Integrality of twisted L-values of elliptic curves

joined work with Hanneke Wiersema
and Julie Tavernier

Let $E/\mathbb{Q}$ be an elliptic curve.

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6, \quad a_i \in \mathbb{Q}$$

Let $\chi \neq 1$ be a primitive Dirichlet character

m its conductor and $\qquad \chi(a) \in \mathbb{Z}[\zeta_d]$

d its order.

THEOREM:

i) If $c_1 = 1$ and no bad prime divides m, then $\mathcal{L}(E,\chi) \in \mathbb{Z}[\zeta_d]$

ii) If $c_0 = 1$ and no additive prime divides m, then $\mathcal{L}(E,\chi) \in \mathbb{Z}[\zeta_d]$

What is $\mathcal{L}(E,\chi)$?

$\mathcal{L}(E,\chi)$ is the algebraic L-value:

$$\mathcal{L}(E,\chi) = \frac{L(E,\chi,1)}{\text{"period"}}$$

It is known that $\mathcal{L}(E,\chi) \in \mathbb{Q}(\zeta_d)$

Birch-Swinnerton-Dyer conjecture $\Rightarrow$

For $K/\mathbb{Q}$ a finite abelian extension

could zero

$$\mathcal{L}(E/K) = \frac{L(E/K, s=1)}{\text{"period"}} \overset{?}{=} \frac{\text{some integer}}{|E(K)_{tors}|^2}$$

Artin formalism: $L(E/K,s) = \prod_{\chi : \mathrm{Gal}(K/\mathbb{Q}) \to \mathbb{C}^\times} L(E,\chi,s)$

$$\mathcal{L}(E/K) = \prod_{\chi} \mathcal{L}(E,\chi)$$

$\mathcal{L}(E,\mathbb{1}) \longrightarrow \mathrm{BSD}/\mathbb{Q}$

The denominator of $\mathcal{L}(E,\chi)$ is linked to

$$E(K)_{\mathrm{tors}} \supsetneq E(\mathbb{Q})_{\mathrm{tors}}$$

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χ, $E_1 \neq E_2$ $\qquad \mathcal{L}(E_1,\chi) \neq \mathcal{L}(E_2,\chi)$

diffs by a unit.

Iwasawa theory: Extend $\chi \longmapsto \mathcal{L}(E,\chi)$ to a p-adic L-function. $\in \mathbb{Z}_p[\![T]\!]$

DEFINITION OF $L(E,\chi,s)$

Motivic definition:

$$L(V,s) = \prod_p \det\left(1 - Fr_p^{-1} \cdot p^{-s} \mid V^{I_p}\right)^{-1}$$

For $L(E,s)$, take V = dual of $(T_\ell E)\otimes \mathbb{Q}$

For $L(E,\chi,s)$, take $V = \quad \otimes \chi$

$L(E,s)$ $\qquad$ $L(E,\chi,s)$

p good red:

$$\boxed{\frac{1}{1 - a_p \cdot p^{-s} + p^{1-2s}}} \qquad \frac{1}{1 - a_p\,\overline{\chi(p)}\,p^{-s} + p^{1-2s}}$$

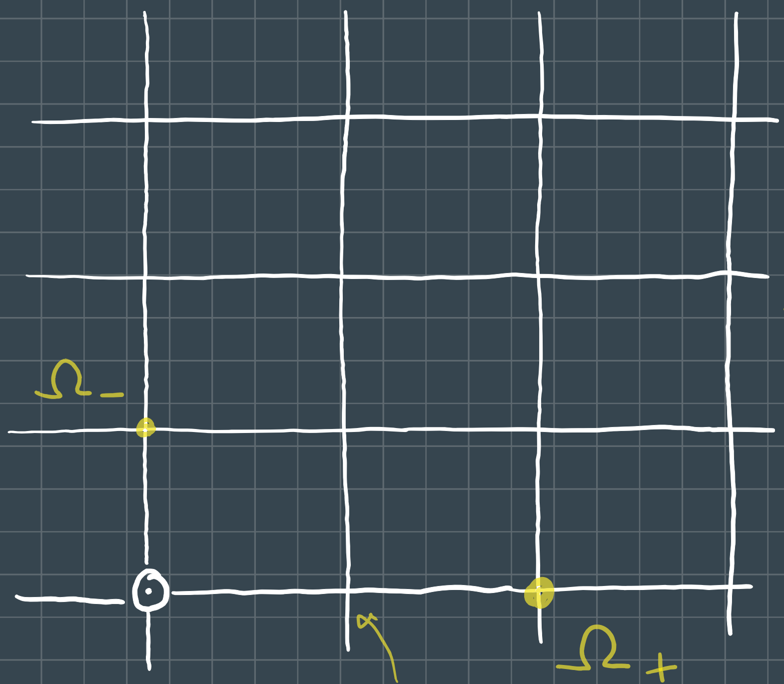
$L(E,s) = \sum_{n\geq 1} \frac{a_n}{n^s}$ related to $\sum_{n\geq 1} \frac{a_n \overline{\chi}(n)}{n^s} =: L^a(E,\chi,s)$

