# The degree of functions in the Johnson and $q$-Johnson schemes

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joint work with Jonathan Mannaert and Alfred Wassermann

## Introductory remarks

- Joint work with
  Jonathan Mannaert and Alfred Wassermann.
- Despite title

  "The degree of functions
  in the Johnson and $q$-Johnson schemes"

  No association schemes in this talk!
- Motivation (next slide) is geometric.

  Indeed: Topic close to design theory.
  Studied objects are "dual designs".

## Cameron-Liebler line classes

- Cameron, Liebler 1982:
  "Special" set $\mathcal{L}$ of lines in $\mathrm{PG}(3, q)$.
- Defined by the following equivalent properties:
  - Algebraic property:
    $\chi_{\mathcal{L}} \in \mathbb{R}$-row space of the point-line incidence matrix.
  - Geometric property:
    Constant intersection with any line spread of $\mathrm{PG}(3, q)$.

## In literature: Various directions of generalization

- Ambient space $\mathrm{PG}(n, q)$.
- lines $\longrightarrow$ $k$-spaces.
- Allow $q = 1$ (set case).
- points $\longrightarrow$ spaces of degree $t$.

## Goal

Coherent theory of all above generalizations.

# Subset and subspace lattices

- Fix $q = 1$ (set case) or prime power $q \geq 2$ ($q$-analog case).
- Fix $n$ non-negative integer.
- Let $V$ be a $\begin{cases} \text{set of size } n \\ \mathbb{F}_q\text{-vector space of dimension } n \end{cases}$
- Let $\mathcal{L}(V)$ be the lattice of all $\begin{cases} \text{subsets of } V \\ \mathbb{F}_q\text{-subspaces of } V \end{cases}$
- For $U \in \mathcal{L}(V)$ let $\operatorname{rk}(U) = \begin{cases} \#U \\ \dim(U) \end{cases}$
- Let $\begin{bmatrix} V \\ k \end{bmatrix} = \{U \in \mathcal{L}(V) \mid \operatorname{rk}(U) = k\}$.
  Set case: $\#\begin{bmatrix} V \\ k \end{bmatrix} = \binom{n}{k} = \begin{bmatrix} n \\ k \end{bmatrix}_1$ Binomial coefficient.
  $q$-analog case: $\#\begin{bmatrix} V \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q$ Gaussian coefficient.
- Always: Use algebraic dimension!
  (Except in established symbols like $\mathrm{PG}(n, q)$).

## Algebraic property

- Algebraic property of Cameron-Liebler line classes: $\chi_{\mathcal{L}} \in \mathbb{R}$-row space of the point-line incidence matrix.
- Straightforward generalization:
  - Let $W^{(tk)}$ incidence matrix of $t$-spaces vs. $k$-spaces.
  - Let $V_t$ be the $\mathbb{R}$-row space of $W^{(tk)}$.
  - Function $f : \begin{bmatrix} V \\ k \end{bmatrix} \to \mathbb{R}$ has algebraic property $A_t$ if $f \in V_t$.

## Baby example

- Let $q = 1$, $V = \{1, 2, 3, 4, 5, 6\}$ (so $n = 6$), $k = 3$, $t = 2$.
- Let $\mathcal{F} = \{\{1, 2, 3\}, \{4, 5, 6\}\} \subseteq \begin{bmatrix} V \\ 3 \end{bmatrix}$.
- Claim: Set $\mathcal{F}$ has algebraic property $A_2$, i. e. its characteristic function $\chi_{\mathcal{F}} : \begin{bmatrix} V \\ 3 \end{bmatrix} \to \mathbb{R}$ has prop. $A_2$.

# Baby example (cont.)

$W^{(23)} =$

| | 123 | 124 | 125 | 126 | 134 | 135 | 136 | 145 | 146 | 156 | 234 | 235 | 236 | 245 | 246 | 256 | 345 | 346 | 356 | 456 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 12 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1/3 |
| 13 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1/3 |
| 14 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1/6 |
| 15 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1/6 |
| 16 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1/6 |
| 23 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1/3 |
| 24 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | −1/6 |
| 25 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | −1/6 |
| 26 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | −1/6 |
| 34 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | −1/6 |
| 35 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | −1/6 |
| 36 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | −1/6 |
| 45 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1/3 |
| 46 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1/3 |
| 56 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1/3 |

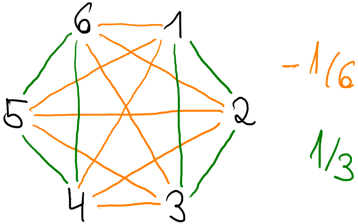
$$\chi_F = (1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1)$$

## Baby example (cont.)

- $\mathcal{F} = \{\{1,2,3\},\{4,5,6\}\}$.
- We found: $f$ has property $A_2$
  and the vector of 2-weights of $\mathcal{F}$ is

$$\mathrm{wt}_{\mathcal{F}}^{(2)} = \left(\tfrac{1}{3}, \tfrac{1}{3}, -\tfrac{1}{6}, -\tfrac{1}{6}, -\tfrac{1}{6}, \tfrac{1}{3}, -\tfrac{1}{6}, -\tfrac{1}{6}, -\tfrac{1}{6}, -\tfrac{1}{6}, -\tfrac{1}{6}, -\tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right).$$

- Visualization.




