TORSORS, DESCENT AND THE BRAUER GROUP - LECTURE BY ALEXEI SKOROBOGATOV

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1. What is a torsor?

Let k be a field of characteristic 0, and let $\overline{k}$ denote an algebraic closure. Let G be an algebraic group over k, and let X be a smooth variety.

Definition 1.1. (First approach.) An X-*torsor* under the group G is a surjective morphism $f : Y \to X$, where Y is equipped with an action of G which preserves the fibres of f, and which is simply transitive on the fibres.

Equivalently, G acts freely on Y, and X is the space of orbits Y/G. Note however that 'freely' should be understood in the scheme-theoretic sense, see Mumford's "Geometric Invariant Theory". The above definition is valid as stated if G is finite, or if G and Y are affine.

Definition 1.2. (Another approach.) A *torsor* is a morphism $f : Y \to X$ together with a group action of G on Y such that "locally in étale topology", Y is isomorphic as a scheme over X to the "trivial torsor" $X \times G$. More precisely, this means that there exists a family of étale (quasi-finite, unramified) maps $\pi_i : U_i \to X$ whose images cover X, such that

$$Y \times_X U_i \cong G \times U_i \,,$$

where each isomorphism respects the action of G.

When $Y \to X$ is finite (equivalently, G is finite), the definition amounts to saying that the map

$$Y \times G \to Y \times_X Y : (y, g) \mapsto (y, gy)$$

is an isomorphism.

1.1. **Examples of torsors.** 1) Let X be a point, $X = \mathrm{Spec}(k)$. "Y is a k-torsor (or $Spec(k)$-torsor) under G" means that G acts on Y in such a way that over $\overline{k}$, this is isomorphic to G acting on itself by translation. For instance Y is a curve of genus 1 and G the Jacobian of Y. Or, G is the 1-dimensional torus given by $x^2 - ay^2 = 1$ and Y is given by $x^2 - ay^2 = c$, for $a, c \in k^\times$.

2) Suppose that Y is a smooth, proper and geometrically irreducible variety, a connected reductive group G acts on Y (freely on an open subset of Y), and there exists a G-linearized ample invertible sheaf on Y. Let Y^s denote the stable points of Y, in the sense of Geometric Invariant Theory. Then there is a morphism $Y^s \to X$ which is an X-torsor under G; the fibres of this morphism are the orbits of G.

3) Given an extension of algebraic groups $1 \to G \to H \to F \to 1$, then $H \to F$ is an F-torsor under the group G.

4) Let E be an elliptic curve. An n-covering $C \to E$ is an E-torsor under $G = E[n]$. Further, if D is an mn-covering of E and the covering map factors as $D \to C \to E$, then $D \to C$ is a C-torsor under $G = E[m]$ (and $C = D/E[m]$).

5) Let X and Y be the affine curves defined by

$$X : y^2 = p_1(x)p_2(x) \text{ for polynomials } p_1 \text{ and } p_2,$$
$$Y : \begin{cases} y_1^2 = \alpha p_1(x) \\ y_2^2 = \dfrac{1}{\alpha} p_2(x) \end{cases} \quad \text{for } \alpha \in k^\times.$$

Then the degree 2 map $Y \to X : (y_1, y_2, x) \mapsto (y_1 y_2, x)$ is a torsor under $G = \mathbb{Z}/2$. Similarly, if

$$X : y^2 - az^2 = p_1(x)p_2(x)$$
$$Y : \begin{cases} y_1^2 - az_1^2 = \alpha p_1(x) \\ y_2^2 - az_2^2 = \dfrac{1}{\alpha} p_2(x) \end{cases}$$

then the obvious map $Y \to X$ is a torsor under $G : y^2 - az^2 = 1$.

2. What is descent?

Torsors are useful in number theory for doing descent.

Given an algebraic group G,

$$\{k\text{-torsors under } G\}/\text{iso} \quad \longleftrightarrow \quad \mathrm{H}^1(k, G) := \mathrm{H}^1_{cont}(\mathrm{Gal}(\overline{k}/k), G(\overline{k}))\,,$$

where $G(\overline{k})$ is given discrete topology.

Let $f : Y \to X$ be a torsor under G, and let $[\alpha] \in \mathrm{H}^1(k, G)$. Then we can form the *twist* of Y by α, denoted $f_\alpha : Y_\alpha \to X$, which can be described as follows. Note that to give a quasi-projective variety over k is the same as to give a variety over $\overline{k}$ together with an action of $\mathrm{Gal}(\overline{k}/k)$ on it. For Y_α, we have $\overline{Y_\alpha} = \overline{Y}$, and the twisted Galois action is $y \mapsto \alpha(\gamma)\gamma y$ for $\gamma \in \mathrm{Gal}(\overline{k}/k)$. The fact that this is a group action amounts to the cocycle condition.