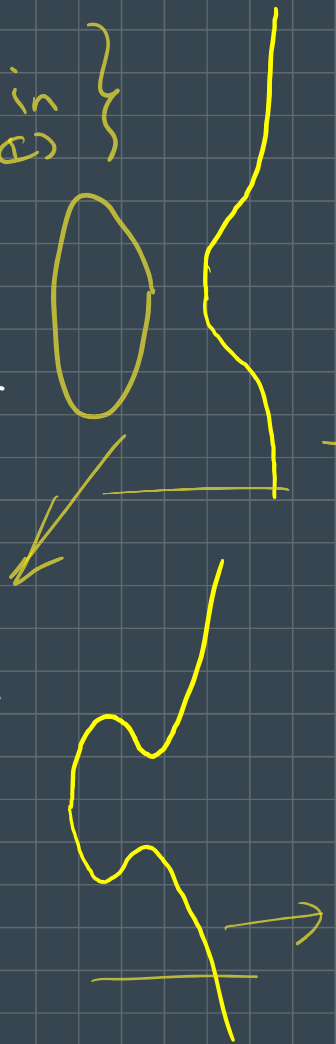
PERIODS

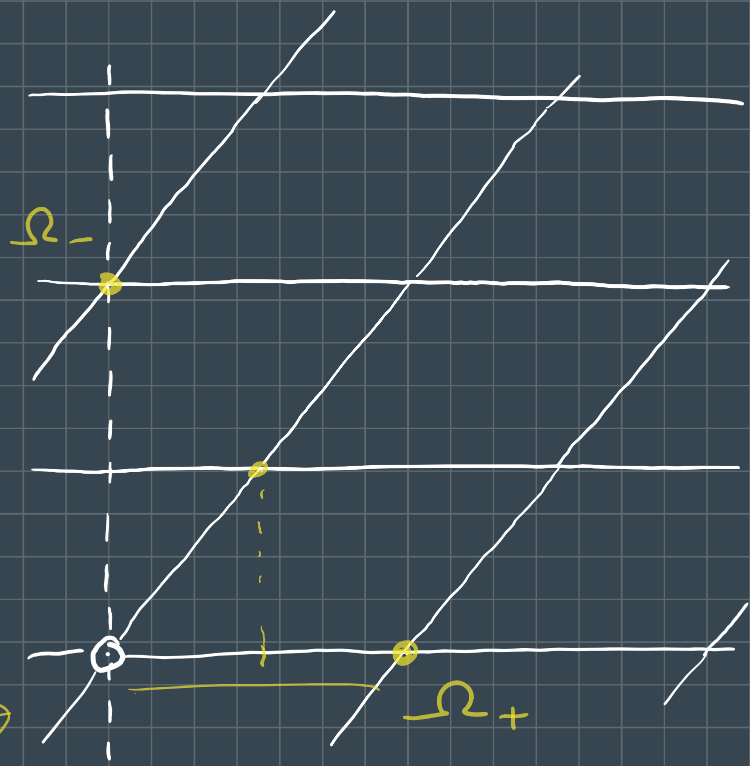
Set $\omega = \dfrac{dx}{2y + a_1 x + a_3}$ on a global minimal equation

$$\Lambda = \left\{ \int_\gamma \omega \;\middle|\; \gamma \text{ loops in } E(\mathbb{C}) \right\}$$

Néron lattice







Define

$$\boxed{\mathcal{L}(E,\chi) := \frac{L(E,\chi,1)\cdot m}{G(\chi)\cdot \Omega_{\chi(-1)}}}$$

where

$$G(\chi) = \sum_{a \bmod m} \chi(a)\cdot e^{2\pi i a/m}$$

is a Gauss sum.

MANIN CONSTANT

Let $f = \sum a_n q^n$ be the newform associated to the isogeny class of E.