- Exercise.

  $\mathcal{F}$ does not have $A_1$.

## Geometric property

- Geometric property of Cameron-Liebler line classes: Constant intersection with any line spread of $\mathrm{PG}(3, q)$
- Generalization? – Not so clear.
- Observation:

$$\begin{aligned} & \text{line spread of } \mathrm{PG}(3, q) \\ = \quad & \text{set of lines in } \mathrm{PG}(3, q) \text{ covering every point exactly once} \\ = \quad & \text{simple } 1\text{-}(4, 2, 1)_q \text{ subspace design} \end{aligned}$$

- $\rightsquigarrow$ use designs!

## Definition: Simple design

A set $\mathcal{D} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ is called a simple $t$-$(n, k, \lambda)_q$ design, if every $T \in \begin{bmatrix} V \\ t \end{bmatrix}$ is contained in exactly $\lambda$ elements of $\mathcal{D}$.

- set case $q = 1$: combinatorial design
- $q$-analog case $q \geq 2$: subspace design

## Example

- Let $q = 1$, $V = \{1, 2, 3, 4, 5, 6\}$ (so $n = 6$), $k = 3$, $t = 2$.
- Let

$$\mathcal{D} = \{\{1,2,3\}, \{1,2,4\}, \{1,3,6\}, \{1,4,5\}, \{1,5,6\}, \{2,4,6\}, \{2,5,6\}, \{2,3,5\}, \{3,4,5\}, \{3,4,6\}\} \subseteq \begin{bmatrix} V \\ 3 \end{bmatrix}.$$

- Check design condition for $t = 2$.
  - $T = \{1, 2\}$ is contained in blocks $\{1, 2, 3\}$ and $\{1, 2, 4\}$.
  - $T = \{1, 3\}$ is contained in blocks $\{1, 2, 3\}$ and $\{1, 3, 6\}$.
  - $\ldots$
  - $T = \{5, 6\}$ is contained in blocks $\{1, 5, 6\}$ and $\{2, 5, 6\}$.
- $\implies \mathcal{D}$ is simple $2$-$(6, 3, 2)_1$ design.

## Example (Trivial simple designs)

- $\emptyset$ is empty $t$-$(v, k, 0)_q$ design.
- $\begin{bmatrix} V \\ k \end{bmatrix}$ is complete $t$-$(v, k, \lambda_{\max})_q$ design where $\lambda_{\max} := \begin{bmatrix} n-t \\ k-t \end{bmatrix}$.

## Definition: Simple design (repeated)

A set $\mathcal{D} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ is called a simple $t$-$(n, k, \lambda)_q$ design,
if every $T \in \begin{bmatrix} V \\ t \end{bmatrix}$ is contained in exactly $\lambda$ elements of $\mathcal{D}$.

- set case $q = 1$: combinatorial design
- $q$-analog case $q \geq 2$: subspace design

## Reformulation in characteristic functions

- Let $\boldsymbol{x}_T$ be characteristic function of pencil $\{K \in \begin{bmatrix} V \\ k \end{bmatrix} \mid T \subseteq K\}$.
- For $f, g : \begin{bmatrix} V \\ k \end{bmatrix} \to \mathbb{R}$ fix standard inner product $\langle f, g \rangle = \sum_{K \in \begin{bmatrix} V \\ k \end{bmatrix}} f(K)g(K)$.
- Note that $\#(\mathcal{F} \cap \mathcal{G}) = \langle \chi_{\mathcal{F}}, \chi_{\mathcal{G}} \rangle$ for $\mathcal{F}, \mathcal{G} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$.
- $\mathcal{D}$ is simple $t$-$(n, k, \lambda)_q$ design
  $$\iff \langle \boldsymbol{x}_T, \chi_{\mathcal{D}} \rangle = \lambda \text{ for all } T \in \begin{bmatrix} V \\ t \end{bmatrix}.$$
- $\rightsquigarrow$ generalization to real designs.

## Generalized definition: Real design

A function $f : \begin{bmatrix} V \\ k \end{bmatrix} \to \mathbb{R}$ is called a real $t$-$(n, k, \lambda)_q$ design,
if $\langle \boldsymbol{x}_T, f \rangle = \lambda$ for all $T \in \begin{bmatrix} V \\ t \end{bmatrix}$.

- $f$ null design or trade if $\lambda = 0$.
- $f$ signed design if $\operatorname{im}(f) \subseteq \mathbb{Z}$.
- $f$ design or possibly non-simple design if $\operatorname{im}(f) \subseteq \mathbb{N}$.
  (Idea: simple design, but with possibly repeated blocks)
- $f$ (characteristic function of) simple design
  $\iff \operatorname{im}(f) \subseteq \{0, 1\} \iff f$ Boolean.

