Number of Graphical Partitions

Axel Kohnert

Bayreuth University
kohnert@uni-bayreuth.de
www.mathe2.uni-bayreuth.de

Partition

A partition is a weakly decreasing sequence of non-negative integers, where allmost all numbers are zero.

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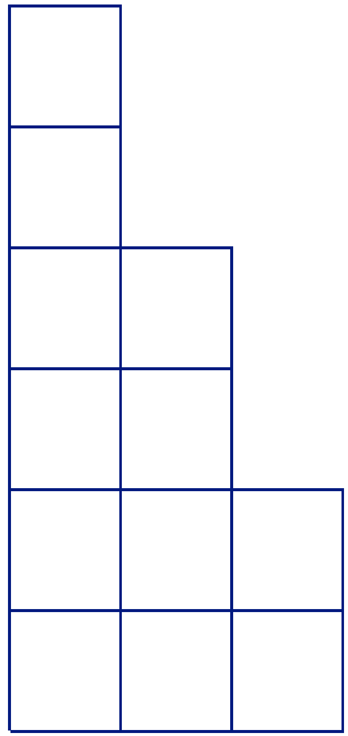
The length of a partition is the number of nonzero parts.

$$l(\lambda) = 6$$

Ferrers Diagram

Partitions are visualized by left adjusted boxes in the first quadrant.



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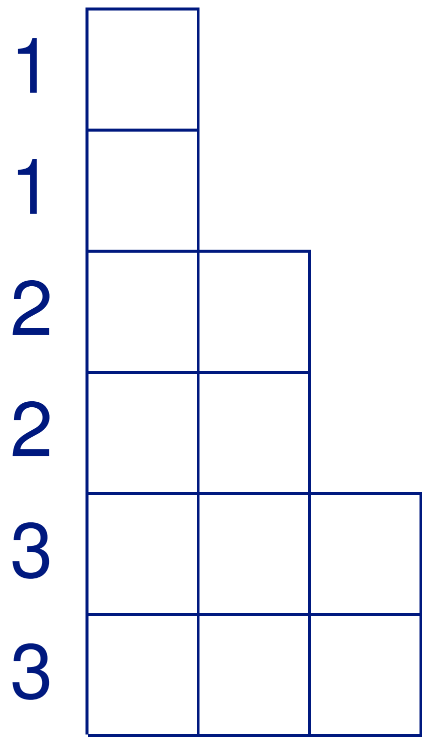
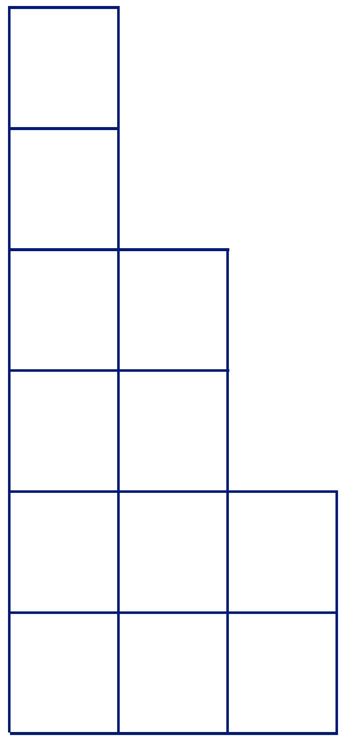


Conjugate Partition

The conjugate partition λ' is the sequence of numbers of boxes in the columns.



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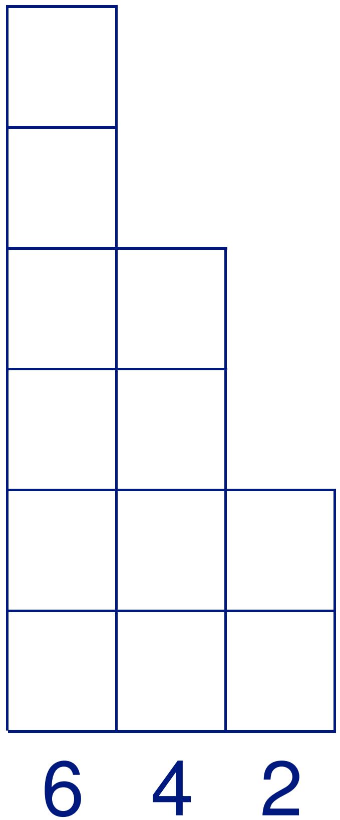

