Let $C$ be $\sum x_i y^j z^{4-i-j} - y^2 z^2 - 2z^4 = 0$. This has points

$p_0: (-1 : 1 : 1)$  
$p_1: (1 : -1 : 1)$  
$p_2: (1 : 1 : -1)$  
$p_3: (25 : -17 : 31)$  
$p_4: \{x^2 + 2z^2, y + z = 0\}$  
$p_5: \{3x^2 + 2y^2 - 3yz - 2z^2 = 0$

$3xy + 2y^2 + 3yz + z^2 = 0$

$3xz - 5y^2 + 3yz - z^2 = 0$

$5y^3 - y^2 z + 4yz^2 + z^3 = 0\}$

Define

$g_1 := [p_2 - p_0]$

$g_2 := [p_4 - 2p_0]$

$g_3 := [p_5 - 3p_0$.

In terms of these, we have

$[p_1 - p_0] := 3g_1 + 2g_2 - 2g_3$

$[p_2 - p_0] := g_1$

$[p_3 - p_0] := 2g_2$.

**Theorem 0.1.** Subject to GRH, $\langle g_1, g_2, g_3 \rangle$ has finite odd index in $J_C(\mathbb{Q}) \simeq \mathbb{Z}^3$.

Strategy:

almost $2J(\mathbb{Q}) \dashrightarrow J(\mathbb{Q}) \dashrightarrow \frac{L^x}{L^{x2}\mathbb{Q}^x}$

almost $2J(\mathbb{Q}_p) \dashrightarrow J(\mathbb{Q}_p) \dashrightarrow \frac{L^p}{L^{p2}\mathbb{Q}_p^x}$

where $L_p := L \otimes \mathbb{Q}_p$.  

*Date:* July 23, 2007.
Here the group $\frac{L^\times}{L^\times \mathbf{Q}^\times}$ is a substitute for $H^1(\mathbf{Q}, J[2])$. We may impose the conditions that cohomology classes are unramified outside a finite set $S$ to replace $\frac{L^\times}{L^\times \mathbf{Q}^\times}$ by a finite subgroup $L(2, S)$ essentially generated by $S$-units:

\[
\begin{array}{cccccc}
\text{almost } 2J(\mathbf{Q}) & \rightarrow & J(\mathbf{Q}) & \rightarrow & L(2, S) & \rightarrow \\
& & \downarrow & & \downarrow & \\
\text{almost } 2J(\mathbf{Q}_p) & \rightarrow & J(\mathbf{Q}_p) & \rightarrow & \frac{L^\times_p}{L^\times_p \mathbf{Q}^\times_p}
\end{array}
\]

We compute the image of $J(\mathbf{Q}_p) \rightarrow \frac{L^\times_p}{L^\times_p \mathbf{Q}^\times_p}$ for each $p \in S$. In the example, $S = \{\infty, 2, 5, 402613\}$.

1. Description of $L$

The genus-3 curve is in $\mathbb{P}^2$ with coordinates $x, y, z$. In the dual projective space $\mathbb{P}^2$ with coordinates $u, v, w$, the set of bitangents corresponds to a reduced 0-dimensional subscheme of degree 28. Project this to a line, to get $\text{Spec } L$, where $L = \mathbf{Q}[t]/(g(t))$ where $g(t)$ is a polynomial of degree 28.

The general bitangent is given by

$$
\lambda_\theta: u_{\theta}x + v_{\theta}y + w_{\theta}z = 0.
$$

The map

$$
J(\mathbf{Q}) \rightarrow \frac{L^\times}{L^\times \mathbf{Q}^\times}
$$

$$
\sum n_P P \mapsto \prod_P (u_{\theta}x(P) + v_{\theta}y(P) + w_{\theta}z(P))^{n_P}.
$$

2. Identification of the image of Galois

Identify $\text{Gal}(g(t))$ as a subgroup of $\text{Sp}_6(\mathbb{F}_2) \subset \mathfrak{S}_{28}$ up to conjugacy. GAP or Magma can list the conjugacy classes of subgroups of $\text{Sp}_6(\mathbb{F}_2)$, and the orbit lengths of the elements.

