightarrow \mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \omega_C)$$

($\mathcal{E}\mathrm{nd}\,(\mathcal{E})$ has an inner product). Try to get subcomplexes.

We have the following rules:

– We may replace S by an open subset or by an étale cover;

– We may replace C by a finite cover $C' \to C$ with C' smooth (here the inner product gets multiplied by the degree of the covering).

Now let's play the game:

Let η be the generic point *(of C?)*, $K = k(\eta)$. We have C_K, the function field $K(C)$, and $\mathcal{E}$ becomes a vector space with an endomorphism ϑ. Let $\vartheta = \vartheta_s + \vartheta_n$ be the JORDAN decomposition of ϑ. Then $\ker \vartheta_n$, $\ker \vartheta_n^2$, and so on define sub-vector spaces, so that we get ϑ–stable sub-bundles

$$0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}$$

over C_K.

Everything holds over some open subset of S, hence without loss of generality already over S. Then we get

$$\begin{array}{ccc} \mathcal{T}_S & \longrightarrow & H^1(\mathcal{E}\mathrm{nd}^{\,\mathrm{filt}}(\mathcal{E}) \xrightarrow{Ad(\vartheta)} \mathcal{E}\mathrm{nd}^{\,\mathrm{filt}}(\mathcal{E}) \otimes \omega_C) \\ & & \downarrow \\ & & H^1(\bigoplus_i \mathcal{E}\mathrm{nd}\,(gr_i(\mathcal{E})) \longrightarrow \bigoplus_i \mathcal{E}\mathrm{nd}\,(gr_i(\mathcal{E})) \otimes \omega_C) \end{array}$$

The inner product becomes the pull-back.

We may replace $\mathcal{E}$ by $gr_i(\mathcal{E})$, hence we may assume $\vartheta_n = 0$.

Choose a covering $C' \longrightarrow C$ such that all eigenvalues of $\vartheta = \vartheta_s$ lie in $K(C')$ (we also *have to take* a covering of S).

Then we get a complete ϑ–stable flag of (subspaces in the generic point and) sub-bundles

$$0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

This reduces us to the case $\operatorname{rk} \mathcal{E} = 1$. In this case,

$$\mathrm{Higgs} = \mathrm{Pic}(C') \times \underbrace{\Gamma(C', \omega_{C'})}_{=\mathrm{Char}},$$

and the inner product is given by ($\mathcal{E} = \mathcal{L}$ is a line bundle)

$$\begin{array}{ccc} \mathcal{E}\mathrm{nd}\,(\mathcal{L}) & \xrightarrow{Ad(\vartheta)} & \mathcal{E}\mathrm{nd}\,(\mathcal{L}) \otimes \omega_C \\ \| & & \| \\ \mathcal{O}_C & \xrightarrow{0} & \omega_C \end{array}$$

The complex splits, whence the assertion. □

Why is this interesting?

We have the following dimensions:

$$\begin{array}{ll} \text{vector bundles:} & r^2(g-1)+1 \\ \text{char. variety:} & \sum_{i=1}^{r} \dim \Gamma(C;\omega^{\otimes i}) \\ & \quad = 1 + \sum_{i=1}^{r}(i(2g-2)+1-g) \\ & \quad = 1 + r^2(g-1) \\ \text{Higgs:} & 2(r^2(g-1)+1) \end{array}$$

Corollary 5 *The map* $\mathrm{Higgs}^{\mathrm{s}} \longrightarrow \mathrm{Char}$ *is flat, and the fibers are maximal isotropic (i.e.* LAGARANG*ian).*

Fibers over generic points of Char:

Claim: a_i generic $\Longrightarrow$ D smooth and irreducible.

Corollary 6 $\mathrm{Jac}(D)$ *(irred.)* $\cong$ fiber $\leftrightarrow$ *line bundle on* D *(of deg ...).*

PROOF: (Claim) Locally, $D \subset C \times \mathbb{A}^1$ is given by

$$P(T) = T^r - a_1 T^{r-1} + \cdots \pm a_r = 0\,.$$

Look at points in $C \times \mathbb{A}^1 \times \mathrm{Char}$ where $P = \dfrac{\partial P}{\partial T} = \dfrac{\partial P}{\partial \xi} = 0$ (ξ a local coordinate in C).

Check: For a given point in $C \times \mathbb{A}^1$, this has codimension 3 in Char.

For $r \geq 3$: Have sections with zero-order 0 and 1.

$P = \dfrac{\partial P}{\partial \xi} = 0$ gives 2 conditions

$\dfrac{\partial P}{\partial T} = 0$ gives 1 condition on a_{r-1}.

For $r = 2$: Hyperelliptic ones cause some problems (WEIERSTRASS points).

$r = 1$: never occurs.

D irreducible:

Otherwise $D = D_1 \cup D_2$ is a disjoint union. Then $P = P_1 \cdot P_2$ with polynomials P_1, P_2 of degrees r_1 and r_2, resp., and cofficients in $\Gamma(C, \omega^{\otimes i})$. We get for the dimension:

$$(1 + r_1^2(g-1)) + (1 + r_2^2(g-1)) - 1 = 1 + (r_1^2 + r_2^2)(g-1)\,,$$

but $r^2 = r_1^2 + 2r_1r_2 + r_2^2 > r_1^2 + r_2^2$, a contradiction. $\square$

June 30, 1995

Last time we have obtained the HIGGS fibration

$$\mathcal{T}^*_{\mathfrak{M}^s} \subset \mathrm{Higgs}^{\,\mathrm{ss}} \xrightarrow{f} \mathrm{Char} = \bigoplus_{i=1}^{r} \Gamma(C, \omega^{\otimes i}) = \mathbb{A}^N$$

(N is the dimension of moduli space), where the complement of the cotangent bundle of $\mathfrak{M}^{\mathrm{s}}$ in $\mathrm{Higgs}^{\,\mathrm{ss}}$ has codimension ≥ 2.

