DEL PEZZO SURFACES OF DEGREE 1

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1. Del Pezzo surfaces

Let K be a number field.

Definition 1.1. X/K is a *del Pezzo surface* if X is smooth, projective, geometrically integral, $\dim X = 2$, and $-K_X$ is ample.

Theorem 1.2 (Segre-Manin)**.** *Let X/K be a del Pezzo surface of degree ≥ 2 such that X has a K-point not on any exceptional curve. Then $X(K)$ is Zariski dense.*

2. dP1s

Theorem 2.1. *A dP1 is a smooth sextic in $\mathbb{P}(1,1,2,3)$, and conversely.*

Let x, y, z, w be the variables on $\mathbb{P}(1,1,2,3)$. If $\operatorname{char} k \neq 2, 3$, then a dP1 can be given an equation

$$w^2 = z^3 + G(x,y)z + F(x,y)$$

where G and F are binary homogeneous forms of degrees 4 and 6.

We will focus on the case $G \equiv 0$. Then X is smooth if and only if F has no square factors.

Theorem 2.2. *Let $X/\mathbb{Q}$ be a dP1 given by*

$$w^2 = z^3 + Ax^6 + By^6$$

in $\mathbb{P}_{\mathbb{Q}}(1,1,2,3)$ where A, B are nonzero integers. Assume (Ш)*: that every elliptic curve $E/\mathbb{Q}$ with $j(E) = 0$ there exists a prime p of good ordinary reduction such that* $\text{Ш}(E,\mathbb{Q})[p^\infty] < \infty$ *(this implies the parity conjecture for E, by work of Nekovar[1]). Also assume that if $A/B = 3a^2/b^2$ for $a, b \in \mathbb{Z}$ with $b \neq 0$ and $(a,b) = 1$, then $\gcd(A,B) = 1$. Then $X(\mathbb{Q})$ is Zariski dense.*

3. Elliptic fibrations

dP1s have a canonical rational point $P_{\text{can}} := [0:0:1:1]$. The anticanonical map is

$$X \dashrightarrow \mathbb{P}^1$$
$$[x:y:z:w] \mapsto [x:y].$$

Its indeterminacy at P_{can} is resolved by taking $\tilde{X} := \mathrm{Bl}_{P_{\text{can}}} X$: We get an elliptic surface $\tilde{X} \to \mathbb{P}^1$, whose fiber above $(m:n) \in \mathbb{P}^1$ is isomorphic to the elliptic curve $E_{m,n}\colon y^2 = x^3 + Am^6 + Bn^6$.

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[1] Recent work by the Dokchitser brothers should allow us to remove the words "of good ordinary reduction." See arxiv:math/0612054

We hope to show that infinitely many of these curves $E_{m,n}$ have infinitely many rational points, since then $X(\mathbb{Q})$ is Zariski dense in X.

4. Root numbers

Given a CM elliptic curve $E/\mathbb{Q}$ of conductor N, we know that $L(E,s)$ has an analytic continuation and functional equation: $\Lambda(E,s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(E,s)$ satisfies $\Lambda(E,s) = \pm\Lambda(E, 2-s)$. Let $W(E)$ be the sign of the functional equation above, so $W(E) = (-1)^{r_{\mathrm{an}}(E)}$. Hypothesis Ш plus work of Nekovar implies that $W(E) = (-1)^{\operatorname{rank} E(\mathbb{Q})}$ for our E's.

We need to compute $W(E)$. It turns out that $W(E) = \prod_{p\leq\infty} W_p(E)$ where $W_p(E) = \pm 1$ is defined in terms of ϵ-factors of representations of the Weil-Deligne of $\mathbb{Q}_p$.

Theorem 4.1 (Rohrlich, Halberstadt, Rizzo)**.** *Let $E/\mathbb{Q}$ be an elliptic curve in Weierstrass form. Let c_4, c_6, ∞ be the usual quantities. Then*

(i) $W_\infty(E) = -1$.
(ii) $W_p(E) = +1$ *if p is a prime of good reduction.*
(iii) *Suppose that E has additive potentially good reduction at $p > 3$. Let $e = 12/\gcd(v_p(\Delta), 12)$. Then*

$$W_p(E) := \begin{cases} 1, & \text{if } e = 1; \\ \left(\frac{-1}{p}\right), & \text{if } e = 2 \text{ or } e = 6; \\ \left(\frac{-3}{p}\right), & \text{if } e = 3; \\ \left(\frac{-2}{p}\right), & \text{if } e = 4. \end{cases}$$