For the example at hand, we find $\text{Gal}(g(t)) = \text{Sp}_6(\mathbb{F}_2)$; this is as hard as it gets.

3. Cassels kernel

\[
\begin{array}{cccccc}
0 & \rightarrow & J[2](\mathbf{Q}) & \rightarrow & R_{27}^\vee(\mathbf{Q}) & \rightarrow & R_{21}^\vee(\mathbf{Q}) & \rightarrow & H^1(\mathbf{Q}, J[2]) & \rightarrow & H^1(\mathbf{Q}, R_{27}^\vee) \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
& & \frac{J(\mathbf{Q})}{2J(\mathbf{Q})} & \rightarrow & \frac{L^\times}{L^\times \mathbf{Q}^\times} & \rightarrow & \frac{L^\times}{L^\times \mathbf{Q}^\times}
\end{array}
\]
The construction of $R_{28} = (\mathbb{Z}/2\mathbb{Z})^S$ is straightforward. There is a unique $R_{27}$ in $R_{28}$, and a unique $R_{21}$ in $R_{28}$. View $J[2]$ as $R_{27}/R_{21}$. Magma shows that $J[2](\mathbb{Q}), R_{27}^\vee(\mathbb{Q}), R_{21}^\vee(\mathbb{Q})$ are all 0. Therefore
\[
\frac{J(\mathbb{Q})}{2J(\mathbb{Q})} \to L^\times \overset{L^\times2\mathbb{Q}^\times}{\longrightarrow}
\]
is injective.

When we projected, we were working over $\mathbb{Q}$, but to get the ring of integers of $L$, we should use possibly more than one projection over $\mathbb{Z}$.

4. Computing $L(2, S)$

This requires $\text{Cl}(\mathcal{O}_L)$, and GRH is required to verify this computation. In our example, $\text{Cl}(\mathcal{O}_L)$ is trivial (assuming GRH).

5. Local computation

We have
\[
\frac{\#J(\mathbb{Q}_p)}{2J(\mathbb{Q}_p)} = \frac{\#J[2](\mathbb{Q}_p)}{|2|_p^3}.
\]

For $p = 2$, we have
\[
L \otimes \mathbb{Q}_2 = \mathbb{Q}_2 \oplus \mathbb{Q}_2 \oplus (\text{deg 2}) \oplus (\text{deg 8}) \oplus (\text{deg 16}).
\]

One finds
\[
\dim J[2](\mathbb{Q}_2) = 1 \\
\dim R_{27}^\vee(\mathbb{Q}_2) = 4 \\
\dim R_{21}^\vee(\mathbb{Q}_2) = 3.
\]

Thus there is no Cassels kernel. Also, by the formula above,
\[
\dim \frac{J(\mathbb{Q}_2)}{2J(\mathbb{Q}_2)} = 1 - (-3) = 4.
\]

To find enough generators of $\frac{J(\mathbb{Q}_2)}{2J(\mathbb{Q}_2)}$, we intersect $C$ with random lines $\ell$ and hope that $C.\ell$ decomposes over $\mathbb{Q}_2$.

For $p = 5$, we find
\[
\dim J[2](\mathbb{Q}_5) = 1 \\
\dim R_{27}^\vee(\mathbb{Q}_5) = 5 \\
\dim R_{21}^\vee(\mathbb{Q}_5) = 4.
\]

We find
\[
\dim \frac{J(\mathbb{Q})}{2J(\mathbb{Q})} \leq 3.
\]

This completes the proof that $J(\mathbb{Q})$ has rank 3.
Remark 5.1. We did not need the information from the prime 402613, which is lucky since

\[
\begin{align*}
\dim J[2](\mathbb{Q}_{402613}) &= 2 \\
\dim R^{(2)}_{27}(\mathbb{Q}_{402613}) &= 7 \\
\dim R^{(2)}_{21}(\mathbb{Q}_{402613}) &= 6,
\end{align*}
\]

leaving the possibility of a nontrivial Cassels kernel.