Algorithms for the generation of integer Heronian triangles

Sascha Kurz

University of Bayreuth, Department of Mathematics, D-95440 Bayreuth, Germany

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Abstract

We describe two $\mathcal{O}(n^{2+\varepsilon})$ algorithm for the generation of integer Heronian triangles with diameter at most n.

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1 Introduction

The greek mathematician Heron of Alexandria (c. 10 A.D. - c. 75 A.D.) was probably the first who proved a relation between the side lengths a, b, and c and the area A of a triangle,

$$A=\sqrt{s(s-a)(s-b)(s-c)} \qquad \text{where } s=\frac{a+b+c}{2}\,.$$

If the area and the side lengths are rational then it is called a Heronian triangle. Triangles with integer sides and rational area where considered by the Indian mathematician Brahmagupta (598-668 A.D.) who gives the parametric solution

$$\begin{aligned} a &= \frac{p}{q}h(i^2+j^2)\\ b &= \frac{p}{q}i(h^2+j^2)\\ c &= \frac{p}{q}(i+h)(ih-j^2) \end{aligned}$$

Email address: `sascha.kurz@uni-bayreuth.de` (Sascha Kurz).
URL: `www.mathe2.uni-bayreuth.de/sascha` (Sascha Kurz).

for positive integers p, q, h, i, and j fulfilling $ih > j^2$ and $gcd(p, q) = gcd(h, i, j) = 1$.

Much has been written [1,2,4,5,6,7] on the determination of those integral triangles, but still less is known about the generation of integer Heronian triangles with diameter $n = \max(a, b, c)$. Our aim is to develop a fast algorithm for the generation of the complet set of integer Heronian triangles with diameter n.
In this context an extensive search on those triangles was made by Randall L. Rathbun. He simply checked the 7,818,928,282,738 integer triangles with diameter at most 2^{17} and received $5,801,746$ primitive, i.e. those with $gcd(a, b, c) = 1$, integer Heronian triangles with rational area.

2 A new parameterization

The obstacle for a computational use of Brahmagupta's parametric solution is the denominator q. So we first prove a few lemmas on q.

Lemma 2.1 *We can assume that* $q = w_1w_2w_3w_4$ *with pairwise coprime integers* w_1, w_2, w_3, w_4 *and*

$$\begin{array}{llll}
(w_1, h) = w_1 & (w_1, i) = w_1 & (w_1, i^2 + j^2) = 1 & (w_1, h^2 + j^2) = 1 \\
(w_2, h) = w_2 & (w_2, i) = 1 & (w_2, i^2 + j^2) = 1 & (w_2, h^2 + j^2) = w_2 \\
(w_3, h) = 1 & (w_3, i) = w_3 & (w_3, i^2 + j^2) = w_3 & (w_3, h^2 + j^2) = 1 \\
(w_4, h) = 1 & (w_4, i) = 1 & (w_4, i^2 + j^2) = w_4 & (w_4, h^2 + j^2) = w_4
\end{array}$$

where (x, y) *abbreviates* $gcd(x, y)$.

Proof. Suppose $q = \frac{q_1q_2}{gcd(q_1,q_2)}$ with $q_1|h$ and $q_2|i^2 + j^2$. Now we let r be a prime divisor of $gcd(q_1, q_2) \Longrightarrow r|h$, $r|i^2 + j^2$. With $b = \frac{pi(h^2+j^2)}{q}$ and $gcd(p, q) = 1$ we also have $r|i$ or $r|h^2 + j^2$. In the first case we have $r|i^2 + j^2 \Longrightarrow r|j^2 \Longrightarrow r|gcd(h, i, j)$ $\Longrightarrow r = 1$. In the second case we can use $r|h^2 + j^2$ and $r|h$ to conclude $r|j^2$. With this and $r|i^2 + j^2$ we also get $r|i^2$ and so $r|gcd(h, i, j) = 1 \Longrightarrow r = 1$. So we know $gcd(q_1, q_2)$=1.

