DEL PEZZO SURFACES OF DEGREE 1

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1. Del Pezzo surfaces

Let $K$ be a number field.

Definition 1.1. $X/K$ is a del Pezzo surface if $X$ is smooth, projective, geometrically integral, $\dim X = 2$, and $-K_X$ is ample.

Theorem 1.2 (Segre-Manin). Let $X/K$ be a del Pezzo surface of degree $\geq 2$ such that $X$ has a $K$-point not on any exceptional curve. Then $X(K)$ is Zariski dense.

2. dP1s

Theorem 2.1. A dP1 is a smooth sextic in $\mathbb{P}(1,1,2,3)$, and conversely.

Let $x, y, z, w$ be the variables on $\mathbb{P}(1,1,2,3)$. If char $k \neq 2, 3$, then a dP1 can be given an equation

$$w^2 = z^3 + G(x, y)z + F(x, y)$$

where $G$ and $F$ are binary homogeneous forms of degrees 4 and 6.

We will focus on the case $G \equiv 0$. Then $X$ is smooth if and only if $F$ has no square factors.

Theorem 2.2. Let $X/\mathbb{Q}$ be a dP1 given by

$$w^2 = z^3 + Ax^6 + By^6$$

in $\mathbb{P}_\mathbb{Q}(1,1,2,3)$ where $A, B$ are nonzero integers. Assume (III): that every elliptic curve $E/\mathbb{Q}$ with $j(E) = 0$ there exists a prime $p$ of good ordinary reduction such that $\text{III}(E, \mathbb{Q})[p^\infty] < \infty$ (this implies the parity conjecture for $E$, by work of Nekovar$^1$). Also assume that if $A/B = 3a^2/b^2$ for $a, b \in \mathbb{Z}$ with $b \neq 0$ and $(a, b) = 1$, then $\gcd(A, B) = 1$. Then $X(\mathbb{Q})$ is Zariski dense.

3. Elliptic fibrations

dP1s have a canonical rational point $P_{\text{can}} := [0 : 0 : 1 : 1]$. The anticanonical map is

$$X \dashrightarrow \mathbb{P}^1$$

$$[x : y : z : w] \mapsto [x : y].$$

Its indeterminacy at $P_{\text{can}}$ is resolved by taking $\tilde{X} := \text{Bl}_{P_{\text{can}}} X$: We get an elliptic surface $\tilde{X} \rightarrow \mathbb{P}^1$, whose fiber above $(m : n) \in \mathbb{P}^1$ is isomorphic to the elliptic curve $E_{m,n}: y^2 = x^3 + Am^6 + Bn^6$.

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$^1$Recent work by the Dokchitser brothers should allow us to remove the words “of good ordinary reduction.”

See arxiv:math/0612054

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We hope to show that infinitely many of these curves $E_{m,n}$ have infinitely many rational points, since then $X(\mathbb{Q})$ is Zariski dense in $X$.

4. Root numbers

Given a CM elliptic curve $E/\mathbb{Q}$ of conductor $N$, we know that $L(E,s)$ has an analytic continuation and functional equation: $\Lambda(E,s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(E,s)$ satisfies $\Lambda(E,s) = \pm\Lambda(E,2-s)$. Let $W(E)$ be the sign of the functional equation above, so $W(E) = (-1)^{\text{rank}(E)}$. Hypothesis III plus work of Nekovar implies that $W(E) = (-1)^{\text{rank}(E)}$ for our $E$'s.

We need to compute $W(E)$. It turns out that $W(E) = \prod_{p \leq \infty} W_p(E)$ where $W_p(E) = \pm 1$ is defined in terms of $c$-factors of representations of the Weil-Deligne of $\mathbb{Q}_p$.

**Theorem 4.1** (Rohrlich, Halberstadt, Rizzo). Let $E/\mathbb{Q}$ be an elliptic curve in Weierstrass form. Let $c_4, c_6, \infty$ be the usual quantities. Then

(i) $W_\infty(E) = -1$.
(ii) $W_p(E) = +1$ if $p$ is a prime of good reduction.
(iii) Suppose that $E$ has additive potentially good reduction at $p > 3$. Let $e = 12/\text{gcd}(v_p(\Delta), 12)$. Then

$$W_p(E) := \begin{cases} 1, & \text{if } e = 1; \\
\left(\frac{-1}{p}\right), & \text{if } e = 2 \text{ or } e = 6; \\
\left(\frac{-3}{p}\right), & \text{if } e = 3; \\
\left(\frac{-2}{p}\right), & \text{if } e = 4. \end{cases}$$

(iv) $W_2(E)$ and $W_3(E)$ can be computed from knowledge of $c_4$, $c_6$, $\Delta$.