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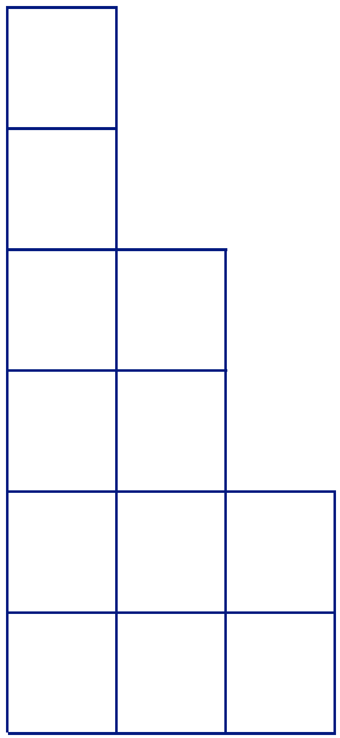

6 4 2 $=(3, 3, 2, 2, 1, 1)'$

Graphical Partitions

A partition λ is called graphical, if there is a simple (undirected, no loops, no multi-edges) graph whose vertex degree sequence equals λ.

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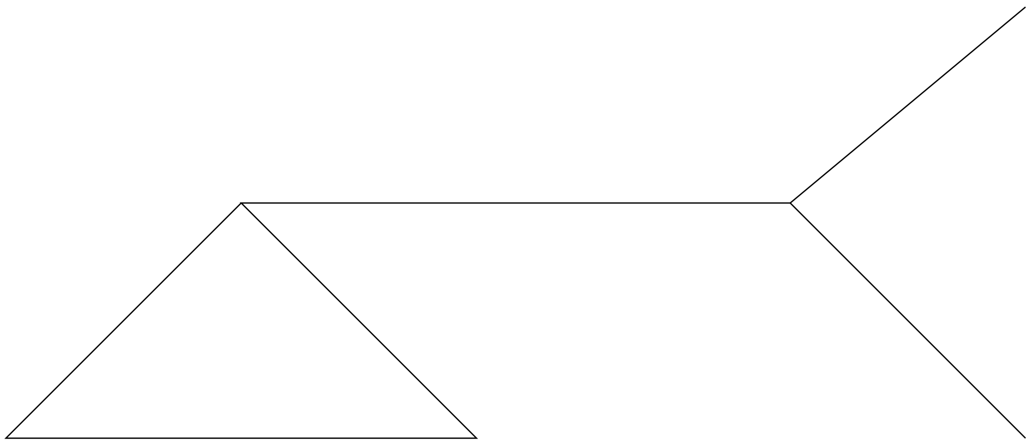
- graphical partitions only exist for even weight
- not all even weight partitions are graphical

Example



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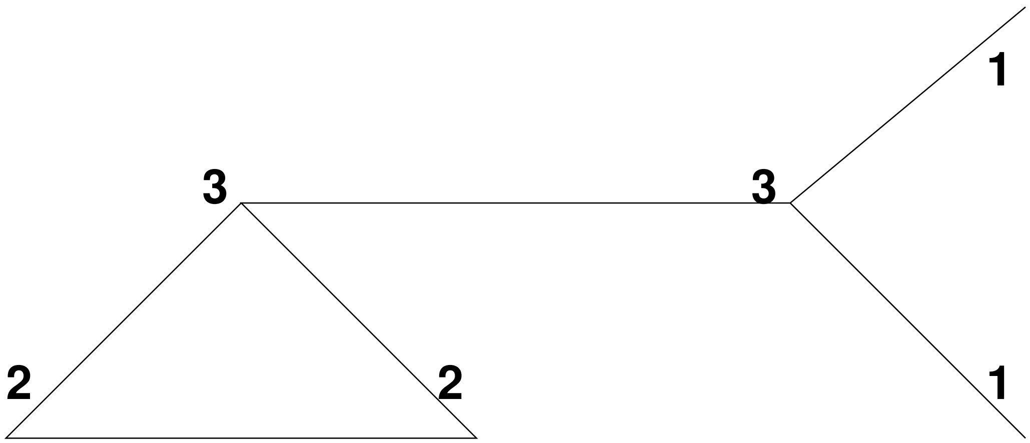