$\omega_f = 2\pi i f(\tau)\,d\tau$ on $\mathcal{H} = \{\tau \mid \mathrm{Im}(\tau) > 0\}$

Modularity: There are morphisms of curves over $\mathbb{Q}$

$\Gamma_0(N) \backslash \mathcal{H}$ $\cup$ $\mathbb{P}^1(\mathbb{Q})$

$$\begin{tikzcd} \varphi_0 : & X_0(N)_{\mathbb{Q}} \arrow[r] & E \\ \varphi_1 : & X_1(N)_{\mathbb{Q}} \arrow[u] \arrow[r] & E \end{tikzcd}$$

(minimal degree, sending $\infty \mapsto O$).

$$\varphi_0^*(\omega) = c_0 \, \omega_f$$

$$\varphi_1^*(\omega) = c_1 \, \omega_f$$

Known: $c_0, c_1 \in \mathbb{Z}$

Steven's conjecture: $c_1 = 1$

Manin's conjecture: $c_0 = 1$ for at least one curve in the isogeny class

THEOREM :

i) If $c_1 = 1$ and no bad prime divides m,
then $\mathcal{L}(E, \chi) \in \mathbb{Z}[\zeta_d]$

ii) If $c_0 = 1$ and no additive prime divides m,
then $\mathcal{L}(E, \chi) \in \mathbb{Z}[\zeta_d]$

Examples : • $E_1 = 11a1$ $\quad c_0 = c_1 = 1$

$\mathcal{L}(E, \chi)$ is integral $\forall \chi$.

• $E_2 = 11a3$ $\quad c_1 = 1 \quad c_0 = 5$

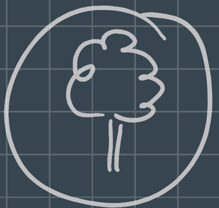
$m = 11 \quad d = 5 \quad K = \mathbb{Q}(\zeta_{11})^+$

$$\mathcal{L}(E, \chi) = \frac{1}{5}\left(2 + 4\zeta_5 + \zeta_5^2 + 3\zeta_5^3\right) \notin \mathbb{Z}[\zeta_5]$$

A Proof

Modular symbols: $\lambda(r) = \int_{i\infty}^{r} \omega_f \qquad r \in \mathbb{Q}$

Birch's formula:



$$\mathcal{L}^a(E,\chi) = \frac{L^a(E,\bar{\chi},1)\cdot m}{G(\chi)\,\Omega_{\chi(-1)}} = \frac{1}{\Omega_{\chi(-1)}} \sum_{a \bmod m} \chi(a)\cdot\lambda\left(\frac{a}{m}\right)$$

Set $D = (m,N)$.

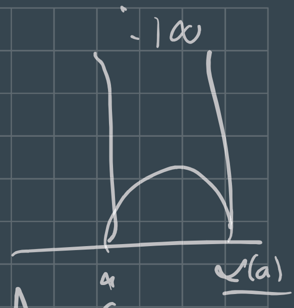
Lemma: If $c_1 = 1$, $D \neq 2$ and $D \neq m$, then $\mathcal{L}^a(E,\chi) \in \mathbb{Z}[\zeta_d]$

Proof:

For any a set $\alpha(a)$ = least residue of a mod D

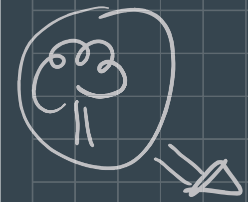
$$\mu\left(\frac{a}{m}\right) = \lambda\left(\frac{a}{m}\right) - \lambda\left(\frac{\alpha(a)}{m}\right)$$

Geometry : $\dfrac{a}{m} \sim_{\Gamma_1(N)} \dfrac{\alpha(a)}{m}$ $\qquad X_1(N)$



$$\mu\left(\frac{a}{m}\right) = \int_{\alpha(a)/m}^{a/m} \omega_f = \int_{\gamma} \omega_f = \int_{\varphi_1(\gamma)} \omega \in \Lambda$$

$\Gamma_1(N)$ – equiv $\qquad$ loop on $X_1(N)$ $\qquad$ loop on E



$$\Omega_{\chi(-1)} \cdot \mathcal{L}^a(E,\chi) = \sum_{a \bmod m} \chi(a)\,\mu\left(\frac{a}{m}\right) + \sum_{a \bmod m} \chi(a)\,\lambda\left(\frac{\alpha(a)}{m}\right)$$

$-D/2 \qquad D/2$

$$= \sum_{\substack{a=1\\(a,m)=1}}^{\left[\frac{m-1}{2}\right]} \left(\chi(a)\,\mu\left(\frac{a}{m}\right) + \chi(-a)\,\mu\left(-\frac{a}{m}\right)\right) + \sum_{-D/2 \le \alpha \le D/2} \lambda\left(\frac{\alpha}{m}\right) \cdot \sum_{\substack{a\\ \alpha \equiv a \bmod D}} \chi(a)$$