Analog we get $q = q_3q_4$ with $gcd(q_3, q_4) = 1$, $q_3|i$, and $q_4|h^2 + j^2$.

Now we set $q_1 = w_1w_2$, $q_2 = w_3w_4$, $q_3 = w_1w_3$, and $q_4 = w_2w_4$. With $gcd(q_1, q_2) = gcd(q_3, q_4) = 1$ we can conclude the 4 divisibility conditions for each w_i and that the w_i are pairwise coprime. $\square$

Lemma 2.2

$$w_4|2(i+h)\,.$$

Proof. We consider $ai-bh=\frac{pih(i+h)(i-h)}{q}$ and conclude $w_4|(i-h)(i+h)$. Now we consider a prime factor r with $r|(i-h)$ and $gcd(r,i+h)=1$. Because $r|w_4|a,b,c$ we get $r|(i^2+j^2)+(h^2+j^2)+2(ih-j^2)=(i+h)^2$, a contradiction to $gcd(r,i+h)=1$. The proof is completed by $gcd(i+h,i-h)|2$. □

Lemma 2.3

$$w_4 \leq 8n\,.$$

Proof. To prove the lemma we will show $w_4|8c$. From $w_4|2(i+h)$ we conclude $w_4|2(i^2+j^2)+2(h^2+j^2)-2(i+h^2)=4(j^2-ih)$ and thus $w_4|2(i+h)\frac{4(ih-j^2)}{w_4}=8c$.
□

The next step is to find a parameterization of the set of solutions which is more suitable for computational purposes. Therefore we set

$$w_2=st^2$$

and

$$w_3=uv^2$$

with squarefree integers s and u. Because $w_2|h^2+j^2$, $w_2|h$, $w_3|i^2+j^2$, $w_3|i$, and $gcd(w_2,w_3)=1$ we have $stuv|j$. Thus we can set

$$\begin{aligned}
h &= \alpha w_1 st^2,\\
i &= \beta w_1 uv^2,\\
j &= \gamma stuv
\end{aligned}$$

with integers α, β, and γ.
With this we can give the following parameterization of the set of integer Heronian triangles.

$$\begin{aligned}
a &= \frac{p\alpha u[(\beta w_1 v)^2+(\gamma st)^2)]}{w_4},\\
b &= \frac{p\beta s[(\alpha w_1 t)^2+(\gamma uv)^2]}{w_4},\\
c &= \frac{p(\beta uv^2+\alpha st^2)(\beta\alpha w_1^2-\gamma^2 su)}{w_4}.
\end{aligned}$$

3 A first algorithm

In this section we will give an algorithm to generate all integer Heronian triangles with diameter at most n. The main idea is to run through all possible values for a and w_4 and then determine the possible parameters p, w_1, s, t, u, v, α, β and γ. Without loss of generality we can assume that $a \geq b$ and thus $n \leq 2a-1$. Then by lemma 2.3 we have $w_4 \leq 8n \leq 16a$.

Algorithm 3.1 ***(Generation of integer Heronian triangles I)***

```
determine the prime factorization of all integers at most $16n$
determine the solutions of $z = x^2 + y^2$ for all $z \leq 16n$
for $a$ from $1$ to $n$
  for $w_4$ from $1$ to $16a$
    run trough all quadruples $(p, \alpha, u, z)$ with $p\alpha uz$=$aw_4$
      run trough all pairs $(x, y)$ with $x^2 + y^2 = z$
        run trough all triples $(\beta, w_1, v)$ with $\beta w_1 v = x$
          run trough all triples $(\gamma, s, t)$ with $\gamma st = y$
            calculate and output $a$, $b$, $c$
```

In order to prove the running time $\mathcal{O}(n^{2+\varepsilon})$ of the above algorithm we cite two results from number theory.