## Further reformulation

- Observation:
  Functions $\boldsymbol{x}_T$ (interpreted as vectors)
  are the rows of incidence matrix $W^{(tk)}$.
- Therefore:
  $f$ real $t$-$(n, k, \lambda)_q$ design $\iff W^{(tk)} f = \lambda \mathbf{1}$.
- In particular:
  $f$ real $t$-$(n, k, 0)_q$ null design $\iff W^{(tk)} f = \mathbf{0}$

## Geometric property, basic version

- For $\lambda \in \mathbb{R}$ let $U_\lambda$ :=set of real $t$-$(n, k, \lambda)_q$ design.
- Just seen: $U_0 = \ker W^{(tk)}$.
- Set of functions with $\mathrm{A}_t$ was $V_t = \operatorname{rowsp} W^{(tk)}$.

$$\implies V_t = U_0^\perp$$

## What did we get?

- Established a connection to designs.
- Concept known as Delsarte's design orthogonality.
- Compared to prototype
  "constant intersection with all spreads":
  Want similar property for $\lambda \neq 0$!

# Geometric property, version II

- Fix $\lambda \in \mathbb{R}$.
- Scaled complete design $\frac{\lambda}{\lambda_{\max}} \cdot \mathbf{1}$ is real $t$-$(n, k, \lambda)_q$ design.
- As solution of linear equation system $W^{(tk)} f = \lambda \mathbf{1}$:

$$U_\lambda = \frac{\lambda}{\lambda_{\max}} \cdot \mathbf{1} + \underbrace{\ker W^{(tk)}}_{= U_0 = V_t^\perp}.$$

- $\implies$

$$U_\lambda = \left\{ \delta : \begin{bmatrix} V \\ k \end{bmatrix} \to \mathbb{R} \mid \langle f, \delta \rangle = \frac{\lambda}{\lambda_{\max}} \cdot \# f \text{ for all } f \in V_t \right\} \quad \text{and}$$

$$V_t = \left\{ f : \begin{bmatrix} V \\ k \end{bmatrix} \to \mathbb{R} \mid \langle f, \delta \rangle = \frac{\lambda}{\lambda_{\max}} \cdot \# f \text{ for all } \delta \in U_\lambda \right\} \text{ Vers. II}$$

(with $\# f = \sum_{K \in \left[\substack{V \\ k}\right]} f(K) = \langle f, \mathbf{1} \rangle$, motivated by $\# \mathcal{F} = \# \chi_{\mathcal{F}}$)

- Still room for improvement:
  - Not happy about “For all real ... designs”.
    $\rightsquigarrow$ enough to look at basis of $U_\lambda$.
  - Allow mixed values of $\lambda$.

## Example

- $q = 1$, $n = 6$, $k = 3$, $t = 2 \rightsquigarrow \lambda_{\max} = \begin{bmatrix} 6-2 \\ 3-2 \end{bmatrix} = 4$.
- Baby example: $\mathcal{F} = \{\{1, 2, 3\}, \{4, 5, 6\}\}$, seen: $\chi_{\mathcal{F}} \in V_2$.
- Geometric property $\implies$ For each 2-$(6, 3, 2)_1$ design:

$$\langle \chi_{\mathcal{F}}, \delta \rangle = \frac{\lambda}{\lambda_{\max}} \cdot \# \chi_{\mathcal{F}} = \frac{2}{4} \cdot 2 = 1.$$

- $\implies$ Each simple 2-$(6, 3, 2)_1$ design $\mathcal{D}$ contains exactly one of the blocks $\{1, 2, 3\}$ and $\{4, 5, 6\}$.
- $\rightsquigarrow \mathcal{D}$ is anti-complementary.
- Can also be shown using intersection numbers.

## Geometric property, toolbox version

- $U_*$ := set of all $t$-$(v,k,\lambda)_q$ designs with arbitrary value $\lambda \in \mathbb{R}$.
- By scaled complete designs: $U_* = U_0 + \langle \mathbf{1} \rangle_{\mathbb{R}}$.
- Lemma (Toolbox version of geometric property).
  Let $\Delta \subseteq U_*$. Then

$$V_t = \left\{ f : \begin{bmatrix} V \\ k \end{bmatrix} \to \mathbb{R} \mid \langle f, \delta \rangle = \frac{\lambda_\delta}{\lambda_{\max}} \cdot \# f \text{ for all } \delta \in \Delta \right\}$$
$$\iff \langle \Delta \cup \{\mathbf{1}\} \rangle_{\mathbb{R}} = U_*$$

  Proof. Dimension argument. Use that $W^{(tk)}$ has full rank (Set case: Gottlieb 1966, $q$-analog case: Kantor 1972)
- Question: Suitable sets $\Delta$?

## Lemma

*Let* $\Delta$ *be*

(a) *the set of all signed* $t$*-*$(n, k, 0)_q$ *null designs or*

(b) *the set of all possibly non-simple* $t$*-*$(n, k, \lambda)_q$ *designs*

*Then* $U_* = \langle \Delta \cup \{\mathbf{1}\} \rangle_{\mathbb{R}}$*.*

## Proof.

Part (a).