**Proposition 4.2.** Let $\alpha \in \mathbb{Z}$. Let $E_\alpha$ be the elliptic curve $y^2 = x^3 + \alpha$. Let $W(\alpha) = W(E_\alpha)$. Then

$$W(\alpha) = - \left[ W_2(E_\alpha) \left(\frac{-1}{\alpha_{\text{odd}}}\right) W_3(\alpha)(-1)^{v_3(\alpha)} \prod_{p > 5 \atop p^2 \mid \alpha} \left(\frac{1}{p}\right) \right] \prod_{p^2 \mid \alpha} \left(\frac{-3}{p}\right),$$

if $v_p(\alpha) \equiv 0, 1, 3, 5 \pmod{6}$, and

$$\prod_{p^2 \mid \alpha} \left(\frac{-2}{p}\right),$$

if $v_p(\alpha) \equiv 2, 4 \pmod{6}$.

Moreover, if $\alpha$ and $\beta$ satisfy $v_2(\alpha) = v_2(\beta) = r_2$ and $v_3(\alpha) = v_3(\beta) = r_3$, and if $\alpha \equiv \beta \pmod{2^{r_2+2}3^{r_3+2}}$, then the product in brackets for $W(\alpha)$ and $W(\beta)$ coincide.

**Corollary 4.3** (Flipping). Suppose $\alpha, \beta \in \mathbb{Z} - \{0\}$ are values of $Am^6 + Bn^6$ with $\text{gcd}(A,B) = 1$. Assume that

- $\alpha \equiv \beta \pmod{36}$.
- $\alpha$ is squarefree.
- $\beta = p^{2+6k} \eta$ with $\eta$ squarefree, with $p$ prime, $p > 3$, $p \equiv 2 \pmod{3}$, $k \in \mathbb{Z}_{\geq 0}$.

Then $W(\alpha) = -W(\beta)$.

To prove Zariski density (at least for $\text{gcd}(A,B) = 1$), we need two families $\mathcal{F}_1$ and $\mathcal{F}_2$ of pairs of relatively prime integers $(m,n)$ such that

(i) $Am^6 + Bn^6$ is of the form $\alpha$ as in the corollary for $(m,n) \in \mathcal{F}_1$
(ii) $Am^6 + Bn^6$ is of the form $\beta$ as in the corollary for $(m,n) \in \mathcal{F}_2$. 
5. **Sieving**

**Theorem 5.1** (Greaves, Gouvea-Mazur, Varilly). Let $F(m, n)$ be a homogeneous binary form in $\mathbb{Z}[x, y]$ with nonzero discriminant. Assume that no irreducible factor of $F$ has degree > 6. Fix a modulus $M$ and integers $a, b$ such that $\gcd(a, b, M) = 1$. Let $S$ be a finite set of distinct primes $p_1, \ldots, p_r$. Let $T$ be a finite set of nonnegative integers $t_1, \ldots, t_r$ (of the same cardinality as $S$). Let $N(x)$ be the number of $(m, n) \in \mathbb{Z}^2$ with $\gcd(m, n) = 1$ such that $0 \leq m, n \leq x$ and $m \equiv a \pmod{M}$ and $n \equiv b \pmod{M}$ and $F(m, n) \equiv p_1^{t_1} \cdots p_r^{t_r} \alpha$ where $\alpha$ is squarefree and $v_{p_i}(\alpha) = 0$ for $i = 1, \ldots, r$. Then $N(x) = Cx^2 + O(x^2/(\log x)^{1/3})$.

**Remark 5.2.** The constant $C$ can be 0.

For $F(m, n) = Am^6 + Bn^6$ with $\gcd(A, B) = 1$, we have $C = 0$ if and only if there exists $i$ such that for all $1 \leq m, n \leq p_i^{t_i+1}$ we have $v_{p_i}(F(m, n)) \neq t_i$.

For $F_1$, we take $S = \emptyset$ and $a, b$ arbitrary and $M = 36$.

For $F_2$, we take $S = \{p\}$ and $T = \{2 + 6k\}$ and $M = 36$. This time $C$ can be 0: in fact, $C = 0$ when $A/B$ is of the form $3a^2/b^2$.

**Example 5.3.** For $y^2 = x^3 + 27m^6 + 16n^6$, $W(E_{m,n}) = +1$. But we have sections: e.g., $(x, y) = (-3m^2, 4n^3)$.

But there are others:

$$y^2 = x^3 + 6(3m^6 + n^6)$$

where $W(E_{m,n}) = +1$ and there are no sections.