Theorem 3.1 ***(317 [3])*** *For $\varepsilon > 0$ and $n > n_0(\varepsilon)$*

$$\tau(n) < 2^{(1+\varepsilon)\frac{\log n}{\log\log n}}$$

where $\tau(n)$ denotes the number of divisors of n.

So for each $\varepsilon > 0$ and each $f \leq 16n^2$ there are only $\mathcal{O}(n^{\varepsilon})$ quadruples (f_1, f_2, f_3, f_4) with $f_1 f_2 f_3 f_4 = f$.

Theorem 3.2 *The equation $z = x^2 + y^2$ has at most 2 solutions in positive integers for each prime z. Each solution of $z = x^2 + y^2$ can be derived of the solutions of $\tilde{z} = \tilde{x}^2 + \tilde{y}^2$ for $\tilde{z}|z$ by the composition formula*

$$z = (a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2 = (ac - bd)^2 + (ad + bc)^2$$

and

$$z = f^2(a^2 + b^2)\,.$$

Thus for each $\varepsilon > 0$ and each $z \leq 16n^2$ there are only $\mathcal{O}(n^{\varepsilon})$ integer solutions of $z = x^2 + y^2$ so that for the above algorithm there exists an implementation with running time $\mathcal{O}(n^{2+\varepsilon})$.

4 Further algorithms

For the derivation of a second algorithm for the determination of integer Heronian triangles we consider

$$16A^2 = (a+b+c)(a+b-c)(a-b+c)(-a+b+c) = (p-c)(p+c)(c^2-q^2)$$

where we set $p = a+b$ and $q = a-b$.

The idea is to run trough all possible values for $16A^2$ and then determine a, b and c by factoring $16A^2$.

Algorithm 4.1 ***(Generation of integer Heronian triangles II)***

```
determine the first primes at most $3n$
run through all $m$ and its prime factorization with $1 \leq m = 4A^2 \leq 2n^2$
and with the greatest prime divisor at most $3n$
  run trough all $p-c$, $p+c$, $c^2-q^2$ with $m^2 = (p-c)(p+c)(c^2-q^2)$
    determine $a$, $b$, and $c$
      if $a$, $b$, and $c$ are integers then output $a$, $b$, and $c$
```

Due to Theorem 3.1 this algorithm can be implementated with running time $\mathcal{O}(n^{2+\varepsilon})$. For completeness we would also like to give the pseudo code of the algorithm mentioned in the introduction.

Algorithm 4.2 ***(Generation of integer Heronian triangles III)***

```
for $a$ from $1$ to $n$
  for $b$ from $\lceil \frac{a+1}{2} \rceil$ to $a$
    for $c$ from $a+1-b$ to $b$
      if $(a+b+c)(a+b-c)(a-b+c)(-a+b+c)$ is the square of an integer
      then output $a$, $b$, and $c$
```

The running time of this algorithm is $\mathcal{O}(n^3)$.

5 Conclusion

Now we would like to compare the practical running times of the described algorithms.

Table 1
Comparison of running times

n	Algorithm 3.1	Algorithm 4.1	Algorithm 4.2
100	2.67 s	1.08 s	0.02 s
200	17.0 s	4.85 s	0.11 s
400	104 s	25.7 s	0.80 s
800	620 s	140 s	6.47 s
1600	3599 s	756 s	51.2 s
3200	20403 s	3987 s	403 s

If we extrapolate the running times then we can state that for $n \geq 80,000$ Algorithm 4.1 beats Algorithm 4.2. The advantage of Algorithm 3.1 is that on can deduce that there are at most $\mathcal{O}(n^{1+\varepsilon})$ integer Heronian triangles with diameter n. Perhaps this estimation and the complexity of Algorithm 3.1 can be lowered by refined number theoretic conditions on w_4.

Unfortunately we were not able to find estimations on the number of integer Heronian triangles in the literature so that we can not give a lower bound for the complexity of generating integer Heronian triangles.

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