NONABELIAN DESCENT ON ENRIQUES SURFACES

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Let k be a number field. Fix an algebraic closure $\overline{k}$.

1. A family of Enriques surfaces (geometry)

Let D_1, D_2 be curves of genus 1, say

$$D_1\colon y_1^2 = d_1(x^2-a)(x^2-ab^2)$$
$$D_2\colon y_2^2 = d_2(t^2-a)(t^2-ac^2)$$

where $b, c, d_1, d_2 \in k^\times$ and $a \in k^\times - k^{\times 2}$, and $b, c \neq \pm 1$.

Let E_i be the Jacobian of D_i for $i = 1, 2$. The elliptic curves E_1 and E_2 have $E_i(\overline{k})[2] \subset E_i(k)$. We have the involution -1 on D_1 and on D_2. Let Y be the Kummer surface obtained as the minimal desingularization of $(D_1 \times D_2)/(-1)$. This is a K3 surface.

Choose rational points $P \in E_1[2]$ and $Q \in E_2[2]$. We have a fixed-point-free involution $\sigma\colon Y \to Y$ induced by $(x, y) \mapsto (x + P, -y + Q)$ for $x \in D_1$ and $y \in D_2$. An *Enriques surface* is an étale quotient of a K3 surface by a fixed-point-free involution. So $X := Y/\sigma$ is an Enriques surface. The variety Y is the minimal smooth projective model of

$$y^2 = d(x^2-a)(x^2-ab^2)(t^2-a)(t^2-ac^2),$$

and $\sigma(x, y, t) = (-x, -y, -t)$.

We have $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$, but $\overline{X} := X \times_k \overline{k}$ is not rational, since it has a $\mathbb{Z}/2$ étale covering $\overline{Y}$. In fact, since a K3 surface is simply connected, we have $\pi_1(\overline{X}) = \mathbb{Z}/2$.

Proposition 1.1. *Under very mild conditions on b, c, the elliptic curves $\overline{E}_1$ and $\overline{E}_2$ are not isogenous.*

Proof. Check that $j(\overline{E}_1)$ is not integral over $\mathbb{Z}[j(\overline{E}_2)]$. □

Assume from now on that $\overline{E}_1$ and $\overline{E}_2$ are not isogenous. Then $\operatorname{Pic}(\overline{D}_1 \times \overline{D}_2) \simeq \operatorname{Pic} \overline{D}_1 \times \operatorname{Pic} \overline{D}_2$.

We define 24 lines (by which we mean rational curves) on $\overline{Y}$. Number the points $(\pm\sqrt{a}, 0)$ and $(\pm b\sqrt{a}, 0)$ on D_1 as $0, 1, 2, 3$. Number the points $(\pm\sqrt{a}, 0)$ and $(\pm c\sqrt{a}, 0)$ on D_2 as $0, 1, 2, 3$. Let ℓ_{ij} be the exceptional curve on $\overline{Y}$ corresponding to the blow-up of $(i, j) \in (\overline{D}_1 \times \overline{D}_2)/(-1)$: this gives 16 lines. Let ℓ_i be the proper transform of $(i \times \overline{D}_2)/(-1)$, and let s_j be the proper transform of $(\overline{D}_1 \times j)/(-1)$. Let $U' = (D_1 - \{y_1 = 0\}) \times (D_2 - \{y_2 = 0\})$ and $V' = U'/(-1)$. Then V' is the complement of the 24 lines on Y.

Proposition 1.2. *We have* $\operatorname{Pic} \overline{V}' = 0$ *(so* $\operatorname{Pic} \overline{Y}$ *is generated by the 24 lines).*

Proof. Use Proposition 1.1 and the Hochschild-Serre spectral sequence associated to $\overline{U}' \to \overline{V}'$. □

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Let $L = k(\sqrt{a})$.

Remark 1.3. The 24 lines are defined over L, and the action of $\mathrm{Gal}(L/k)$ coincides with the action of σ.

2. A counterexample to weak approximation

Let $k = \mathbb{Q}$. Let b be a prime number p with $\left(\frac{a}{p}\right) = -1$. Let a be another prime number, with $a \equiv 1 \pmod 4$. Let $c \in \mathbb{Z}$ such that $p \nmid c(c^2 - 1)$. Let $d_1 = d_2 = 1$.

