p-adic Analysis in Arithmetic Geometry

Problem Sheet 7

Winter Semester 2015/16

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(1) Show that d and ρ as defined in Definition 5.8 are really metrics.

(2) Let $a, b \in \mathbb{C}_p$ with $a \neq 0$. The automorphism given by $x \mapsto ax + b$ of $\mathbb{C}_p[x]$ induces an automorphism of $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ (compare Problem (1) on Sheet 6).

How do $d(\xi, \xi')$ and $\rho(\xi, \xi')$ change under this map?

(3) Let $\mathscr{D}$ be the set of all closed disks $D \subset \mathbb{C}_p$ such that $0 \notin D$. Show that

$$\mathbb{C}_p \setminus \{0\} \longrightarrow \mathbb{C}_p \setminus \{0\}, \qquad z \longmapsto z^{-1},$$

induces an involution of $\mathscr{D}$ that can be extended to a continuous involution of $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p} \setminus \{0\}$ (where we identify a closed disk with a point of type 1, 2 or 3).

How does the big metric behave under this map?

Solutions

(1) Let ξ, ξ', ξ'' be points in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ (with ξ, ξ', ξ'' not of type 1 when considering ρ). The statement for the small metric is

$$d(\xi, \xi'') \leq d(\xi, \xi') + d(\xi', \xi''),$$

which is equivalent to

$$\operatorname{diam}(\xi \vee \xi'') \leq \operatorname{diam}(\xi \vee \xi') + \operatorname{diam}(\xi' \vee \xi'') - \operatorname{diam}(\xi').$$

It suffices to prove this when ξ, ξ', ξ'' are points of type 2 or 3; for points of types 1 and 4, it then follows by continuity. So assume that $\xi = \zeta_{a,r}$, $\xi' = \zeta_{a',r'}$, $\xi'' = \zeta_{a'',r''}$. Then

$$\operatorname{diam}(\xi \vee \xi') = \max\{r, r'', |a - a''|\} \leq \max\{r, r'', |a - a'|, |a' - a''|\}$$

and

$$\operatorname{diam}(\xi \vee \xi') = \max\{r, r', |a - a'|\}, \quad \operatorname{diam}(\xi' \vee \xi'') = \max\{r', r'', |a' - a''|\}, \quad \operatorname{diam}(\xi') = r'.$$

We have

$$\begin{aligned} r &= r + r' - r' \leq \operatorname{diam}(\xi \vee \xi') + \operatorname{diam}(\xi' \vee \xi'') - \operatorname{diam}(\xi') \\ r'' &= r' + r'' - r' \leq \operatorname{diam}(\xi \vee \xi') + \operatorname{diam}(\xi' \vee \xi'') - \operatorname{diam}(\xi') \\ |a - a'| &= |a - a'| + r' - r' \leq \operatorname{diam}(\xi \vee \xi') + \operatorname{diam}(\xi' \vee \xi'') - \operatorname{diam}(\xi') \\ |a' - a''| &= r' + |a' - a''| - r' \leq \operatorname{diam}(\xi \vee \xi') + \operatorname{diam}(\xi' \vee \xi'') - \operatorname{diam}(\xi') \end{aligned}$$

and the claim follows.

For the big metric, the argument is the same, replacing diam by $-c \log \operatorname{diam}$.

(2) Let $a, b \in \mathbb{C}_p$ with $a \neq 0$. The automorphism given by $x \mapsto ax + b$ of $\mathbb{C}_p[x]$ induces an automorphism of $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ (compare Problem (1) on Sheet 6).

How do $d(\xi, \xi')$ and $\rho(\xi, \xi')$ change under this map?

We have to determine how $\operatorname{diam}(\xi)$ changes. Consider $\xi = \zeta_{\alpha,r}$. Its image under $x \mapsto ax + b$ corresponds to the seminorm given by

$$\|f\| = \|f(ax+b)\|_\xi = |f(a(x+\alpha)+b)|_r = |f(ax+(a\alpha+b))|_r = |f(x+(a\alpha+b))|_{|a|r} = \|f\|_{a\alpha+b,|a|r}.$$

(Note that if $f = f_0 + f_1 x + \ldots + f_n x^n$, then

$$|f(ax)|_r = \max_j r^j |f_j a^j| = \max_j (|a|r)^j |f_j| = |f|_{|a|r}.)$$

So the image of $\zeta_{\alpha,r}$ is $\zeta_{a\alpha+b,|a|r}$ and the diameter gets multiplied by $|a|$. This remains true for points of type 4 by continuity. Since the ordering of the points is defined in terms of the seminorms, it is preserved under automorphisms.