- entries of $W^{(tk)}$ are in $\mathbb{Q}$.
- $\implies U_0 = \ker W^{(tk)}$ has rational basis.
- Multiply by common denominators $\rightsquigarrow$ integral basis $B$.
- $\implies B \subseteq \Delta$ and $\langle B \cup \{\mathbf{1}\} \rangle_{\mathbb{R}} = U_*$.

Part (b).

- Start with $B$.
- Add suitable integral multiples of $\mathbf{1}$
  $\rightsquigarrow$ non-negative integral set $B'$.
- $\implies B' \subseteq \Delta$ and $\langle B' \cup \{\mathbf{1}\} \rangle_{\mathbb{R}} = U_*$.

We arrive at:

### Theorem

*Let $f : \begin{bmatrix} V \\ k \end{bmatrix} \to \mathbb{R}$. The following are equivalent.*

(i) *Algebraic property: $f \in V_t$.*

*Geometric properties:*

(ii) *There is a constant $c \in \mathbb{R}$ such that $\langle f, \delta \rangle = \lambda_\delta c$ for all real $t$-$(n, k, *)_q$ designs $\delta$ with $\lambda \in \mathbb{R}$.*

(iii) $\langle f, \delta \rangle = 0$ *for all signed $t$-$(n, k, 0)_q$ null designs $\delta : \begin{bmatrix} V \\ k \end{bmatrix} \to \mathbb{Z}$.*

(iv) *There is a constant $c \in \mathbb{R}$ such that $\langle f, \delta \rangle = \lambda_\delta c$ for all possibly non-simple $t$-$(n, k, *)_q$ designs $\delta : \begin{bmatrix} V \\ k \end{bmatrix} \to \mathbb{N}$.*

*The constant in properties (ii) and (iv) necessarily equals $c = \frac{1}{\lambda_{\max}} \cdot \# f$.*

## Geometric property: Discussion

- Tempting: Is the following a suitable geometric property?

  "There is a constant $c \in \mathbb{R}$ such that $\langle f, \delta \rangle = \lambda c$ for all simple $t$-$(n, k, *)_q$ designs"

- By toolbox version: If and only if $\langle \{\text{simple } t\text{-}(n, k, *)_q \text{ designs}\} \rangle_{\mathbb{R}} = U_*$ (richness cond)
- Unfortunately: Not always true.

  Counterexample. $q = 1$, $n = 10$, $k = 5$, $t = 4$.
  By integraliy conditions: All simple 4-$(10, 5, *)_1$ are trivial.
  $\implies \dim\langle \{\text{simple } 4\text{-}(10, 5, *)_1 \text{ designs}\} \rangle_{\mathbb{R}} = 1$, too small!
- Research problem. (probably hard!)

  Classify the parameters $(q, n, k, t)$ where the richness condition holds.

## The Degree

- Fix $k \in \{0, \ldots, n\}$ and $f : \begin{bmatrix} V \\ k \end{bmatrix} \to \mathbb{R}$.
- Lemma.

$$\{\mathbf{1}\} = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_k = V.$$

  Proof. $W^{(ij)} W^{(jk)} \sim W^{(ik)}$ for $0 \leq i \leq j \leq k$.
- Definition.
  Degree $\deg(f) :=$ smallest $t$ such that $f \in V_t$.

## Example

- Functions $f$ of degree 0
  are the scalar functions $f = \lambda\mathbf{1}$ with $\lambda \in \mathbb{R}$.
- Baby example $\mathcal{F} = \{\{1, 2, 3\}, \{4, 5, 6\}\}$.
  In $V = \{1, 2, 3, 4, 5, 6\}$ we have $\deg(\mathcal{F}) := \deg(\chi_{\mathcal{F}}) = 2$.
  - Seen: $\chi_{\mathcal{F}} \in V_2$.
  - Exercise: $\chi_{\mathcal{F}} \notin V_1$.
- In $V = \{1, 2, 3, 4, 5, 6, 7\}$ we have $\deg(\mathcal{F}) = 3$.
  $\Longrightarrow$ Ambient space $V$ matters!

## The Degree (cont.)

- Remember. Rows of $W^{(tk)}$ are the $t$-pencils $\boldsymbol{x}_T$.
- $\rightsquigarrow$ Alternative characterization of degree.

  $\deg(f)$ is smallest $t$
  such that $f$ is a linear combination of $t$-pencils $\boldsymbol{x}_T$.

  The (unique) coefficients are called $t$-weights $\mathrm{wt}_f(T)$ of $f$:

$$f = \sum_{T \in \binom{V}{t}} \mathrm{wt}_f(T) \boldsymbol{x}_T$$

### Lemma

(a) $\deg(\lambda f) \leq \deg(f)$ *with equality iff* $\lambda \neq 0$.

(b) $\deg(f+g) \leq \max(\deg(f), \deg(g))$.

(c) $\deg(fg) \leq \deg(f) + \deg(g)$.

