NONABELIAN DESCENT ON ENRIQUES SURFACES

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Let $k$ be a number field. Fix an algebraic closure $\overline{k}$.

1. A family of Enriques surfaces (geometry)

Let $D_1, D_2$ be curves of genus 1, say

$$D_1: y_1^2 = d_1(x^2-a)(x^2-ab^2)$$
$$D_2: y_2^2 = d_2(t^2-a)(t^2-ac^2)$$

where $b, c, d_1, d_2 \in k^\times$ and $a \in k^\times - k^x$, and $b, c \neq \pm 1$.

Let $E_i$ be the Jacobian of $D_i$ for $i = 1, 2$. The elliptic curves $E_1$ and $E_2$ have $E_i(\overline{k})[2] \subset E_i(k)$. We have the involution $-1$ on $D_1$ and on $D_2$. Let $Y$ be the Kummer surface obtained as the minimal desingularization of $(D_1 \times D_2)/(-1)$. This is a K3 surface.

Choose rational points $P \in E_1[2]$ and $Q \in E_2[2]$. We have a fixed-point-free involution $\sigma: Y \to Y$ induced by $(x, y) \mapsto (x + P, -y + Q)$ for $x \in D_1$ and $y \in D_2$. An Enriques surface is an étale quotient of a K3 surface by a fixed-point-free involution. So $X := Y/\sigma$ is an Enriques surface. The variety $Y$ is the minimal smooth projective model of

$$y^2 = d(x^2-a)(x^2-ab^2)(t^2-a)(t^2-ac^2),$$

and $\sigma(x, y, t) = (-x, -y, -t)$.

We have $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$, but $\overline{X} := X \times_k \overline{k}$ is not rational, since it has a $\mathbb{Z}/2$ étale covering $\overline{Y}$. In fact, since a K3 surface is simply connected, we have $\pi_1(\overline{X}) = \mathbb{Z}/2$.

Proposition 1.1. Under very mild conditions on $b, c$, the elliptic curves $\overline{E}_1$ and $\overline{E}_2$ are not isogenous.

Proof. Check that $j(\overline{E}_1)$ is not integral over $\mathbb{Z}[j(\overline{E}_2)]$. □

Assume from now on that $\overline{E}_1$ and $\overline{E}_2$ are not isogenous. Then $\text{Pic}(\overline{D}_1 \times \overline{D}_2) \simeq \text{Pic} \overline{D}_1 \times \text{Pic} \overline{D}_2$.

We define 24 lines (by which we mean rational curves) on $\overline{Y}$. Number the points $(\pm \sqrt{a}, 0)$ and $(\pm b\sqrt{a}, 0)$ on $D_1$ as 0, 1, 2, 3. Number the points $(\pm \sqrt{a}, 0)$ and $(\pm c\sqrt{a}, 0)$ on $D_2$ as 0, 1, 2, 3. Let $\ell_{ij}$ be the exceptional curve on $\overline{Y}$ corresponding to the blow-up of $(i, j) \in (\overline{D}_1 \times \overline{D}_2)/(-1)$: this gives 16 lines. Let $\ell_i$ be the proper transform of $(i \times \overline{D}_2)/(-1)$, and let $s_j$ be the proper transform of $(\overline{D}_1 \times j)/(-1)$. Let $U' = (D_1 - \{y_1 = 0\}) \times (D_2 - \{y_2 = 0\})$ and $V' = U'/(-1)$. Then $V'$ is the complement of the 24 lines on $Y$.

Proposition 1.2. We have $\text{Pic}\overline{V}' = 0$ (so $\text{Pic}\overline{Y}$ is generated by the 24 lines).

Proof. Use Proposition 1.1 and the Hochschild-Serre spectral sequence associated to $\overline{U} \to \overline{V}'$. □

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Let \( L = k(\sqrt{a}) \).

Remark 1.3. The 24 lines are defined over \( L \), and the action of \( \text{Gal}(L/k) \) coincides with the action of \( \sigma \).

2. A COUNTEREXAMPLE TO WEAK APPROXIMATION

Let \( k = \mathbb{Q} \). Let \( b \) be a prime number \( p \) with \( \left( \frac{a}{p} \right) = -1 \). Let \( a \) be another prime number, with \( a \equiv 1 \pmod{4} \). Let \( c \in \mathbb{Z} \) such that \( p \nmid c(p^2 - 1) \). Let \( d_1 = d_2 = 1 \).