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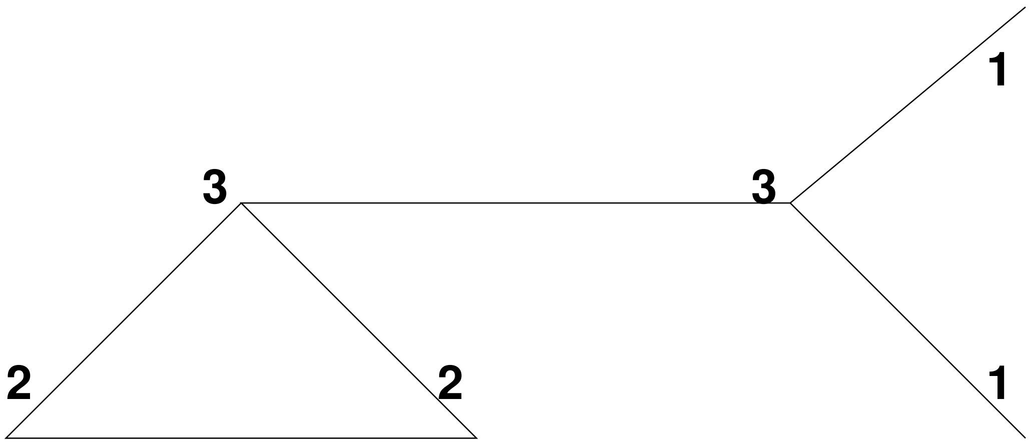

$$\downarrow$$

$$3, 3, 2, 2, 1, 1$$

Open Questions

The number $g(n)$ of graphical partitions of weight n is smaller than $p(n)$, the number of partitions.

$$lim_{n\to\infty}\frac{g(n)}{p(n)} =?$$

Known: $lim < 0.25$

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For the number of partitions there is a generating function, which allows the fast computation of $p(n)$.
For $g(n)$: missing

Criteria for being Graphical

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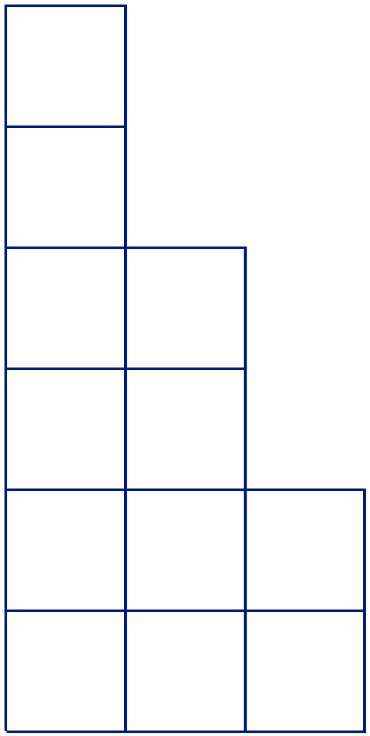
- Hässelbarth: for all $k \leq$ Durfee size

$$\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} (\lambda_i' - 1)$$

Durfee



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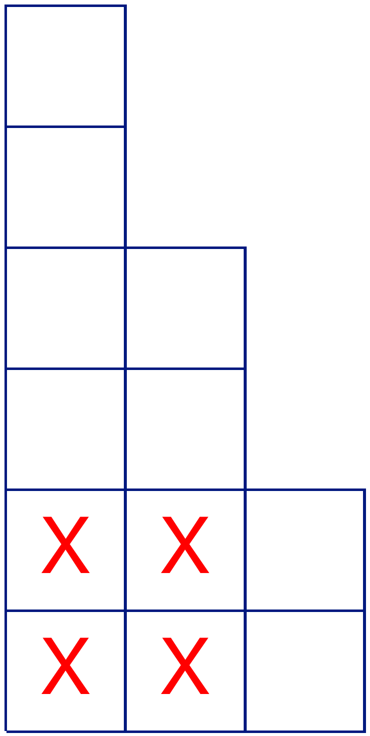
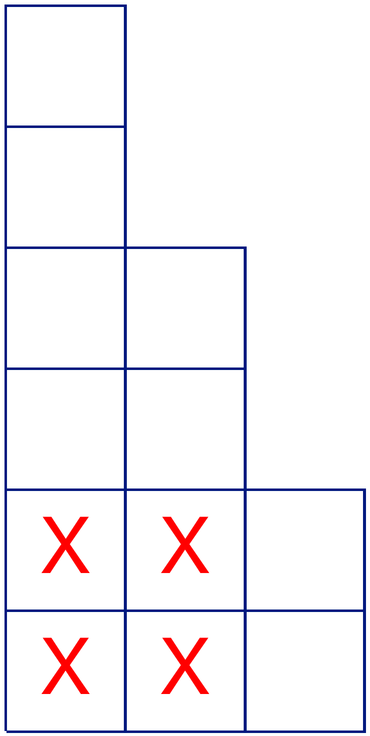