Higgs classifies pairs $(\mathcal{E}, \vartheta)$ with $\vartheta \in \Gamma(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \omega_C)$, and (semi-) stability of HIGGS bundles is defined with respect to ϑ–invariant sub-bundles.

For some fixed degree, the generic fiber of f is a connected ABELian variety ($= \mathrm{Pic}(D)$, where $D \to C$ is some covering).

From now on, we will assume that $\det \mathcal{E} = \mathcal{O}$ (i.e., trivial), so that we can replace $\mathbf{GL}_r$ by $\mathbf{SL}_r$ (which simplifies things, since there is no $\mathbb{G}_{\mathrm{m}}$–factor). Correspondingly we restrict to $\vartheta \in \Gamma(C, \mathfrak{sl}(\mathcal{E}) \otimes \omega_C)$, i.e., $\mathrm{tr}\,\vartheta = 0$. (This implies that we must drop the summand $\Gamma(C, \omega)$ in Char—the dimension goes down by g.)

REMARK: We may replace $\mathbf{SL}_r$ by any semi-simple group G, e.g. $\mathbf{SO}_r$ or $\mathbf{Sp}_r$ (bundles with a non-degenerate symmetric or symplectic inner product). We then have to take $\vartheta \in \Gamma(C, \mathfrak{g}(\mathcal{E}) \otimes \omega_C)$, where $\mathfrak{g}$ is the LIE algebra of G.

(The proof that D is irreducible can be difficult for $G \neq \mathbf{SL}_r$.)

f is proper, *hence* $f_* \mathcal{O}_{\mathrm{Higgs}^{ss}} = \mathcal{O}_{\mathrm{Char}}$, so that we get the same global functions:

$$\bigoplus_{n \geq 0} \Gamma(\mathfrak{M}^{\mathrm{s}}, S^n \mathcal{T}_{\mathfrak{M}^s}) = \Gamma(\mathrm{Higgs}^{\,\mathrm{ss}}, \mathcal{O}_{\mathrm{Higgs}^{ss}}) = \Gamma(\mathrm{Char}, \mathcal{O}_{\mathrm{Char}}) = \bigoplus_{n \geq 0} S^n \Big(\bigoplus_{i=2}^{r} \Gamma(C, \omega_C^{\otimes i})^* \Big).$$

This is an isomorphism of *graded* vector spaces if we give $\Gamma(C, \omega_C^{\otimes i})^*$ degree i.

Corollary 7 *$\Gamma(\mathfrak{M}^{\mathrm{s}}, \mathcal{O}_{\mathfrak{M}^s}) = 0$.*

It is difficult to determine the higher cohomology, because we don't know much about R^1 in this situation. But we can show that certain classes in H^1 are non-zero.

How do we get classes in H^1 ?

We have an ample line bundle

$$\mathcal{L} = \left(\det H^\bullet(C, \mathcal{E} \otimes \mathcal{F})\right)^{\otimes(-1)}$$

on $\mathfrak{M}$ and on Higgs. (This depends only on $\operatorname{rk} \mathcal{F}$, since $\det \mathcal{E}$ is trivial.)

$\mathcal{L}$ has a first CHERN class $c_1(\mathcal{L}) \in H^1(\mathfrak{M}, \Omega^1_{\mathfrak{M}})$, and we get a map

$$c_1(\mathcal{L}) \cup \cdot : \Gamma(\mathfrak{M}, S^n \mathcal{T}_{\mathfrak{M}}) \longrightarrow H^1(\mathfrak{M}, \Omega^1_{\mathfrak{M}} \otimes S^n \mathcal{T}_{\mathfrak{M}}) \longrightarrow H^1(\mathfrak{M}, S^{n-1} \mathcal{T}_{\mathfrak{M}})$$

where the second arrow is some kind of derivative (use that Ω is dual to $\mathcal{T}$ and contract once).

Claim: If $n > 0$, this map is injective.

PROOF: The idea is to restrict to the generic fiber:

Recall the HIGGS fibration:

$$\begin{array}{ccccc} \mathcal{T}^*_{\mathfrak{M}} & \subset & \text{Higgs} & \xrightarrow{f} & \text{Char} \\ \downarrow & & & & \\ \mathfrak{M} & & & & \end{array}$$

We lift $\mathcal{L}$ from $\mathfrak{M}$ to $\mathcal{T}^*_{\mathfrak{M}}$ and push it to Higgs. Since the complement has codimension ≥ 2, we get an injection on H^1. *We claim that there is a map α such that the following diagram commutes:*

$$\begin{array}{ccc} H^1(\text{Higgs}, \mathcal{O}_{\text{Higgs}}) & \hookrightarrow & \bigoplus_{n\geq 0} H^1(\mathfrak{M}, S^n \mathcal{T}_{\mathfrak{M}}) \\ \alpha \uparrow & & \uparrow c_1(\mathcal{L})\cup\cdot \\ \Gamma(\text{Higgs}, \mathcal{O}_{\text{Higgs}}) & = & \bigoplus_{n\geq 0} \Gamma(\mathfrak{M}, S^n \mathcal{T}_{\mathfrak{M}}) \end{array}$$

Let $\varphi \in \Gamma(\text{Char}, \mathcal{O}_{\text{Char}})$ be of positive degree. Then $c_1(\mathcal{L}) \cup \varphi$ is some class in $H^1(\mathfrak{M}, S^\bullet \mathcal{T}_{\mathfrak{M}})$. To show that this is non-zero, restrict to generic fibers of f.

Restrict $\mathcal{L}$ to generic fiber ($= \operatorname{Pic}^0(D)$): There, $(\det H^\bullet(C, \mathcal{E}))^{\otimes(-1)} = (\det H^\bullet(D, \mathcal{E}))^{\otimes(-1)}$ is known to give an ample line bundle (θ–divisor) on $\operatorname{Pic}^0(D)$.