### Proof.

Parts (a), (b): easy. Part (c): Use weights & $\deg \boldsymbol{x}_T \leq \mathrm{rk}\, T$. □

## Dualization

- Fix anti-isomorphism $\perp$ of the lattice $\mathcal{L}(V)$.
  - Set case: Set complement.
  - $q$-analog case: Perp wrt non-degenerate bilinear form.
- Induces dual map of $f : \begin{bmatrix} V \\ k \end{bmatrix} \to \mathbb{R}$:

$$f^{\perp} : \begin{bmatrix} V \\ n-k \end{bmatrix} \to \mathbb{R}, \quad U \mapsto f(U^{\perp})$$

- Effect of dualization on the degree?

### Theorem

(a) $\deg f^\perp = \deg f$.

(b) *For $i \in \{0, \ldots, \deg f\}$, the i-weight distribution of $f^\perp$ is*

$$\mathrm{wt}^{(i)}_{f^\perp}(J) = \sum_{I \in \begin{bmatrix} V \\ i \end{bmatrix}} \gamma(n-k, i, \mathrm{rk}(I^\perp \cap J))\ \mathrm{wt}^{(i)}_f(I)$$

*where*

$$\gamma(k, i, z) := \begin{cases} \delta_{z,k} & \text{if } i = k, \\ (-1)^{i-z} \dfrac{1}{q^{(k-i)(i-z)+\binom{i-z}{2}}} \dfrac{\begin{bmatrix} k-i \\ 1 \end{bmatrix}}{\begin{bmatrix} k-z \\ 1 \end{bmatrix}} \dfrac{1}{\begin{bmatrix} k \\ z \end{bmatrix}} & \text{otherw.} \end{cases}$$

### Proof.

- Enough to look at pencils $f = \boldsymbol{x}_J$.
- Set up linear equation system for the weights of $f^\perp$, assuming that $\mathrm{wt}(I)$ only depends on $\mathrm{rk}(I \cap J)$.
- Equation system matrix is triangular with non-zero diagonal $\implies$ invertible $\implies$ Part (a).
- Apply negation formula & $q$-Vandermonde formula for Gaussian coefficients $\rightsquigarrow$ compute solution $\rightsquigarrow$ Part (b).

## Change of ambient space

Two elementary ways to shrink the ambient space $V$.

- $V \to H$ $\quad$ ($H \in \begin{bmatrix} V \\ n-1 \end{bmatrix}$ hyperplane)
- $V \to V/P$ $\quad$ ($P \in \begin{bmatrix} V \\ 1 \end{bmatrix}$ point)

Implication on the degree?

We start with $V \to V/P$.

## Theorem

*Let* $1 \le k \le n$ *and* $P \in \begin{bmatrix} V \\ 1 \end{bmatrix}$. *Then*

$$\Phi : \mathbb{R}^{\begin{bmatrix} V/P \\ k-1 \end{bmatrix}} \to \mathbb{R}^{\begin{bmatrix} V \\ k \end{bmatrix}}, \qquad \Phi(f) : K \mapsto \begin{cases} f(K/P) & \text{if } P \subseteq K, \\ 0 & \text{if } P \not\subseteq K \end{cases}$$

*is an injective* $\mathbb{R}$*-linear map with*

$$\operatorname{im}(\Phi) = \{g \in \mathbb{R}^{\begin{bmatrix} V \\ k \end{bmatrix}} \mid \operatorname{supp} g \subseteq \begin{bmatrix} V \\ k \end{bmatrix}|_P\} \quad \text{and}$$

$$\deg_V \Phi(f) = \begin{cases} 0 & \text{if } f = 0, \\ \min(\overbrace{\deg_{V/P}(f) + 1}^{\text{main case}}, n-k) & \text{otherwise.} \end{cases}$$

## Proof.

- Straightforward, except "$\deg_V \Phi(f) \ge \deg_{V/P}(f) + 1$".
- Lemma. In main case
  For all $g \in \operatorname{im} \Phi$: $\quad P \not\le T \implies \operatorname{wt}_g(T) = 0.$
  Proof. Incidence matrices of certain attenuated geometries are of full rank. (Guo, Li, Wang, 2014.)

## Theorem

*Let* $1 \le n-k \le n$ *and* $H \in \begin{bmatrix} V \\ n-1 \end{bmatrix}$. *Then*

$$\Psi : \mathbb{R}^{\left[{H \atop k}\right]} \to \mathbb{R}^{\left[{V \atop k}\right]}, \qquad \Psi(f) : K \mapsto \begin{cases} f(K) & \textit{if } K \subseteq H, \\ 0 & \textit{if } K \not\subseteq H \end{cases}$$

*is an injective* $\mathbb{R}$*-linear map with*

$$\operatorname{im}(\Psi) = \{g \in \mathbb{R}^{\left[{V \atop k}\right]} \mid \operatorname{supp} g \subseteq \begin{bmatrix} H \\ k \end{bmatrix}\} \quad \textit{and}$$

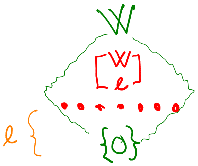
$$\deg_V \Psi(f) = \begin{cases} 0 & \textit{if } f = 0, \\ \min(\deg_H(f) + 1, \ k) & \textit{otherwise.} \end{cases}$$

## Proof.

Follows from the previous theorem by dualization. □

## Example (Basic sets)

- Start with "complete set" $\begin{bmatrix} W \\ \ell \end{bmatrix}$ of degree 0.



