On the maximum number of planes in PG($\mathbb{F}_2^6$) whose pairwise interection is at most a point

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$\mathbb{F}_q$11
July 23, 2013
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joint work with Sascha Kurz

Constant dimension network codes

- prime power q
- v-dim. $\mathbb{F}_q$-vector space V
- $\begin{bmatrix} V \\ k \end{bmatrix}$: set of all k-dim. subspaces of V.
- $C \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$: constant dimension code (cdc)
- Distance between $B, B' \in C$:

$$\begin{aligned} d(B, B') &= \text{distance in subspace lattice of } V \\ &= 2(k - \dim B \cap B') \end{aligned}$$

- Minimum distance $d(C) = \min_{B \neq B' \in C} d(B, B')$
- C is called a $(v, d(C); k)_q$ constant dimension code.

Problem in network coding

Given $q, v, k, d(C)$, find maximum possible size

$$\#C = A_q(v, d(C); k).$$

Smallest open case

$$A_2(6,4;3) = ?$$

Geometrically

Find the maximum number of planes in $\mathrm{PG}(\mathbb{F}_2^6) = \mathrm{PG}(5,2)$ such that the pairwise intersection is at most a point.

Known bounds

$$77 \leq A_2(6,4;3) \leq 81$$

Goal of this talk

Close the gap!

Closer look at the upper bound

- Let $V = \mathbb{F}_2^6$ and C be a $(6,4;3)_2$ cdc.
- Consider point $P \in \begin{bmatrix} V \\ 1 \end{bmatrix}$.
- Define weight $w(P) = \#\{B \in C \mid P < B\}$.
- Shortened code in P

$$\{B + P \mid B \in C, P < B\} \quad \subseteq \quad \begin{bmatrix} V/P \\ 2 \end{bmatrix}$$

  is $(5,4;2)_2$ cdc of size $w(P)$.
  Geometrically: Set of disjoint lines in $\mathrm{PG}(\mathbb{F}_2^5)$ (partial spread)
- Beutelspacher 1975: $A_2(5,4;2) = 9$

$$\implies \quad w(P) \leq 9$$

- Double count flags $(P, B) \in \begin{bmatrix} V \\ 1 \end{bmatrix} \times C$ with $P < B$:

$$7 \cdot \#C \leq 63 \cdot 9 \quad \implies \quad \#C \leq 81$$

Definition

A 9-configuration is a $(6,4;3)_2$ cdc C' with

- $\#C' = 9$ and
- $\bigcap_{B \in C'} \in \begin{bmatrix} V \\ 1 \end{bmatrix}$.
  (Blocks of C pass through a common point.)

Lemma

Let C be a $(6,4;3)_2$ cdc with $\#C \geq 73$.
Then there is a point P with $w(P) = 9$.
Equivalently: Then C contains a 9-configuration.

Proof.

- Assume not.
- Double count the flags $(P, B) \in \begin{bmatrix} V \\ 1 \end{bmatrix} \times C$ with $P < B$:

$$7 \cdot \#C \leq 63 \cdot 8 \quad \Longrightarrow \quad \#C \leq 72$$

  Contradiction.

First attempt

- Leonard Soicher 2000: Classification of complete spreads in $\mathrm{PG}(\mathbb{F}_2^5)$
  $\rightsquigarrow$ 4 isomorphism types of 9-configurations.
- Idea: Computationally check the four 9-configurations for extendibility to a $(6, 4; 3)_2$ cdc of size ≥ 77.
- Problem: Search space way too large.
- Conclusion: We need another intermediate step!

Definition

A 17-configuration is a $(6,4;3)_2$ cdc C' with

- $\#C' = 17$
- containing two 9-configurations.

Lemma

Let C be a $(6,4;3)_2$ cdc with $\#C \geq 74$.
Then there is a block $B \in C$ containing two points P, P' with $w(P) = w(P') = 9$.
Equivalently: Then C contains a 17-configuration.

Proof.

- By first Lemma: There is a point P with $w(P) = 9$.
- The 9 blocks through P cover $9 \cdot 6 = 54$ points $P' \neq P$.
- Assume the Lemma is wrong. Then $w(P') \leq 8$ for all P'.
- Double counting:

$$7 \cdot \#C \leq 54 \cdot 8 + (63 - 54) \cdot 9 \quad \Longrightarrow \quad \#C \leq 513/7 \approx 73.29.$$

Computer classification

- For each of the four 9-configurations:
  Compute all extensions to 17-configurations.
- Use canonizer program of Thomas Feulner
  to filter out isomorphic copies
  $\rightsquigarrow$ 12770 Isomorphism types of 17-configurations.
- For each 17-configuration:
  Compute all extensions to a $(6, 4; 3)_2$ cdc of size ≥ 77.
  $\approx$ 10 minutes per case.

Result:

Theorem

$$A_2(6, 4; 3) = 77$$

Filtering out isomorphic copies $\rightsquigarrow$

Theorem

There are exactly 5 *isomorphism types of* $(6, 4; 3)_2$ *cdc's of size* 77*:*

- *self-dual,* $\# \operatorname{Aut} = 168$*,*
- *self-dual,* $\# \operatorname{Aut} = 48$*,*
- *self-dual,* $\# \operatorname{Aut} = 2$*,*
- *dual pair with* $\# \operatorname{Aut} = 2$*.*

Open problems

- Give computer-free constructions
  (talk of Thomas Honold)
- Generalize to a family of large cdc's
- Understand upper bound ≤ 77 without a computer.
  Possible approach:
  Prove that there is a "light" plane E of weight ≤ 41:

$$\sum_{P \in \left[{E \atop 1}\right]} w(P) \leq 41.$$

- Attack $(7, 4; 3)_2$.
  Is there a 2-analog of the Fano plane?