For example, take $a = 5$, $b = 13$, $c = 2$. In this case, Y is given by

$$y^2 = (x^2 - a)(x^2 - ap^2)(t^2 - a)(t^2 - ac^2).$$

There is an obvious rational point $M \in Y(k)$, given by $x = t = 0$ and $y = a^2pc$.

Define an adelic point $(M_v) \in \prod_v Y(k_v)$ where $M_v = M$ for v real and for $v \neq p$, and M_p given by $x = t = p^{-1}$, $y = p^{-4}\alpha$ where $\alpha \in \mathbb{Z}_p^\times$ with $\alpha \equiv 1 \pmod p$.

Proposition 2.1. *Define $Q_v = f(M_v)$. Then (Q_v) is not in the closure of $X(k)$ in $\prod_v X(k_v)$.*

Idea: We can find a 1-dimensional k-torus T and an Y-torsor Z under T such that Z is also an X-torsor under a k-group G fitting into an exact sequence

$$1 \to T \to G \to \mathbb{Z}/2 \to 1.$$

In other words, we have

$$\begin{tikzcd} & Z \arrow[dl, "T"'] \arrow[ddl, "G"] \\ Y \arrow[d, "\mathbb{Z}/2"'] & \\ X. & \end{tikzcd}$$

The étale cohomology set $H^1(X, G)$ classifies X-torsors under G. We have $[Z] \in H^1(X, G)$.

Fact: $[Z](Q_v) \in \prod_v H^1(k_v, G)$ does not belong to the diagonal image of $H^1(k, G)$. This shows that $(Q_v) \notin \overline{X(k)}$, because of the Borel-Serre finiteness theorem.

3. Computations of Brauer groups

Goal: Show that (Q_v) is in the Brauer-Manin set of X: i.e., that for all $\alpha \in \mathrm{Br}\, X$,

$$\sum_v j_v(\alpha(Q_v)) = 0. \tag{1}$$

Let f be the map $Y \to X$. Recall that $\mathrm{Br}_1 X$ is the kernel of $\mathrm{Br}\, X \to \mathrm{Br}\, \overline{X}$.

Proposition 3.1. *The group $f^*(\mathrm{Br}_1 X)$ is contained in the image of $\mathrm{Br}\, k \to \mathrm{Br}\, Y$.*

Proof. If k is a number field, then $\mathrm{Br}_1 X / \mathrm{Br}\, k = H^1(k, \mathrm{Pic}\, \overline{X})$. Similarly, $\mathrm{Br}_1 Y / \mathrm{Br}\, k = H^1(k, \mathrm{Pic}\, \overline{Y})$. Since $\mathrm{Pic}\, \overline{Y}$ is torsion-free, it is sufficient to show that $H^1(k, (\mathrm{Pic}\, \overline{X})/\mathrm{tors}) = 0$. The spectral sequence for $\overline{Y} \to \overline{X}$ gives

$$0 \to \mathbb{Z}/2 \to \mathrm{Pic}\, \overline{X} \to (\mathrm{Pic}\, \overline{Y})^\sigma \to H^2(\mathbb{Z}/2, \overline{k}^\times)$$

and $(\mathrm{Pic}\, \overline{X})/\mathrm{tors} = \mathbb{Z}^r$ with trivial Galois action. $\square$

Theorem 3.2. *If $-d$ and $-ad$ are not squares, then* $\operatorname{Br}_1 X = \operatorname{Br} X$. *(Note that* $\operatorname{Br} \overline{X} = \mathbb{Z}/2$.*)*

Proof of (1). Take $\alpha \in \operatorname{Br} X = \operatorname{Br}_1 X$. Then

$$\sum_v j_v(\alpha(Q_v)) = \sum_v j_v(f^*(\alpha)(M_v)),$$

which is constant by Proposition 3.1, so it is 0. □

Conclusion: "The Brauer-Manin obstruction to weak approximation is not the only one for Enriques surfaces."

Remark 3.3. The important facts we used were:

- G is not commutative
- G is not connected.

If one of these failed, the obstruction would be explained by the Brauer-Manin obstruction.

Question 3.4. The map $\operatorname{Br} \overline{X} \to \operatorname{Br} \overline{Y}$ is injective for this family. Is it true in general for every Enriques surface?

Conjecture 3.5. The Brauer-Manin obstruction to the Hasse principle is not the only one for Enriques surfaces.