- $i$-fold application of $\Phi$ and $j$-fold application of $\Psi$

$$\rightsquigarrow \text{basic set } \mathcal{F}(I, J) = \{K \in \begin{bmatrix} V \\ k \end{bmatrix} \mid I \subseteq K \subseteq J\}.$$

- By theorems: $\deg \mathcal{F}(I, J) = i + j$.

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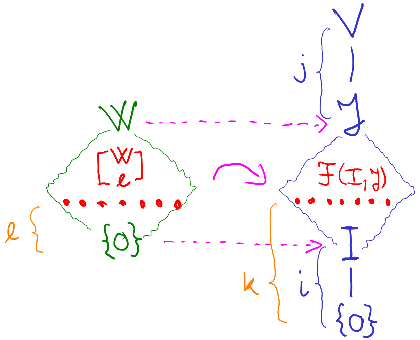

- $i$-fold application of $\Phi$ and $j$-fold application of $\Psi$

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- By theorems: $\deg \mathcal{F}(I, J) = i + j$.

## Example (Basic sets (cont.))

- Basic sets $\mathcal{F}(I, J)$ include pencils ($j = 0$) and dual pencils ($i = 0$).

  In particular $\deg \boldsymbol{x}_I = \operatorname{rk} I$.
- Geometric property of $\mathcal{F}(I, J)$

  $$\longleftrightarrow$$

  design property of $i$-fold derived and $j$-fold residual design.

## Sets and Boolean functions

- Of particular interest: Sets $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ of low degree.
- Via characteristic functions:
  Sets correspond to Boolean functions $\begin{bmatrix} V \\ k \end{bmatrix} \to \{0, 1\}$.

## Boolean degree 1 functions

- Set case:

  Filmus, Ihringer 2019:
  Only basic functions.
  $\implies$ only pencils and dual pencils (since $t = 1$).
- $q$-analog case:

  Boolean degree 1 function $=$ Cameron-Liebler set of $(k-1)$-spaces in $\mathrm{PG}(n-1, q)$.

  Non-basic examples do exist.

  Classification: Hard research problem.

# Computer classification

**Goal.**

For $q = 1$ and small $n, k$, classify all sets $\mathcal{F}$ of degree $t = 2$.

**Strategy.**

- Use "basic" geometric property:

$$\deg \chi_{\mathcal{F}} \leq t \iff \chi_{\mathcal{F}} \in \ker W^{(tk)}.$$

  $\rightsquigarrow$ Want to find all $\{0, 1\}$-vectors in $\ker W^{(tk)}$.
- Find integral basis of $\ker W^{(tk)}$.
  - either: computationally
  - or: Use literature like
    Khosrovshahi, Ajoodani-Namini (1990):
    *A new basis for trades*
- $\rightsquigarrow$ system of linear Diophantine equations.
- Solve using SOLVEDIOPHANT (A. Wassermann)
- Filter out isomorphic copies.
  (action of symmetric group $\mathfrak{S}_n$)

# Results

| $n$ | $k$ | size distribution | $\Sigma$ |
|---|---|---|---|
| 6 | 3 | $2\ 4^3 6^5 8^8 10^{10} 12^8 14^5 16^3 18$ | 44 |
| 7 | 3 | $5^2 10^6 15^{12} 20^{12} 25^6 30^2$ | 40 |
| 8 | 3 | $6\ 8\ 11\ 12\ 14\ 15^2 16\ 17\ 18\ 20^2 21^2 22\ 23\ 24^3 25\ 26^4 27\ 28^2 \ldots$ | 52 |
| 9 | 3 | $7\ 14^2 21^5 28^5 35^5 42^{11} 49^5 56^5 63^5 70^2 77$ | 47 |
| 10 | 3 | $8\ 16\ 20\ 24^2 28^3 32^2 36^4 40^2 44^2 48^2 52^2 56^5 60^5 64^5 68^2 72^2 \ldots$ | 59 |
| 8 | 4 | $10\ 15^2 20^3 30^6 35^4 40^6 50^3 55^2 60$ | 28 |
| 9 | 4 | $21^2 35^3 56^5 70^5 91^3 105^2$ | 20 |
| 10 | 4 | $28\ 42\ 56^2 70\ 84^2 98^3 112^3 126^2 140\ 154^2 168\ 182$ | 20 |
| 11 | 4 | $36\ 78\ 84^2 120^2 126\ 162^3 168^3 204\ 210^2 246^2 252\ 294$ | 20 |
| 12 | 4 | $45\ 120^2 135\ 165^2 210\ 240^3 255^3 285\ 330^2 360\ 375^2 450$ | 20 |

blue $=$ sizes of basic sets

Goal. Explain divisibility pattern of the sizes!