For example, take \( a = 5 \), \( b = 13 \), \( c = 2 \). In this case, \( Y \) is given by

\[
y^2 = (x^2 - a)(x^2 - ap^2)(t^2 - a)(t^2 - ac^2).
\]

There is an obvious rational point \( M \in Y(k) \), given by \( x = t = 0 \) and \( y = a^2p^2c \).

Proposition 2.1. Define \( Q_v = f(M_v) \). Then \( (Q_v) \) is not in the closure of \( X(k) \) in \( \prod_v X(k_v) \).

Idea: We can find a 1-dimensional \( k \)-torus \( T \) and an \( Y \)-torsor \( Z \) under \( T \) such that \( Z \) is also an \( X \)-torsor under a \( k \)-group \( G \) fitting into an exact sequence

\[
1 \to T \to G \to \mathbb{Z}/2 \to 1.
\]

In other words, we have

\[
\begin{array}{ccc}
Z & \xrightarrow{T} & Y \\
\downarrow & & \downarrow \sigma \\
X & \xrightarrow{G} & \mathbb{Z}/2
\end{array}
\]

The étale cohomology set \( H^1(X, G) \) classifies \( X \)-torsors under \( G \). We have \([Z] \in H^1(X, G)\).

Fact: \([Z](Q_v) \in \prod_v H^1(k_v, G)\) does not belong to the diagonal image of \( H^1(k, G) \). This shows that \((Q_v) \notin \overline{X}(k)\), because of the Borel-Serre finiteness theorem.

3. COMPUTATIONS OF BRAUER GROUPS

Goal: Show that \((Q_v)\) is in the Brauer-Manin set of \( X \): i.e., that for all \( \alpha \in \text{Br} X \),

\[
\sum_v j_v(\alpha(Q_v)) = 0.
\]

Let \( f \) be the map \( Y \to X \). Recall that \( \text{Br}_1 X \) is the kernel of \( \text{Br} X \to \text{Br} \overline{X} \).

Proposition 3.1. The group \( f^*(\text{Br}_1 X) \) is contained in the image of \( \text{Br} k \to \text{Br} Y \).

Proof. If \( k \) is a number field, then \( \text{Br}_1 X/\text{Br} k = H^1(k, \text{Pic} \overline{X}) \). Similarly, \( \text{Br}_1 Y/\text{Br} k = H^1(k, \text{Pic} \overline{Y}) \). Since \( \text{Pic} \overline{Y} \) is torsion-free, it is sufficient to show that \( H^1(k, (\text{Pic} \overline{X})/\text{tors}) = 0 \).

The spectral sequence for \( \overline{Y} \to \overline{X} \) gives

\[
0 \to \mathbb{Z}/2 \to \text{Pic} \overline{X} \to (\text{Pic} \overline{Y})^\sigma \to H^2(\mathbb{Z}/2, \overline{k}^\times)
\]

and \((\text{Pic} \overline{X})/\text{tors} = \mathbb{Z}^r \) with trivial Galois action. \( \square \)
Theorem 3.2. If \(-d\) and \(-ad\) are not squares, then \(\text{Br}_1 X = \text{Br} X\). (Note that \(\text{Br} \overline{X} = \mathbb{Z}/2\).)

Proof of (1). Take \(\alpha \in \text{Br} X = \text{Br}_1 X\). Then

\[
\sum_v j_v(\alpha(Q_v)) = \sum_v j_v(f^*(\alpha)(M_v)),
\]

which is constant by Proposition 3.1, so it is 0. \(\Box\)

Conclusion: “The Brauer-Manin obstruction to weak approximation is not the only one for Enriques surfaces.”

Remark 3.3. The important facts we used were:

- \(G\) is not commutative
- \(G\) is not connected.

If one of these failed, the obstruction would be explained by the Brauer-Manin obstruction.

Question 3.4. The map \(\text{Br} \overline{X} \rightarrow \text{Br} \overline{Y}\) is injective for this family. Is it true in general for every Enriques surface?

Conjecture 3.5. The Brauer-Manin obstruction to the Hasse principle is not the only one for Enriques surfaces.