Aside: Higgs is a symplectic manifold ($\mathcal{T}_{\text{Higgs}} = H^\bullet(\mathfrak{sl}(\mathcal{E}) \xrightarrow{Ad(\vartheta)} \mathfrak{sl}(\mathcal{E}) \otimes \omega_C)$, and the form induced by this restricts to the canonical symplectic form on $\mathcal{T}^*_{\mathfrak{M}}$). *This gives a canonical identification* $\Omega^1_{\text{Higgs}} \cong \mathcal{T}_{\text{Higgs}}$. Thus for a local function $\varphi \in \mathcal{O}_{\text{Higgs}}$, we have $d\varphi \in \Omega^1_{\text{Higgs}} \cong \mathcal{T}_{\text{Higgs}}$, i.e., we get a vector field H_φ from φ (the HAMILTONian vector field of φ).

If φ is induced from $\mathcal{O}_{\text{Char}}$, we have $d\varphi = 0$ on the fibers *(since φ is constant along fibers)*, and H_φ is tangential to the (generic) fibers (since the tangent spaces of the fibers are maximal isotropic).

Claim: The map α is given by $\varphi \mapsto c_1(\mathcal{L}) \cup H_\varphi \in H^1(\text{Higgs}, \mathcal{O}_{\text{Higgs}})$.

PROOF: Later. □

To see that $\alpha(\varphi) \neq 0$: Take 'generic fiber' $f^{-1}(\eta) = A$ (an ABELian variety). Since $\mathcal{L}$ is ample on A, one knows that

$$c_1(\mathcal{L}) \in H^1(A, \Omega^1_A) = H^1(A, \mathcal{O}_A) \otimes t^*_A$$

induces a perfect duality *between* $H^1(A, \mathcal{O}_A)$ *and* t_A. *(Is $t_A = \Gamma(A, \mathcal{T}_A)$?)* Hence if $c_1(\mathcal{L}) \cup H_\varphi = 0$, then $H_\varphi = 0$ on $f^{-1}(\eta)$. Since H_φ is tangential to $f^{-1}(\eta)$, this means that $H_\varphi = 0$ everywhere, hence $d\varphi = 0$, and φ is constant, contradicting our choice of φ.

REMARK: Looking at functions and H^1 doesn't note removal of bits of codimension ≥ 2. Hence it doesn't matter whether we look at Higgs, Higgs$^{\text{ss}}$, Higgs$^{\text{s}}$, $\mathcal{T}^*_{\mathfrak{M}}$, $\mathcal{T}^*_{\mathfrak{M}^s s}$ or $\mathcal{T}^*_{\mathfrak{M}^s}$.

□

What is $c_1(\mathcal{L})$?

Facts on CHERN *classes and the* CHERN *character can be found, for example, in chapter 1 of [*HIRZEBRUCH, BERGER, JUNG: *Manifolds and Modular Forms].*

We take a base space (or stack) S and let π be the projection $\pi : C \times S \longrightarrow S$. Let $\mathcal{E}$ be a vector bundle on $C \times S$. *(E.g. $S = \mathfrak{M}$ or $S =$* Higgs *with universal bundle.)* Then $\mathcal{L} = \det R^\bullet\pi_*\mathcal{E}$ is a line bundle on S and we have $c_1(\mathcal{L}) = c_1(R^\bullet\pi_*\mathcal{E})$. [HBJ, p. 11].

If $\mathcal{E}$ is an $\mathbf{SL}_r$–bundle, which we will assume, then $c_1(\mathcal{E}) = 0$.

Recall the CHERN character of a (virtual) vector bundle $\mathcal{F}$:

$$ch(\mathcal{F}) = \operatorname{rk}\mathcal{F} + c_1(\mathcal{F}) + \tfrac{1}{2}(c_1(\mathcal{F})^2) - c_2(\mathcal{F}) + \cdots$$

and the *total* TODD *class* of $\mathcal{F}$:

$$Td(\mathcal{F}) = 1 + \tfrac{1}{2}c_1(\mathcal{F}) + \tfrac{1}{12}(c_1(\mathcal{F})^2 + c_2(\mathcal{F})) + \cdots$$

By GROTHENDIECK–HIRZEBRUCH–RIEMANN–ROCH, we have (using $c_1(\mathcal{E}) = 0$)

$$\begin{aligned} ? + c_1(\mathcal{L}) + \cdots &= ch(R^\bullet \pi_* \mathcal{E}) \\ &= \pi_*(ch(\mathcal{E}) \cup Td(C)) \\ &= \pi_*((1 - c_2(\mathcal{E}) + \cdots) \cup (1 + \tfrac{1}{2}c_1(\mathcal{T}_C))) \\ &= ? - \pi_*(c_2(\mathcal{E})) + \cdots , \end{aligned}$$

hence (since π_* shifts degrees down by one)

$$c_1(\mathcal{L}) = -\pi_*(c_2(\mathcal{E}))\,.$$

In what cohomology theory do the CHERN classes live?

c_p can be defined in $H^{2p}(\cdot, \tau^{\geq p}\Omega)$ (hyper-cohomology), where

$$\tau^{\geq p}\Omega = 0 \longrightarrow 0 \longrightarrow \ldots \longrightarrow 0 \longrightarrow \Omega^p \xrightarrow{d} \Omega^{p+1} \xrightarrow{d} \ldots$$

is the truncated DE RHAM–complex.

For example, $c_1(\mathcal{L}) \in H^2(0 \longrightarrow \Omega^1 \xrightarrow{d} \Omega^2 \longrightarrow \ldots)$ is the obstruction to have an *integrable connection* on $\mathcal{L}$.

A *connection* on $\mathcal{L}$ is a map $\nabla : \mathcal{L} \longrightarrow \mathcal{L} \otimes \Omega^1$ such that $\nabla(\varphi \cdot l) = \varphi \nabla l + l \otimes d\varphi$ (for $\varphi \in \mathcal{O}$, $l \in \mathcal{L}$) (i.e. a prescription how vector fields act on $\mathcal{L}$).

∇ is called *integrable* if the '*curvature*' ∇^2 vanishes, or equivalently, if the action of vector fields preserves commutators.