Theorem 2.1. *Let $f : Y \to X$ be a torsor under G. Then*

$$X(k) = \coprod_{[\alpha] \in \mathrm{H}^1(k,G)} f_\alpha(Y_\alpha(k))\,.$$

Proof. Suppose $P \in X(k)$. Then $f^{-1}(P) \to P$ is a k-torsor under G. Take the corresponding class $[\alpha] := [f^{-1}(P)] \in \mathrm{H}^1(k, G)$. Then $f_\alpha^{-1}(P)$ contains a k-rational point. □

Note: If X is projective and k is a number field, then only finitely many of the $f_\alpha(Y_\alpha(k))$ are nonempty.

Warning: If G is not abelian, then in general Y_α is not a torsor under G. (In fact Y_α is a torsor under a certain twisted form of G.)

Define

$$\left(\prod_{\text{all } v} X(k_v)\right)^f := \bigcup_{[\alpha] \in \mathrm{H}^1(k,G)} f_\alpha\left(\prod_{\text{all } v} Y_\alpha(k_v)\right) \subseteq \prod_{\text{all } v} X(k_v)$$

(the set of "adelic points that survive descent with respect to $Y \to X$"). By Theorem 2.1 this subset of $\prod_v X(k_v)$ contains $X(k)$; another such subset is the Brauer set $(\prod_v X(k_v))^{Br}$. Thus we have two "competing approaches" to bounding $X(k)$: descent using torsors (the more classical approach), and the Brauer-Manin obstruction. It has been known for some time that the information that can be obtained from torsors under abelian groups G can also be obtained via the Brauer-Manin obstruction. In fact, Colliot-Thélène and Sansuc showed that for any torsor $f : Y \to X$ under an abelian group G,

$$\left(\prod_v X(k_v)\right)^{Br_1} \subset \left(\prod_v X(k_v)\right)^f$$

where $Br_1X := \ker(BrX \to Br\overline{X})$. For curves and for rational varieties we have $Br_1X = BrX$.

In the late 90's examples were found of descents involving torsors under nonabelian G, which go beyond the Brauer-Manin obstruction.

3. From torsors to the Brauer group

Suppose $Y \to X$ is a torsor under an *abelian* group G, and let $[Y/X]$ be its class in $\mathrm{H}^1_{\text{ét}}(X, G)$. Let $\hat{G}$ denote the character group $\mathrm{Hom}(G, \mathbb{G}_m)$. For each $c \in \mathrm{H}^1(k, \hat{G})$ we obtain an element of BrX via the cup product

$$\mathrm{H}^1_{\text{ét}}(X, G) \times \mathrm{H}^1(k, \hat{G}) \longrightarrow \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m) = BrX\,.$$

Examples: 1) Assume that $Pic\overline{X}$ has no divisible part, e.g. is torsion free. Let G be the group dual to $Pic\overline{X}$, i.e. $\hat{G} = Pic\overline{X}$, and let Y/X be a universal torsor. The cup product with the class $[Y/X]$ defines a homomorphism $\mathrm{H}^1(k, Pic\overline{X}) \to Br_1X$, which is a splitting of the exact sequence

$$0 \to Brk \to Br_1X \to \mathrm{H}^1(k, Pic\overline{X}) \to 0.$$

2) Y/X is multiplication by 2 on an elliptic curve $E : y^2 = (x - c_1)(x - c_2)(x - c_3)$, so that $k(Y) = k(X)(\sqrt{x - c_1}, \sqrt{x - c_2})$. Here $G = \hat{G} = E[2]$. From

$$(a_1, a_2) \in k^\times/(k^\times)^2 \times k^\times/(k^\times)^2 \cong \mathrm{H}^1(k, E[2])\,,$$

one obtains the element $(x - c_1, a_1) + (x - c_2, a_2) \in (BrE)[2]$. In fact, every element of $(BrE)[2]$ with trivial value at the origin is of such a form.

3) (Swinnerton-Dyer + A.S.) Consider (the unique minimal smooth projective model of) the surface

$$X : z^2 = (x - c_1)(x - c_2)(x - c_3)(y - d_1)(y - d_2)(y - d_3)\,.$$

X a K3 surface, more precisely the Kummer surface obtained from the product of two elliptic curves

$$u^2 = (x - c_1)(x - c_2)(x - c_3) \quad \text{and} \quad v^2 = (y - d_1)(y - d_2)(y - d_3)\,.$$

Assume that these curves are not isogenous over $\overline{k}$. Then $Br_1X = Brk$. Using Example 2 one shows that $(Br\overline{X})[2] \simeq (\mathbb{Z}/2)^4$ is generated by the elements

$$\mathcal{A}_{ij} = ((x - c_i)(x - c_3), (y - d_j)(y - d_3)) \quad \text{for } i, j \in \{1, 2\}.$$