Durfee square = $(2, 2)$

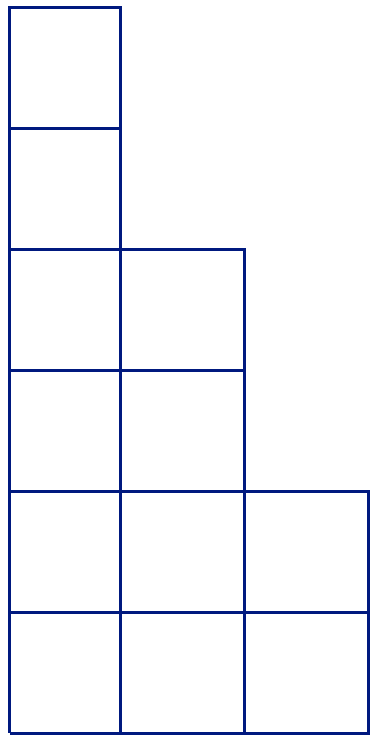
Durfee



Durfee square = $(2, 2)$

Durfee size = 2

Durfee Decomposition



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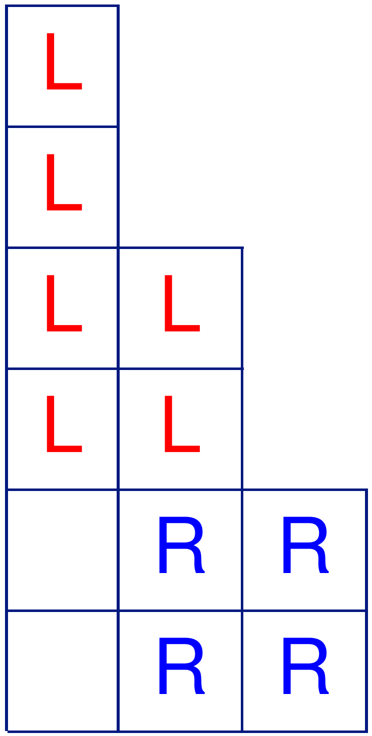

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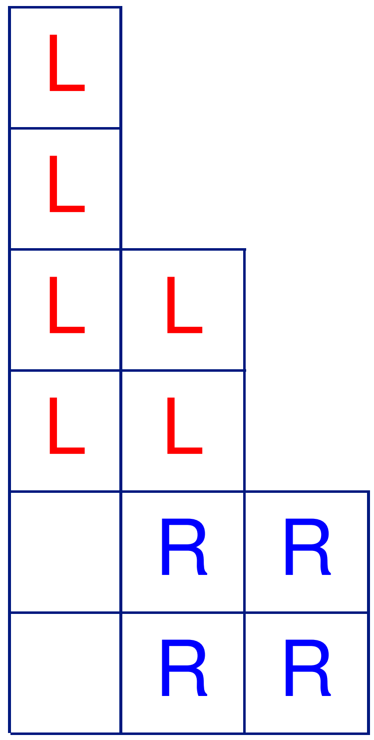

L = $(4, 2)$

R = $(2, 2)$

Dominance Order

The 'natural' order on partitions.