Connections always exist locally. Cover your space by open affines U_i such that there is a connection ∇_i on $\mathcal{L}$ over U_i. Then $\nabla_i - \nabla_j = \alpha_{ij}$ is a

1–form on $U_i \cap U_j$, and $\nabla_i^2 = R_i \wedge \cdot$ for some R_i (also called curvature) with $dR_i = 0$. On $U_i \cap U_j$, we then have $R_i - R_j = \alpha_{ij}\nabla_j + \nabla_j \alpha_{ij} = \pm d\alpha_{ij}$ (one of the signs is correct). This gives us a ČECH–2–cocycle in the ČECH complex associated to $0 \longrightarrow \Omega^1 \xrightarrow{d} \Omega^2 \longrightarrow \ldots$ vanishing iff there is an integrable connection on $\mathcal{L}$ globally.

We have a sequence

$$H^0(\Omega^2)^{d=0} \longrightarrow H^2(0 \longrightarrow \Omega^1 \longrightarrow \Omega^2 \longrightarrow \ldots) \longrightarrow H^1(\Omega^1),$$

where the quotient measures the existence of a global connection (the image of $c_1(\mathcal{L})$ in $H^1(\Omega^1)$ is called the HODGE class of $\mathcal{L}$) and where the subspace measures if the global connection can be made integrable.

An important property of the first ČHERN class of line bundles is that $c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2)$. This means that in chracteristic zero, it doesn't matter if we look at $\mathcal{L}$ or at $\mathcal{L}^{\otimes n}$ (with $n \neq 0$) when we want to know if $c_1(\mathcal{L})$ vanishes.

Knowing c_1 for line bundles, we can define c_i for vector bundles (by the splitting principle).

For $c_1(\mathcal{E})$: Locally define connections ∇_i, then $\nabla_i - \nabla_j = \alpha_{ij} \in \mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \Omega^1$ and so on. A similar construction as for line bundles gives us a class ('ATIYAH class') in $H^1(\mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \Omega^1)$ that is the obstruction to having a global connection on $\mathcal{E}$.

$$0 \longrightarrow \mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \Omega^1 \longrightarrow ? \longrightarrow \mathcal{O} \longrightarrow 0$$

The fiber over $1 \in \mathcal{O}$ is a homogeneous space under $\mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \Omega^1$; it is the space of connections on $\mathcal{E}$. *Look at the long exact cohomology sequence:*

$$0 \longrightarrow \Gamma(\mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \Omega^1) \longrightarrow \Gamma(?) \longrightarrow k \longrightarrow H^1(\mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \Omega^1)$$

The ATIYAH *class $\mathcal{E}$ is the image of $1 \in k$ in $H^1(\mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \Omega^1)$. Hence $\Gamma(?)$ has an element mapping to 1 (i.e. a global connection) iff this class is zero.*

In general, we get the projection of $c_p(\mathcal{E})$ to $H^p(\Omega^p)$ by applying symmetric polynomials *(i.e., the coefficients of the characteristic polynomial $\mathcal{E}\mathrm{nd}\,(\mathcal{E}) \to \mathcal{O}[X]$).*

Let $\mathcal{E}$ be an $\mathbf{SL}_r$–bundle. Then

$$0 \to \mathfrak{sl}(\mathcal{E}) \longrightarrow \mathcal{E}\mathrm{nd}\,(\mathcal{E}) \xrightarrow{\mathrm{tr}} \mathcal{O} \longrightarrow 0$$

We have $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = c_2(\mathcal{E}^*)$ *cf. [HBJ p. 11]*, hence

$$c_2(\mathfrak{sl}(\mathcal{E})) = c_2(\mathcal{E}\mathrm{nd}\,(\mathcal{E})) = c_2(\mathcal{E}^* \otimes \mathcal{E}) = 2rc_2(\mathcal{E})$$

(Look at the Chern *character* $ch(\mathcal{E}) = r - c_2(\mathcal{E}) + \ldots$ *and use* $ch(\mathcal{E}^* \otimes \mathcal{E}) = ch(\mathcal{E}^*)ch(\mathcal{E})$, *cf. [HBJ, p. 9]. For the first equality, note that the sequence above splits (at least if r is prime to the characteristic) by* $\mathcal{O} \to \mathcal{E}\mathrm{nd}\,(\mathcal{E}), x \mapsto (1/r)x\, id_{\mathcal{E}}$*.)*

This means that for many purposes, we can use $\mathfrak{sl}(\mathcal{E})$ in place of $\mathcal{E}$. (This is related to the fact that different representations of $\mathbf{SL}_r$ give Chern classes differing by a constant factor.)

Example: Consider a stable vector bundle $\mathcal{E}$ on a curve C. Then its Atiyah class lives in

$$H^1(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}) \otimes \omega_C) = H^0(C, \mathcal{E}\mathrm{nd}\,(\mathcal{E}))^* = k$$

and is proportional to $\deg \mathcal{E}$. Hence stable $\mathbf{SL}_r$–bundles always have a (global) connection. (Over $\mathbb{C}$, the Theorem of Narasimhan–Seshadri tells us that there is even a unitary one.)