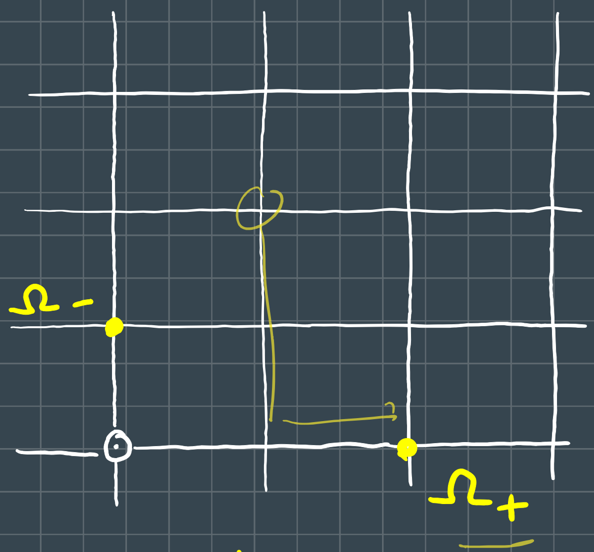
$\chi(-a) = \chi(-1)\cdot\chi(a)$ $\qquad$ 0 $\qquad$ $D \neq 2$ $\qquad$ $D \neq m$

$$\mu(-r) = \overline{\mu(r)}$$

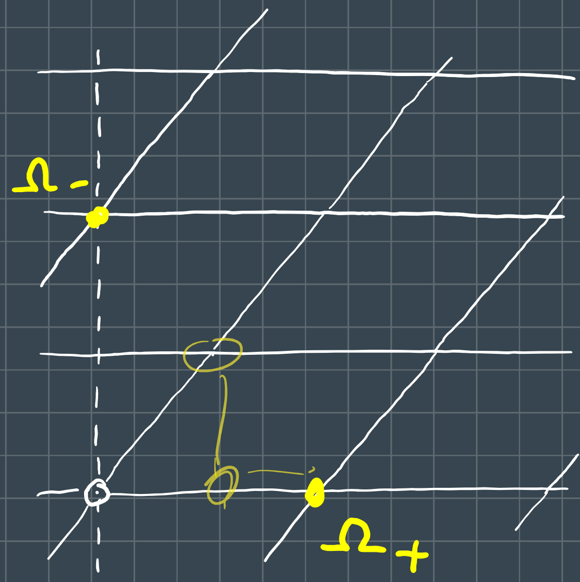
$$\Omega_{\chi(-1)}\,\mathcal{L}^a(E,\chi) = \sum_{a=1}^{\left[\frac{m-1}{2}\right]} \chi(a)\cdot\left(\mu\left(\tfrac{a}{m}\right) + \chi(-1)\,\overline{\mu\left(\tfrac{a}{m}\right)}\right)$$

$\chi(-1) = +1$

$$\Omega_+\,\mathcal{L}^a(E,\chi) = \sum \chi(a)\cdot 2\cdot \mathrm{Re}\left(\mu\left(\tfrac{a}{m}\right)\right) \qquad \mu\left(\tfrac{a}{m}\right) \in \Lambda$$



$\Delta > 0$



$\Delta < 0$

$$\mathcal{L}^a(E,\chi) = \sum \chi(a)\cdot(\text{integer}) \in \mathbb{Z}[\zeta_d]$$

Theorem: Assume $c_1 = 1$. If $\mathcal{L}^a(E,\chi) \notin \mathbb{Z}[\zeta_d]$ then

- Δ_E is a square OR E admits an isogeny over $\mathbb{Q}$
- $m^2 \mid N$ OR $c_0 > 1$ and $m \mid N$
- $K \subset \mathbb{Q}(E[\psi] \mid \psi \text{ isog}/\mathbb{Q},\ E[2^\infty])$

$E_3 = 392f1$ $\quad \Delta = 28^2$ $\quad K = \mathbb{Q}(\zeta_7)^+$ $\quad m = 7$

$\mathcal{L}(E,\chi) = \frac{2+\zeta_3}{2}$

E_4: $99b1$ $\quad K = \mathbb{Q}(\zeta_3)$ $\quad m = 3$ $\quad d = 2$

$\mathcal{L}(E,\chi) = 3/2 \longleftrightarrow \mathcal{L}^a(E,\chi) = 2$

no new torsion points ! $\mathcal{L}(E/K) = \mathcal{L}(E,\chi)\,\mathcal{L}(E,\mathbb{1})$

Examples : • $E_3 =$

• $E_4 =$

Summer research project with JULIE TAVERNIER.