(iv) *$W_2(E)$ and $W_3(E)$ can be computed from knowledge of c_4, c_6, Δ.*

Proposition 4.2. *Let $\alpha \in \mathbb{Z}$. Let E_α be the elliptic curve $y^2 = x^3 + \alpha$. Let $W(\alpha) = W(E_\alpha)$. Then*

$$W(\alpha) = -\left[W_2(E_\alpha)\left(\frac{-1}{\alpha_{\mathrm{odd}}}\right)W_3(\alpha)(-1)^{v_3(\alpha)}\right]\prod_{\substack{p>5\\ p^2|\alpha}}\begin{cases} 1, & \text{if } v_p(\alpha) \equiv 0,1,3,5 \pmod 6 \\ \left(\frac{-3}{p}\right), & \text{if } v_p(\alpha) \equiv 2,4 \pmod 6 \end{cases}.$$

Moreover, if α and β satisfy $v_2(\alpha) = v_2(\beta) =: r_2$ and $v_3(\alpha) = v_3(\beta) =: r_3$, and if $\alpha \equiv \beta \bmod 2^{r_2+2}3^{r_3+2}$, then the product in brackets for $W(\alpha)$ and $W(\beta)$ coincide.

Corollary 4.3 (Flipping)**.** *Suppose $\alpha, \beta \in \mathbb{Z} - \{0\}$ are values of $Am^6 + Bn^6$ with $\gcd(A, B) = 1$. Assume that*

- $\alpha \equiv \beta \pmod{36}$.
- *α is squarefree.*
- *$\beta = p^{2+6k}\eta$ with η squarefree, with p prime, $p > 3$, $p \equiv 2 \pmod 3$, $k \in \mathbb{Z}_{\geq 0}$.*

Then $W(\alpha) = -W(\beta)$.

To prove Zariski density (at least for $\gcd(A, B) = 1$), we need two families $\mathcal{F}_1$ and $\mathcal{F}_2$ of pairs of relatively prime integers (m, n) such that

(i) $Am^6 + Bn^6$ is of the form α as in the corollary for $(m, n) \in \mathcal{F}_1$
(ii) $Am^6 + Bn^6$ is of the form β as in the corollary for $(m, n) \in \mathcal{F}_2$.

5. Sieving

Theorem 5.1 (Greaves, Gouvea-Mazur, Varilly)**.** *Let $F(m,n)$ be a homogeneous binary form in $\mathbb{Z}[x,y]$ with nonzero discriminant. Assume that no irreducible factor of F has degree > 6. Fix a modulus M and integers a,b such that $\gcd(a,b,M)=1$. Let S be a finite set of distinct primes $p_1,\dots,p_r$. Let T be a finite set of nonnegative integers $t_1,\dots,t_r$ (of the same cardinality as S). Let $N(x)$ be the number of $(m,n)\in\mathbb{Z}^2$ with $\gcd(m,n)=1$ such that $0\le m,n\le x$ and $m\equiv a \pmod{M}$ and $n\equiv b \pmod{M}$ and $F(m,n)\equiv p_1^{t_1}\cdots p_r^{t_r}\alpha$ where α is squarefree and $v_{p_i}(\alpha)=0$ for $i=1,\dots,r$. Then $N(x)=Cx^2+O(x^2/(\log x)^{1/3})$.*

Remark 5.2. The constant C can be 0.

For $F(m,n)=Am^6+Bn^6$ with $\gcd(A,B)=1$, we have $C=0$ if and only if there exists i such that for all $1\le m,n\le p_i^{t_i+1}$ we have $v_{p_i}(F(m,n))\ne t_i$.

For $\mathcal{F}_1$, we take $S=\emptyset$ and a,b arbitrary and $M=36$.

For $\mathcal{F}_2$, we take $S=\{p\}$ and $T=\{2+6k\}$ and $M=36$. This time C can be 0: in fact, $C=0$ when A/B is of the form $3a^2/b^2$.

Example 5.3. For $y^2=x^3+27m^6+16n^6$, $W(E_{m,n})=+1$. But we have sections: e.g., $(x,y)=(-3m^2,4n^3)$.

But there are others:

$$y^2=x^3+6(3m^6+n^6)$$

where $W(E_{m,n})=+1$ and there are no sections.