Now look at the universal bundle $\mathcal{E}$ on $C \times \mathfrak{M} \xrightarrow{f} \mathfrak{M}$ and its Atiyah class

$$\alpha = \alpha_C + \alpha_{\mathfrak{M}} \in H^1(C \times \mathfrak{M}, \mathfrak{sl}(\mathcal{E}) \otimes (\Omega_C \oplus \Omega_{\mathfrak{M}}))\,.$$

Since α_C restricted to a fiber of f is trivial, we see that

$$\alpha_C \in H^1(\mathfrak{M}, f_*(\mathfrak{sl}(\mathcal{E}) \otimes \Omega_C)) = H^1(\mathfrak{M}, \Omega_{\mathfrak{M}})$$

(Look at $R^1 f_*$). As to $\alpha_{\mathfrak{M}}$, we have

$$\begin{aligned} \alpha_{\mathfrak{M}} &\in H^1(C \times \mathfrak{M}, \mathfrak{sl}(\mathcal{E}) \otimes \Omega_{\mathfrak{M}}) \\ &\cong \Gamma(\mathfrak{M}, R^1 f_*(\mathfrak{sl}(\mathcal{E})) \otimes \Omega_{\mathfrak{M}}) \\ &= \mathrm{Hom}\,(\mathcal{T}_{\mathfrak{M}}, R^1 f_*(\mathfrak{sl}(\mathcal{E}))) \\ &= \mathrm{Hom}\,(\mathcal{T}_{\mathfrak{M}}, \mathcal{T}_{\mathfrak{M}}) \end{aligned}$$

Claim: $\alpha_{\mathfrak{M}} = \pm id \in \operatorname{Hom}(\mathcal{T}_{\mathfrak{M}}, \mathcal{T}_{\mathfrak{M}})$ (with some choice of sign).

PROOF: Choose an open cover $C = \bigcup U_i$ such that $\mathcal{E}|_{U_i}$ is trivial. On $U_{ij} = U_i \cap U_j$, we have transition functions g_{ij} (parametrized by $\mathfrak{M}$). Take the trivial connection ∇_i on $U_i \times \mathfrak{M}$. Then (taking the derivative in $\mathfrak{M}$–direction)

$$\begin{aligned}\nabla_i - \nabla_j &= dg_{ij} \circ g_{ij}^{-1} \\ &\in \Gamma(U_{ij} \times \mathfrak{M}, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega^1_{\mathfrak{M}}) \\ &= \operatorname{Hom}(\mathcal{T}_{\mathfrak{M}}, \mathcal{E}\mathrm{nd}(\mathcal{E}|_{U_{ij}})).\end{aligned}$$

Now follow the definitions. □

Claim: $\alpha_C \in H^1(\mathfrak{M}, \Omega_{\mathfrak{M}})$ is proportional to $c_1(\mathcal{L})$.

$\mathcal{L}$ is the determinant of cohomology or its inverse.

PROOF: *(?)* $c_2(\mathcal{E}) = \operatorname{tr}((\alpha_C + \alpha_{\mathfrak{M}}) \cup (\alpha_C + \alpha_{\mathfrak{M}})) = \operatorname{tr}(\alpha_C \cup \alpha_{\mathfrak{M}}) \bmod \Omega^2_{\mathfrak{M}}$ in $H^1(C \times \mathfrak{M}, \Omega_C \otimes \Omega_{\mathfrak{M}})$ *(maybe it's H^2 ?)* which maps to $H^1(\mathfrak{M}, \Omega_{\mathfrak{M}})$ *(*SERRE *duality $H^1(C, \Omega_C) = H^0(C, \mathcal{O}_C)^* = k$?)*

Hence (by RIEMANN–ROCH), $c_1(\mathcal{L}) \sim f_* c_2(\mathcal{E}) = \pm\alpha_C$. *I cannot say that this is particularly clear to me.* □

This class in $H^1(\mathfrak{M}, \Omega_{\mathfrak{M}})$ classifies an extension of $\mathcal{O}_{\mathfrak{M}}$ by $\Omega_{\mathfrak{M}}$, which dually is given by

$$0 \longrightarrow \mathcal{O}_{\mathfrak{M}} \longrightarrow \mathcal{D}\mathrm{iff}_{\leq 1}\mathcal{L} \longrightarrow \mathcal{T}_{\mathfrak{M}} \longrightarrow 0,$$

where $\mathcal{D}\mathrm{iff}_{\leq 1}\mathcal{L}$ are the differential operators of order ≤ 1 on $\mathcal{L}$ and where $\mathcal{O}_{\mathfrak{M}}$ is embedded in $\mathcal{D}\mathrm{iff}_{\leq 1}\mathcal{L}$ as the differential operators of order 0. (Differential operators of order 1 correspond to connections.)

This is equal to the obstruction to defining a connection on $\mathcal{E}$ (on $C \times \mathfrak{M}$) relative $\mathfrak{M}$ (i.e. $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_C$). By a Theorem of A. WEIL, ∇ exists locally in $\mathfrak{M}$, and two such ∇'s differ by an element of $f_*(\mathfrak{sl}(\mathcal{E}) \otimes \Omega_C) = \Omega_{\mathfrak{M}}$.

Another formulation is the following: Consider the moduli space $\mathfrak{M}^\nabla$ of stable ($\mathbf{SL}_r$–)bundles $\mathcal{E}$ together with a connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_C$. Then the canonical (forgetful) map $\mathfrak{M}^\nabla \to \mathfrak{M}$ gives a homogeneous space under $\Omega_{\mathfrak{M}}$ with class $c_1(\mathcal{L})$. Hence

$$\mathfrak{M}^\nabla = \operatorname{Spec}\left(S^\bullet \mathrm{Diff}_{\leq 1}\mathcal{L} \,\Big/\, \text{“}1 = 1\text{”}\right)$$

where the relation "$1 = 1$" means that one has to identify $1 \in S^0\mathrm{Diff}_{\leq 1}\mathcal{L} = k$ with $1 \in S^1\mathrm{Diff}_{\leq 1}\mathcal{L} = \mathrm{Diff}_{\leq 1}\mathcal{L}$ (identity operator).

A reference for these things is

G. FALTINGS: *Stable G–bundles* ..., J. Alg. Geom. **2** (1993), 507–586

July 4, 1995

First, a correction/explanation of some arguments from the last lecture.

$$\begin{array}{ccccc} \mathcal{T}^*_{\mathfrak{M}} & \hookrightarrow & \text{Higgs} & \xrightarrow{f} & \text{Char} \\ \big\downarrow & & & & \\ \mathfrak{M} & & & & \end{array}$$

Higgs is symplectic. For $\varphi, \psi \in \mathcal{O}_{\text{Higgs}}$, we have the POISSON bracket $\{\varphi, \psi\} = \langle d\varphi, d\psi \rangle = H_\varphi(\psi)$ (where $\langle \cdot, \cdot \rangle$ is the symplectic form on $\mathcal{T}_{\text{Higgs}}$). It satisfies the JACOBI identity. If φ and ψ are pull-backs from Char, we have $\{\varphi, \psi\} = 0$.