Q: What are the possible denominators of

$$\left[\frac{a}{m}\right]^+ = \frac{\mathrm{Re}\left(\lambda\left(\frac{a}{m}\right)\right)}{\Omega_+} \; ?$$

$11^2 b1 \quad j = -11^2 \quad \Delta = 11^{16}$

$X_0(37)$

$j = -7 \cdot 11^3$

11

Appear : 1, 2, ... 22, 25, 26, (28), 30, 34, 37, 38,

42, 43, 67, 74, 86, 134, 163, 326

maybe also 32, 64, ...

48, 96, ...

40, 80, ...

56, 112, ...

36, 72, ...

50, 100.

Table 1: All non-integral $\mathscr{L}(E,\chi)$ for $N < 100$

Curve	c_0	c_∞	$\square$?	$t(\mathbb{Q})$	$t(K_\chi)$	m	χ	$\mathscr{L}^a$	$\mathscr{L}(E,\chi)$
11a3	5	1	no	5	25	11	$2 \mapsto \zeta_5$		$(2+4\zeta_5+\zeta_5^2+3\zeta_5^3)/5$
14a4	3	1	no	6	18	7	$3 \mapsto \zeta_3$		$(1-\zeta_3)/3$
14a6	3	2	no	6	18	7	$3 \mapsto \zeta_3$		$(1-\zeta_3)/3$
15a3	2	2	yes	8	16	5	$(5/\cdot)$		$1/2$
15a7	2	2	no	4	8	5	$(5/\cdot)$		$1/2$
15a8	4	1	no	4	8	3	$(-3/\cdot)$		$1/2$
15a8	4	1	no	4	16	5	$2 \mapsto i$		$(1+i)/2$
15a8	4	1	no	4	8	5	$(5/\cdot)$		$1/2$
20a2	2	2	no	6	12	5	$(5/\cdot)$		$1/2$
20a4	2	2	no	2	4	5	$(5/\cdot)$		$3/2$
21a4	2	1	no	4	8	3	$(-3/\cdot)$		$1/2$
21a4	2	1	no	4	8	7	$(-7/\cdot)$		$1/2$
24a4	2	1	no	4	8	4	$(-1/\cdot)$		$1/2$
24a4	2	1	no	4	8	3	$(-3/\cdot)$		$1/2$
26a3	3	1	no	3	9	13	$2 \mapsto \zeta_3$		$(2+\zeta_3)/3$
27a1	1	1	no	3	9	3	$(-3/\cdot)$		$1/3$
27a2	1	1	no	1	3	3	$(-3/\cdot)$		$1/3$
27a3	3	1	no	3	9	9	$2 \mapsto \zeta_3$		$(2+\zeta_3)/3$
27a3	3	1	no	3	9	3	$(-3/\cdot)$		$1/3$
27a4	3	1	no	3	9	9	$2 \mapsto \zeta_3$		$(2+\zeta_3)/3$
32a1	1	1	no	4	8	4	$(-1/\cdot)$		$1/2$
32a2	2	2	yes	4	8	4	$(-1/\cdot)$		$1/2$
32a2	2	2	yes	4	8	8	$(2/\cdot)$		$1/2$
32a3	2	2	no	2	4	4	$(-1/\cdot)$		$1/2$
32a4	2	2	no	4	8	8	$(2/\cdot)$		$1/2$
33a2	2	2	no	2	4	3	$(-3/\cdot)$		$1/2$
33a2	2	2	no	2	4	11	$(-11/\cdot)$		$1/2$
35a3	3	1	no	3	9	7	$3 \mapsto \zeta_3$		$(1-\zeta_3)/3$
36a1	1	1	no	6	12	3	$(-3/\cdot)$		$1/2$
36a3	1	1	no	2	12	3	$(-3/\cdot)$		$1/2$
40a3	2	2	no	4	8	5	$(5/\cdot)$		$1/2$
45a1	1	1	no	2	8	3	$(-3/\cdot)$	$1/4$	$3/16$
45a2	1	2	yes	4	8	3	$(-3/\cdot)$	$1/2$	$3/8$
45a3	1	2	no	2	4	3	$(-3/\cdot)$	$1/2$	$3/8$
45a4	1	2	yes	4	8	3	$(-3/\cdot)$	1	$3/4$
45a5	1	2	yes	4	4	3	$(-3/\cdot)$	2	$3/2$