Let μ, ν be two partitions

$$\mu \trianglerighteq \nu :\Leftrightarrow \forall k \geq 1 : \sum_{i=1}^{k} \mu_i \geq \sum_{i=1}^{k} \nu_i$$

New Criterion

Theorem

A partition λ of even weight is graphical

$$\Longleftrightarrow$$

$$L(\lambda) \trianglerighteq R(\lambda)$$

Recursion Formula (1)

$$G(n) \quad := \quad \text{set of graphical partitions of weight } n$$

$$G_i(n) \quad := \quad \text{set of graphical partitions of weight } n \text{ and Durfee size } i$$

$$G(n) = G_1(n) \dot\cup \ldots \dot\cup G_{\lceil \sqrt{n} \rceil}(n)$$

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From the Durfee decomposition a bijection:

$$G_i(n) \longleftrightarrow \left\{ \;(\mu,\nu) \quad with \quad \begin{array}{l} \mu \trianglerighteq \nu \\ l(\mu) \le i, l(\nu) = i \\ |\nu| + |\mu| = n - (i-1)*i \end{array} \right\}.$$

Recursion Formula (2)

$$P(m,k,n,l) \quad := \quad \text{pairs of partitions } (\mu,\nu) \text{ with}$$

$$\mu \trianglerighteq \nu$$

$$l(\mu) = k, |\mu| = m$$

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rewrite above recursion with $r = n - (i-1) * i$:

$$G_i(n) \longleftrightarrow \dot{\bigcup_{\substack{j=1,\dots,i \\ s=0,\dots,r}}} P(s,j,r-s,i)$$

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Recursion Formula (3)

L	L		
L	L		
L	L	L	
L	L	L	
		R	R
		R	R
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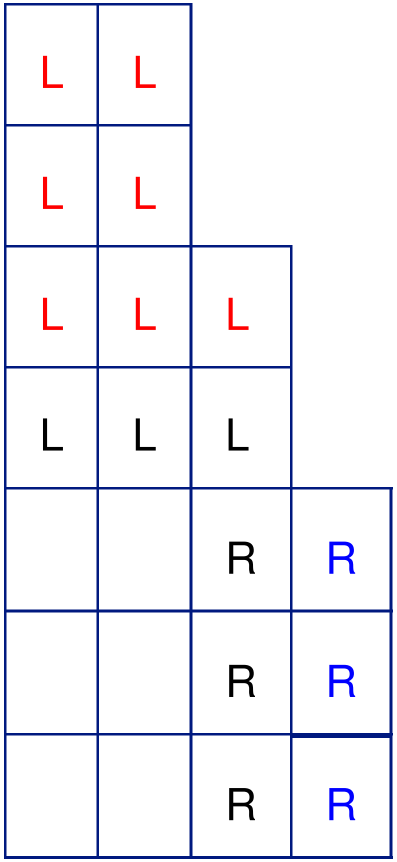

$$P(m,k,n,l) \rightarrowtail \dot{\bigcup_{\substack{i=0,\ldots,k \\ j=0,\ldots,l}}} P(m-k,i,n-l,j)$$

$$p(m,k,n,l) = \sum_{\substack{i=0,\ldots,k \\ j=0,\ldots,l}} p(m-k,i,n-l,j)$$

Telescoping Sum

For $m > n$:

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$$\begin{aligned} &= p(m-1,k-1,n,l) \\ &\quad +p(m,k,n-1,l-1) \\ &\quad -p(m-1,k-1,n-1,l-1) \\ &\quad +p(m-k,k,n-l,l) \end{aligned}$$

Results

n	$g(n)$	$p(n)$	$g(n)/p(n)$
100	69065657	190569292	.3624175
200	1.397805.210533	3.972999.029388	. 3518262
220	7.443670.977177	21.248279.009367	. 3503187

Barnes, Savage 1995

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1000	7.812520.197904 .651287.725407.239942	24.061467.864032 .622473.692149.727991	.3246900

Concluding Remarks

Limiting factors:

memory to store intermediate results
(18GB for n=1000)

time if you do not store intermediate results

References

- Sierksma, Hoogeveen: Seven Criteria for Integer Sequences being Graphic, J. Graph Theory, 1991.
- Barnes, Savage: A Recurrence for Counting Graphical Partitions, EJC, 1995.
- A. Kohnert: Dominance Order and Graphical Partitions, EJC, 2003, accepted.

Thank you very much for your attention.