We have the map

$$\begin{array}{ccccccc} \Gamma(\text{Char}, \mathcal{O}_{\text{Char}}) & \to & \Gamma(\text{Higgs}, \mathcal{O}_{\text{Higgs}}) & \to & \Gamma(\text{Higgs}, \mathcal{T}_{\text{Higgs}}) & \to & H^1(\text{Higgs}, \mathcal{O}_{\text{Higgs}}) \\ \varphi & \mapsto & \varphi = f^*\varphi & \mapsto & H_\varphi & \mapsto & c_1(\mathcal{L}) \cup H_\varphi \end{array}$$

and we have shown that its kernel are the constants.

Let $\mathfrak{M}^\nabla$ be the moduli space of stable $\mathbf{SL}_r$–bundles with connection introduced at the end of last lecture. We have the canonical map $f : \mathfrak{M}^\nabla \longrightarrow \mathfrak{M}$.

$\mathfrak{M}^\nabla = \text{Conn}_{\mathcal{L}} = \text{Spec}_{\mathfrak{M}}(S^\bullet \text{Diff}_{\leq 1}\mathcal{L}/\text{“}1 = 1\text{”})$ classifies connections on $\mathcal{L}$.

Claim: $\Gamma(\mathfrak{M}^\nabla, \mathcal{O}_{\mathfrak{M}^\nabla}) = k$.

We have $\Gamma(\mathfrak{M}^\nabla, \mathcal{O}_{\mathfrak{M}^\nabla}) = \Gamma(\mathfrak{M}, f_*\mathcal{O}_{\mathfrak{M}^\nabla})$, and $f_*\mathcal{O}_{\mathfrak{M}^\nabla}$ is a twisted version of $S^\bullet \mathcal{T}_{\mathfrak{M}}$. It is still filtered $(F_\bullet)$ with graded pieces $gr_n = F_n/F_{n-1} \cong S^n \mathcal{T}_{\mathfrak{M}}$.

Try to lift and look at obstructions.

$$0 \longrightarrow F_{n-1}/F_{n-2} \longrightarrow F_n/F_{n-2} \longrightarrow F_n/F_{n-1} \longrightarrow 0$$

Show that

$$\Gamma(S^n \mathcal{T}_{\mathfrak{M}}) = \Gamma(F_n/F_{n-1}) \longrightarrow H^1(F_{n-1}/F_{n-2}) = H^1(S^{n-1}\mathcal{T}_{\mathfrak{M}})$$

is injective (for $n > 0$). *Then a section of F_n/F_{n-2} must already come from a section of F_{n-1}/F_{n-2}. Induction then shows that all sections must come from $F_0 = k$.*

Claim: This map is given by

$$\begin{array}{ccccc} \bigoplus_{n\geq 0}\Gamma(\mathfrak{M}, S^n\mathcal{T}_{\mathfrak{M}}) & = & \Gamma(\text{Higgs}\,,\mathcal{O}_{\text{Higgs}}) & \overset{c_1(\mathcal{L})\cup\cdot}{\longrightarrow} & \bigoplus_{n\geq 1} H^1(S^{n-1}\mathcal{T}_{\mathfrak{M}}) \\ & & & {\scriptstyle c_1(\mathcal{L})\cup\cdot}\searrow & \cup \\ & & & & H^1(\text{Higgs}\,,\mathcal{O}_{\text{Higgs}}) \end{array}$$

(The injection on the far right comes from the fact that the complement of $\mathcal{T}^*_{\mathfrak{M}}$ in Higgs has codimension ≥ 2.)

EXAMPLE: $n = 1$:

$$0 \longrightarrow \mathcal{O}_{\mathfrak{M}} \longrightarrow \mathcal{D}\text{iff}_{\leq 1}\mathcal{L} \longrightarrow \mathcal{T}_{\mathfrak{M}} \longrightarrow 0$$

Lift sections of $\mathcal{T}_{\mathfrak{M}}$ locally to $\mathcal{D}\text{iff}_{\leq 1}\mathcal{L}$; differences on overlaps (in $\mathcal{O}$) determine a class in $H^1(\mathcal{O})$.

ϑ of degree 1 in $\mathcal{T}^*_{\mathfrak{M}}$ corresponds to $\vartheta \in \Gamma(\mathfrak{M}, \mathcal{T}_{\mathfrak{M}})$. Form $d\vartheta$ and use duality to obtain a vector field on $T^*_{\mathfrak{M}}$ (total space of cotangent bundle).

Let $\mathfrak{M} = \bigcup U_i$ be a local cover trivializing $\mathcal{L}$, with transition functions g_{ij}. Then $c_1(\mathcal{L})$ is represented by $d\log g_{ij} = dg_{ij}\, g_{ij}^{-1}$. We have

$$\langle d\vartheta, dg_{ij}\, g_{ij}^{-1}\rangle = g_{ij}^{-1}\{\vartheta, g_{ij}\} = g_{ij}^{-1}\vartheta(g_{ij})$$

(where g_{ij} is to be considered as a funtion 'upstairs' *(i.e. on $T^*_{\mathfrak{M}}$?)*). This is the obstruction to lifting ϑ to $\mathcal{D}\text{iff}_{\leq 1}\mathcal{L}$.

For $n > 1$, we can write locally $\vartheta = \sum_I a_I\, \vartheta_{i1}\cdots\vartheta_{in}$. By similar computations using the appropriate variant of LEIBNIZ's rule and the case $n = 1$, we get the result.

Varying the curve

We want to vary the curve C. (The curves of genus g are parametrized by a moduli space of dimension $3g - 3$.)