## Theorem (Divisibility theorem)

*Let $f : \begin{bmatrix} V \\ k \end{bmatrix} \to \mathbb{Z}$ be a function of degree $t$. Then*

$$\underbrace{\gcd\left(\begin{bmatrix} n-0 \\ k-0 \end{bmatrix}, \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \ldots, \begin{bmatrix} n-t \\ k-t \end{bmatrix}\right)}_{=:a} \mid \#f.$$

## Proof.

- Algebraic property $\Rightarrow \exists \boldsymbol{x} : \begin{bmatrix} V \\ t \end{bmatrix} \to \mathbb{R}$ with $\boldsymbol{x}^\top W^{(tk)} = f^\top$. (1)
- Complete design: $W^{(tk)} \cdot \mathbf{1} = \lambda_{\max} \cdot \mathbf{1}$ (2)
- Design theory:
  parameters $t$-$(n, k, \lambda_{\min})_q$ with $\lambda_{\min} = \frac{\lambda_{\max}}{a}$ are admissible.
- $\Rightarrow \exists$ signed $t$-$(n, k, \lambda_{\min})_q$ design $\delta \Rightarrow W^{(tk)}\delta = \lambda_{\min} \cdot \mathbf{1}$ (3)
  - Set case: Wilson, "The necessary conditions for $t$-designs are sufficient for something" (1973).
  - $q$-analog case: Ray-Chaudhuri, Singhi (1989).
- Left multiplication of (2) and (3) by $\boldsymbol{x}^\top$, using (1)
  $\Longrightarrow \#f = \lambda_{\max} \cdot \#\boldsymbol{x}$ and $\langle f, \delta \rangle = \lambda_{\min} \cdot \#\boldsymbol{x}$
- $\Longrightarrow \#f = a \cdot \underbrace{\langle f, \delta \rangle}_{\in \mathbb{Z}} \in \mathbb{Z}.$

# Compare with the results

$q = 1, t = 2 \implies a = \gcd(\binom{n}{k}, \binom{n-1}{k-1}, \binom{n-2}{k-2})$.

| $n$ | $k$ | size distribution | $a$ |
|---|---|---|---|
| 6 | 3 | $2\ 4^3 6^5 8^8 10^{10} 12^8 14^5 16^3 18$ | 2 |
| 7 | 3 | $5^2 10^6 15^{12} 20^{12} 25^6 30^2$ | 5 |
| 8 | 3 | $6\ 8\ 11\ 12\ 14\ 15^2 16\ 17\ 18\ 20^2 21^2 22\ 23\ 24^3 25\ 26^4 27\ 28^2 \ldots$ | 1 |
| 9 | 3 | $7\ 14^2 21^5 28^5 35^5 42^{11} 49^5 56^5 63^5 70^2 77$ | 7 |
| 10 | 3 | $8\ 16\ 20\ 24^2 28^3 32^2 36^4 40^2 44^2 48^2 52^2 56^5 60^5 64^5 68^2 72^2 \ldots$ | 4 |
| 8 | 4 | $10\ 15^2 20^3 30^6 35^4 40^6 50^3 55^2 60$ | 5 |
| 9 | 4 | $21^2 35^3 56^5 70^5 91^3 105^2$ | 7 |
| 10 | 4 | $28\ 42\ 56^2 70\ 84^2 98^3 112^3 126^2 140\ 154^2 168\ 182$ | 14 |
| 11 | 4 | $36\ 78\ 84^2 120^2 126\ 162^3 168^3 204\ 210^2 246^2 252\ 294$ | 6 |
| 12 | 4 | $45\ 120^2 135\ 165^2 210\ 240^3 255^3 285\ 330^2 360\ 375^2 450$ | 15 |

Perfect fit!

## Parameter of Cameron-Liebler sets of $k$-spaces

- Consider $q$-analog case $q \geq 2$.
- For sets $\mathcal{F}$ of degree $t = 1$ define parameter $x := \#\mathcal{F} / \left[{n-1 \atop k-1}\right] \in \mathbb{Q}$
- Corollary of divisibility theorem.

$$\frac{q^k - 1}{q^{\gcd(n,k)} - 1} \cdot x \in \mathbb{Z},$$

restricting denominator of fraction $x$ in canceled form.

### Example

- $k \mid n \implies x \in \mathbb{Z}$.
  Already known: Blokhuis, De Boeck, D'haeseleer (2019).
- $n$ and $k$ coprime $\implies (1 + q + \ldots + q^{k-1}) \cdot x \in \mathbb{Z}$.
- $k = 4$, $n \equiv 2 \pmod 4 \implies (1 + q^2) \cdot x \in \mathbb{Z}$.

## The paired construction

- Construction for the set case $q = 1$ only.
- Idea. Disjoint union of two “opposite” basic sets.
- Let $I, J \subseteq V$ be disjoint, not both empty. Define

$$\mathcal{P}(I, J) := \mathcal{F}(I, J^{\complement}) \uplus \mathcal{F}(J, I^{\complement})$$

- Clear:

$$\deg \mathcal{P}(I, J) \leq \min(\#I + \#J, k) \quad \text{(trivial bound).}$$

- Will see: There are cases with a strict “$<$”!

