# Open Problems suggested at Rational Points 2015

Franken-Akademie Schloss Schney, June 29, 2015

**Alexei Skorobogatov**:

Suppose $k$ is a local field (of characteristic 0) and consider elliptic curves $E$ over $k$ for which the Galois module structure of $E[p]$ is fixed ($p$ a prime, not necessarily the residue characteristic). It is well known that the image of

$$E(k)/p \to H^1(k, E[p])$$

is a maximal isotropic subspace with respect to the symmetric bilinear pairing

$$H^1(k, E[p]) \times H^1(k, E[p]) \to H^2(k, \mu_p) \simeq \mathbb{Z}/p$$

induced by the cup product and the Weil pairing.

**Question 1:** Can $E(k)/p$ be any maximal isotropic subspace?
*Bjorn Poonen:* If $p = 2$ there is an additional restriction on the image; it must also be isotropic with respect to corresponding the quadratic form.

**Question 2:** Is there a concrete way to compute this pairing?
*Bartosz Naskręcki and others:* results of Cremona-Fisher-O'Neil-Simon-Stoll on $n$-descents are likely to be relevant and may provide a method for computing the pairing.

**Ronald van Luijk**:

**Question 1**: Does there exist a minimal del Pezzo of degree 1 without a conic bundle (so that Picard number is one) that is unirational?

**Question 2**: Does there exist a minimal del Pezzo of degree 1 without a conic bundle (so that Picard number is one) that is not unirational?

*Comments*:

- When there is a conic bundle structure results of Kollar show that many (perhaps all?) are unirational.
- When there is a rational point, almost all are unirational.

**David Holmes**:

Let $S/\mathbb{Q}$ be a smooth, projective geometrically connected scheme. Given a line bundle $\mathcal{L}$ on $S$ we get a height function,

$$h_{\mathcal{L}} : S(\overline{\mathbb{Q}}) \to \mathbb{R} \qquad \text{(up to } O(1)).$$

For $d \in \mathbb{Z}_{\geq 1}$ and $B \in \mathbb{R}$ let

$$X(d, B) = \left\{ s \in S(\overline{\mathbb{Q}}) \mid [\kappa(s) : \mathbb{Q}] \leq d,\ h_{\mathcal{L}} \leq B \right\}.$$

($\kappa(s)$ is the residue field of $s$.)

**Question 1:** When is $X(d, B)$ not Zariski dense in $S$?

**Question 2:** What if $\mathcal{L}$ is NEF and not $\mathcal{O}_S$?
*Bjorn Poonen*: If $\mathcal{L}$ is numerically trivial, then it is dense.

**Question 3:** What if $\mathcal{L}$ is NEF and has positive degree with respect to every line bundle?

*Comments*:

- If $\mathcal{L}$ is ample, then $X(d, B)$ is finite.
- If $\mathcal{L}$ is semiampe and not $\mathcal{O}_S$, then $X(d, B)$ is not Zariski dense.

**David Harari**:

Suppose $G$ is a finite group scheme over $\mathbb{Q}$ and $G \hookrightarrow \mathrm{SL_n}$. Set $X = \mathrm{SL}_n / G$.

**Question 1**: Is $X(\mathbb{Q})$ dense in $X(\mathbb{R})$? (Borovoi)

*Comments*:

- In terms of Galois cohomology this can be rephrased as asking whether the map $H^1(\mathbb{Q}, G) \to H^1(\mathbb{R}, G)$ is surjective.
- This is independent of the embedding because $\mathrm{SL_n}$ is cohomologically trivial.
- The answer is (trivially) yes if $G$ is constant (i.e. Galois acts trivially on $G(\overline{\mathbb{Q}})$).
- The answer is yes if $G$ is commutative.
  Sketch: use an exact sequence $1 \to G \to R \to T \to 1$ with $R$ a quasi-trivial torus and use weak approximation of the reals for the torus $T$ (Serre).
- There is no Brauer-Manin obstruction to this problem (G. Luccini Arteche)

**Question 2**: Suppose $K$ is a number field with real completions $K_1, \ldots, K_r$ and that $G$ is constant. Is the map

$$H^1(K, G) \to \prod_{i=1}^{r} H^1(K_i, G)$$

surjective?

**Michael Stoll**:

Take $N$ large (say $N \sim 100,000$). Let $\Lambda \subset \mathbb{Z}^N$ be a subgroup of index $\ll 2^N$ and let $a \in \mathbb{Z}^N$. (For the application in mind one has that $\Lambda$ is the kernel of a map $\mathbb{Z}^N \to \prod(\mathbb{Z}/q_i\mathbb{Z})^\times$ and the image of $a$ is of the form $(1, \ldots, 1, -1, \ldots, -1)$.)

**Question 1:** Are there some reasonable conditions (preferably on $\Lambda$ rather than $a$) that would imply

$$(a + \Lambda) \cap \{0, 1\}^N \neq \emptyset\ ?$$

**Question 2:** If $\Lambda$ and $a$ are given explicitly and the intersection is known to be nonempty, is there a reasonably efficient what of finding an element of $(a + \Lambda) \cap \{0, 1\}^N$?