Assume we have a family $\mathsf{C} \longrightarrow B$ of curves of genus g over a base B. Then we get a family $\mathsf{M} \xrightarrow{f} B$ of corresponding moduli spaces (or stacks) and a line bundle $\mathcal{L} = \det H^\bullet$ on M (everything considered is assumed to be smooth).

We want a 'projective connection' on $f_*\mathcal{L}^{\otimes n}$:

Given a vector field Z on B, find a second-order differential operator D_Z on M acting on $\mathcal{L}$. D_Z should be first-order 'along B', and '$D_Z - Z$' should be '$\mathcal{O}_B$–linear' (meaning: we get the same result from commuting functions in $\mathcal{O}_B$ with D_Z or Z).

(A differential operator has order zero if it commutes with functions; it has order n if commuting with functions gives differential operators of order $n-1$.)

Our construction will work for $\mathcal{L}^{\otimes n} = \mathcal{K}^{\sigma}_{\mathsf{M}/B}$ with $\sigma \neq \frac{1}{2}$ ($\mathcal{K}$ is the canonical bundle).

First, we want to solve the related problem obtained by replacing $\mathcal{D}\mathrm{iff}_{\leq 2}\mathcal{L}$ by $S^2(\mathcal{D}\mathrm{iff}_{\leq 1}\mathcal{L})$ (here the exceptional value is $\sigma = 0$).

We have a family of moduli spaces $\mathsf{M}^{\nabla} \longrightarrow B$ as well, and M^{∇} is 'infinitesimally constant': Let Z be a vector field on B.

Claim: Z lifts canonically to L_Z on M^{∇}, respecting commutators.

ARGUMENT: We assume that C is over $R = k[[t_1, \ldots, t_d]]$: C is a formal scheme over R, with a formal Ω^1.

Claim: Restriction induces an eqivalence

$$(\mathcal{E} + \text{ integrable } \nabla \text{ on } C_0) \quad \longleftrightarrow \quad \left(\mathcal{E} + \text{ integrable } \nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_{\mathsf{C}/k} \text{ on } \mathsf{C}\right)$$

(C_0 is the special fiber ($t_i \mapsto 0$) of C).

PROOF: We show that this even holds on any open affine $\mathsf{U} \subset \mathsf{C}$, where $\mathsf{U} \cong U \times_k \operatorname{Spf} R$. The claim is then obtained by gluing.

'$\leftarrow$' above is the restriction, for '$\rightarrow$', we use the trivialization $\mathsf{U} \cong U \times_k \operatorname{Spf} R$ to lift.

Let $(\mathcal{E}, \nabla)$ on U be given. Then

$$\Gamma(U, \mathcal{E}|_U) = \Gamma(\mathsf{U}, \mathcal{E})^{\nabla(\frac{\partial}{\partial t_i})=0}$$

and $\mathcal{E}$ is generated by the right hand side.

E.g. for $d = 1$: Start with some e_0. If e is constructed and is o.k. mod t^{n-1}, then $\nabla(\frac{\partial}{\partial t})(e) = t^{n-1} \cdot ?$. Replace e by $e - \frac{t^n}{n} \cdot ?$ *(we are in characteristic zero, obviously)*, then e is o.k. mod t^n.

For $d > 1$, use a similar argument ($\frac{\partial}{\partial t_i}$ commute).

That is: Let $b \in B \xleftarrow{f} \mathsf{M}^\nabla$. Then $\widehat{\mathsf{M}^\nabla}$, the formal completion of M^∇ along $f^{-1}(b)$, is canonically isomorphic to $f^{-1}(b) \times \hat{B}$ (hence tangent vectors of B lift to $\mathsf{M}^\nabla \rightsquigarrow$ action).

Or better: Consider $B \hookrightarrow B \times B$ the diagonal embedding, $\widehat{B \times B}$ the formal completion along the diagonal. Then $\widehat{pr_1^* \mathsf{M}^\nabla} \cong \widehat{pr_2^* \mathsf{M}^\nabla}$. □

We have the map $\mathsf{M}^\nabla \longrightarrow \mathsf{M}$, where the latter varies with C. The map

$$\mathcal{T}_{\mathsf{M}^\nabla/B} = H^1_{\mathrm{dR}}(\mathfrak{sl}(\mathcal{E})) = H^1(\mathfrak{sl}(\mathcal{E}) \xrightarrow{\nabla} \mathfrak{sl}(\mathcal{E}) \otimes \Omega^1) \longrightarrow H^1(\mathfrak{sl}(\mathcal{E})) = \mathcal{T}_{\mathsf{M}/B}$$

is its tangent map *(fiber-wise?)*.

A vector field $Z \in \mathcal{T}_B \longrightarrow H^1(C, \mathcal{T}_C)$ (the latter is the tangent space of the moduli space of curves at C; the map is the tangent map of $B \to$ moduli space) now induces

$$\Gamma(C, \mathfrak{sl}(\mathcal{E}) \otimes \Omega^1) \xrightarrow{L_Z} H^1_{\mathrm{dR}} \longrightarrow H^1(\mathfrak{sl}(\mathcal{E}))$$

Claim: This is the cup product.

We get the image of Z in $H^1(C, \mathcal{T}_C)$ as follows:

Write $C = \bigcup U_i$ and take $\mathsf{C}/k[\varepsilon]$ with $\mathsf{U}_i = U_i \times k[\varepsilon]$. The gluings *of the constant deformation and of the deformation given by Z* differ by $\varepsilon \cdot \vartheta_{ij}$ with $\vartheta_{ij} \in \Gamma(U_i \cap U_j, \mathcal{T}_C)$; this gives a 1–cocycle *representing the image of Z*.

We work in $\hat{B}$, i.e. over $k[[t_1, \dots, t_d]]$. Then $\mathsf{M}^\nabla = \mathsf{M}^\nabla_0 \times \hat{B}$ (M^∇_0 is the fiber over 0), and we can extend by a constant ∇ on $\hat{B}$.