## Example

- $V = \{1, \dots, 6\}$, $k = 3$, $I = \emptyset$, $J = \{4, 5, 6\}$,

$$\begin{aligned} & \mathcal{P}(\emptyset, \{4,5,6\}) \\ = \quad & \mathcal{F}(\emptyset, \{1,2,3\}) \uplus \mathcal{F}(\{4,5,6\}, \{1,2,3,4,5,6\}) \\ = \quad & \{\{1,2,3\}, \{4,5,6\}\} \quad \text{Baby example!} \end{aligned}$$

- Already seen:
  $\deg \mathcal{P}(\emptyset, \{1,2,3\}) = 2$, beating the trivial bound "$\leq 3$"!

## Example

- $V = \{1, \dots, 7\}$, $k = 3$, $I = \{1\}$, $J = \{6, 7\}$.

$$\begin{aligned} & \mathcal{P}(\{1\}, \{6,7\}) \\ = \quad & \mathcal{F}(\{1\}, \{1,2,3,4,5\}) \uplus \mathcal{F}(\{6,7\}, \{2,3,4,5,6,7\}) \\ = \quad & \{\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \\ & \{2,6,7\}, \{3,6,7\}, \{4,6,7\}, \{5,6,7\}\} \end{aligned}$$

- $\deg(\mathcal{P}(\{1\}, \{6,7\}) = 2$, beating the trivial bound.

## Theorem

*Let* $q = 1$*,* $I, J \subseteq V$ *disjoint,* $i = \#I$*,* $j = \#J$*,* $k \leq \frac{n}{2}$*,* $i \leq k \leq n - i$*,* $j \leq k \leq n - j$*.*

*In the cases*

(a) $i + j \leq k$ *and* $i + j$ *odd;*

(b) $i + j \geq k$ *and* $k$ *odd and* $n = 2k$

*we have*

$$\deg \mathcal{P}(I, J) \leq \min(i + j, k) - 1.$$

## Proof (Idea).

Part (a): Write $\chi_{\mathcal{P}(I,J)}$ as an integer linear combination of basic functions of degree $i + j - 1$.

Part (b):

- Use $\mathcal{P}(X, Y) = \mathcal{P}(X \uplus \{x\}, Y) \uplus \mathcal{P}(X, Y \uplus \{x\})$ (where $X, Y, \{x\}$ are pairwise disjoint)
- Moving elements from $J$ to $I$ $\rightsquigarrow$ $\deg \mathcal{P}(I, J) \leq \deg \mathcal{P}(K, J')$
- $\mathcal{P}(K, J') = \mathcal{P}(K, \emptyset) \implies$ Back in Case (a).

## Theorem

*Let $q = 1$, $I, J \subseteq V$ disjoint, $i = \#I$, $j = \#J$, $k \leq \frac{n}{2}$, $i \leq k \leq n - i$, $j \leq k \leq n - j$.*
*In the cases*

(a) $i + j \leq k$ *and* $i + j$ *odd;*

(b) $i + j \geq k$ *and* $k$ *odd and* $n = 2k$

*we have*

$$\deg \mathcal{P}(I, J) \leq \min(i + j, k) - 1.$$

## Conjecture

Statement of Theorem is best possible.

- In fact always equality

$$\deg \mathcal{P}(I, J) = \min(i + j, k) - 1.$$

- In all cases not covered by (a) and (b), the trivial bound is sharp:

$$\deg \mathcal{P}(I, J) = \min(i + j, k).$$

## Small sets of degree $t$

- **Natural question.**
  Smallest size $m_q(n, k, t)$ of a non-empty set of degree $\leq t$?
- From $\deg \boldsymbol{x}_T = t$ we get

$$m_q(n, k, t) \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}. \qquad (*)$$

- Bound $(*)$ is always sharp for $t = 1$.
  - Set case: Filmus, Ihringer (2019).
  - $q$-analog case: Blokhuis, De Boeck, D'haeseleer (2019).
- For $q = 1$, $n = 2k$, $t \geq 2$ even, $i = 0$ and $j = t + 1$, the paired construction beats bound $(*)$!

### Corollary

*Let $t \in \{0, \dots, k-1\}$ be even. Then*

$$m_1(2k, k, t) \leq 2 \cdot \binom{2k-t-1}{k}.$$

## Open problems

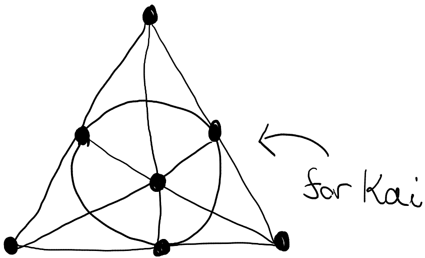
- Many!
- For fixed $(q, n, k, t)$, characterize the sizes of degree $t$ sets.
  - Smallest,
  - second smallest,
  - gaps,
  - etc.
- Further investigate and exploit relationship

  $$\text{degree } t \text{ functions} \longleftrightarrow t\text{-designs.}$$

  Which results can be translated?
- Maybe most important:

  Better name for the studied objects.
  - "dual designs"? $\longrightarrow$ ambiguous.
  - Something involving "Cameron-Liebler"?
  - other ideas?

# Thank you!




Slides will be uploaded at

`https://mathe2.uni-bayreuth.de/michaelk/`