Write $\mathsf{C} = \bigcup \mathsf{U}_i$ with $\mathsf{U}_i \cong U_i \times \hat{B}$ and where $(\mathsf{U}_i \cong \mathsf{U}_j)$ is $(U_i \cong U_j) + \sum t_i \vartheta_i + \cdots$. Additonally, the universal bundle $\mathcal{E}$ on M is trivialized: $\mathcal{E}|_{\mathsf{U}_i} \cong \mathcal{E}|_{U_i} \times \hat{B}$.

Then

$\tau \in \mathcal{T}_{\mathsf{M}^\nabla}$ *mapping to* $0 \in \mathcal{T}_{\mathsf{M}}$

$\longleftrightarrow$ family of vector bundles $+ \nabla$ on $\mathsf{C}[\varepsilon]$

$\longleftrightarrow$ family on $C_0[\varepsilon]$ + constant (w.r.t. $\mathcal{E}$) extension

$\longleftrightarrow$ given by $(\mathcal{E}_0, \nabla_0 + \varepsilon\alpha)$ with $\alpha \in \Gamma(C, \mathfrak{sl}(\mathcal{E}))$

Now apply vector field Z.

Compute the commutator $[L_Z, \tau]$: The original $\mathcal{E}_0$ is given by a 1–cocycle g_{ij} on $U_i \cap U_j$; α is given by α_i on U_i such that $\alpha_i - \alpha_j = d \log g_{ij}$. Extend to a 2–parameter family over $k[[s,t]]$, where τ corresponds to $\frac{\partial}{\partial s}$ and Z corresponds to $\frac{\partial}{\partial t}$: $(\mathcal{E}, \nabla)(s,t)$ is a vector bundle with integrable connection. Take the formal completion in the origin; then $\mathcal{E}$ becomes constant, $\cong \mathcal{E}_0 \times k[[s,t]]$. For the transition functions $g_{ij}(s,t)$ on $U_i \cap U_j$, this means $g_{ij}(s,0) = g_i(s) g_j(s)^{-1}$. Take the t–derivative: $\frac{\partial}{\partial t} \log g_{ij}(s,t) = Z_{ij}(g_{ij}) g_{ij}^{-1}$. s–derivative $\longleftrightarrow$ change connection $\longleftrightarrow$ change the trivialization of $\mathcal{E}$.

$(\dots)$

A plausibility argument is as follows:

M^{∇} parametrizes locally constant sheaves (sheaves that are parametrized by locally constant transition fucntions g_{ij} on $U_i \cap U_j$ (which are therefore insensitive to changes of the gluing)). If we change the connection by α, the transition fucntions now satisfy $g^{-1} \circ \nabla g = \alpha$. To compute the commutator $[L_Z, \tau]$, we compare: Changing the gluing first, then the connection does nothing, but changing the connection first and then the gluing gives us $\langle \alpha, Z \rangle$.

$\square$

We want to apply RIEMANN–ROCH to the universal bundle $\mathcal{E}$ on $\mathsf{C} \times_B \mathsf{M}^{\nabla} \xrightarrow{f} \mathsf{M}^{\nabla}$.

Locally, $\mathcal{E}$ has an (integrable) connection ∇_i lifting its universal connection (mapping to $\mathfrak{sl}(\mathcal{E}) \otimes \Omega^1_C$). We have

$$\begin{aligned} \nabla_i - \nabla_j &: \mathfrak{sl}(\mathcal{E}) \longrightarrow \mathfrak{sl}(\mathcal{E}) \otimes f^*\Omega^1_{\mathsf{M}^{\nabla}} \\ \nabla_i^2 &\in \mathfrak{sl}(\mathcal{E}) \otimes (\Omega_C \wedge f^*\Omega^1_{\mathsf{M}^{\nabla}} \bmod f^*\Omega^2_{\mathsf{M}^{\nabla}}) \end{aligned}$$

This defines a class in

$$H^1(\mathfrak{sl}(\mathcal{E}) \xrightarrow{\nabla} \mathfrak{sl}(\mathcal{E}) \otimes \Omega^1_C) \otimes f^*\Omega^1_{\mathsf{M}^{\nabla}} \cong \mathcal{T}_{\mathsf{M}^{\nabla}/B} \otimes f^*\Omega^1_{\mathsf{M}^{\nabla}} .$$

Claim: This is the canonical map $\mathcal{T}_{\mathsf{M}^{\nabla}} / \mathcal{T}_B \cong \mathcal{T}_{\mathsf{M}^{\nabla}/B}$.

(We have a splitting $\mathcal{T}_{\mathsf{M}^{\nabla}} = \mathcal{T}_B \oplus \mathcal{T}_{\mathsf{M}^{\nabla}/B}$.)

Complex topology argument:

M^∇ parametrizes locally constant (in C–direction) transition functions g_{ij}. The obstruction for a global connection *(on $\mathcal{L}$?)* is given by the dervatives of g_{ij} in M^∇–direction. By locally picking canonical connections (with $\nabla^2 = 0$), we can represent this object by $(d \log g_{ij}, 0)$, giving a class in $H^1(C, \mathfrak{sl}_r(\mathbb{C}))$ ($\mathfrak{sl}_r(\mathbb{C})$ is constant).

Then $c_2(\mathcal{E})$ is represented by a class which mod $f^*\Omega^2_{\mathsf{M}^\nabla}$ is represented by $\pm$ the square of this class. Hence $f_*(c_2(\mathcal{E})) \in \Gamma(\mathsf{M}^\nabla, \Omega^2_{\mathsf{M}^\nabla})$ is a class corresponding to the inner product on $\mathcal{T}_{\mathsf{M}^\nabla/B}$.

Result: The pull-back of $\mathcal{L}$ to M^∇ has a connection with curvature given by the inner product *on $\mathcal{T}_{\mathsf{M}^\nabla}$ induced by $\mathcal{T}_{\mathsf{M}^\nabla} \longrightarrow \mathcal{T}_{\mathsf{M}^\nabla/B}$ and the inner product on $\mathcal{T}_{\mathsf{M}^\nabla/B}$.*