If $\varphi \colon x \mapsto ax + b$ denotes the automorphism, then we obtain

$$\begin{aligned} d(\varphi^*(\xi), \varphi^*(\xi')) &= 2 \operatorname{diam}(\varphi^*(\xi \vee \xi')) - \operatorname{diam}(\varphi^*(\xi)) - \operatorname{diam}(\varphi^*(\xi')) \\ &= 2|a| \operatorname{diam}(\xi \vee \xi') - |a| \operatorname{diam}(\xi) - |a| \operatorname{diam}(\xi') = |a| d(\xi, \xi') \end{aligned}$$

and

$$\begin{aligned} \rho(\varphi^*(\xi), \varphi^*(\xi')) &= c\bigl(2 \log \operatorname{diam}(\varphi^*(\xi \vee \xi')) - \log \operatorname{diam}(\varphi^*(\xi)) - \log \operatorname{diam}(\varphi^*(\xi'))\bigr) \\ &= c\bigl(2 \operatorname{diam}(\xi \vee \xi') + 2 \log |a| - \operatorname{diam}(\xi) - \log |a| - \operatorname{diam}(\xi') - \log |a|\bigr) \\ &= \rho(\xi, \xi'). \end{aligned}$$

(3) We first have to show that when $D \in \mathscr{D}$, then $\{z^{-1} : z \in D\}$ is again a disk in $\mathscr{D}$. So let $D = D(a, r)$; since $0 \notin D$, we have $|a| > r$. For $z \in D$ we therefore have $|z| = \max\{|a|, |z - a|\} = |a|$ and so

$$\Big|\frac{1}{z} - \frac{1}{a}\Big| = \Big|\frac{a - z}{za}\Big| = \frac{|a - z|}{|a|^2} \le \frac{r}{|a|^2}\,.$$

So $\{z^{-1} : z \in D\} \subset D(a^{-1}, r/|a|^2)$. Since $z \mapsto z^{-1}$ is an involution (and $(a^{-1})^{-1} = a$, $(r/|a|^2)/|a^{-1}|^2 = r$), the reverse inclusion follows, and we get the involution

$$\mathscr{D} \longrightarrow \mathscr{D}\,, \qquad D(a, r) \longmapsto D(a^{-1}, r|a|^{-2})\,.$$

This immediately gives rise to an involution ϕ on the set of points of type 1, 2 or 3 corresponding to disks not containing 0. We extend it to all points $\neq 0$ by setting $\phi(\zeta_{0,r}) = \zeta_{0,r^{-1}}$ and $\phi(\xi) = \lim_{n\to\infty} \phi(\xi_n)$ when ξ is a type 4 point that is the limit of the decreasing sequence (ξ_n) (note that any type 4 point is a limit of a decreasing sequence of type 2 or 3 points $\zeta_{a,r}$ with $0 \notin D(a, r)$; otherwise 0 would be in the intersection of the corresponding nested disks). It remains to show that ϕ is continuous.

Since the topology is generated by open and complements of closed Berkovich disks, it suffices to show that the preimage (= image) under ϕ of a closed (resp., open) Berkovich disk is closed (resp., open). So let $\mathcal{D}(a, r)$ be a closed Berkovich disk. There are two cases.

(a) $0 \notin \mathcal{D}(a, r)$, equivalently, $|a| > r$. Then for any $\zeta_{a',r'} \in \mathcal{D}(a, r)$ we have $D(a', r') \in \mathscr{D}$ and $|a' - a| \le r$, $r' \le r$. This implies (in a similar way as above) that $|a'^{-1} - a^{-1}| \le r/|a|^2$ and $r'/|a'|^2 = r'/|a|^2 \le r/|a|^2$, so $\phi(\zeta_{a',r'}) \in \mathcal{D}(a^{-1}, r/|a|^2)$. This extends to type 4 points, and we get $\phi(\mathcal{D}(a, r)) \subset \mathcal{D}(a^{-1}, r/|a|^2)$, and then equality by symmetry. The same argument, but using strict inequalities, shows that $\phi(\mathcal{D}(a, r)^-) = \mathcal{D}(a^{-1}, r/|a|^2)^-$.

(b) $0 \in \mathcal{D}(a, r)$, then w.l.o.g. $a = 0$. We claim that $\phi(\mathcal{D}(0, r))$ is the complement of $\mathcal{D}(0, 1/r)^-$ (which by symmetry implies that $\phi(\mathcal{D}(0, r)^-)$ is the complement of $\mathcal{D}(0, 1/r)$). So let $\zeta_{a',r'} \in \mathcal{D}(0, r)$. This means that $|a'| \le r$ and $r' \le r$. There are two cases again.

   (i) $0 \notin D(a', r')$. Then $r' < |a'| \le r$, so $|a'^{-1}| \ge 1/r$ and $\phi(\zeta_{a',r'}) \notin \mathcal{D}(0, 1/r)$.

   (ii) $0 \in D(a', r')$. Then w.l.o.g. $a' = 0$, and we have $r' \le r$. This implies $1/r' \ge 1/r$, hence $\phi(\zeta_{0,r'}) = \zeta_{0,1/r'} \notin \mathcal{D}(0, 1/r)$.

   Now consider $\zeta_{a',r'} \notin \mathcal{D}(0, r)$. Then $|a'| > r$ or $r' > r$. We again distinguish two cases.

   (i) $0 \notin D(a', r')$. Then $r' < |a'|$. If $r' > r$, then also $|a'| > r$, so we can assume the latter. We then have $|a'^{-1}| < 1/r$ and $r'/|a'|^2 < 1/|a'| < 1/r$, so $\phi(\zeta_{a',r'}) \in \mathcal{D}(0, 1/r)^-$.

   (ii) $0 \in D(a', r')$. Then w.l.o.g. $a' = 0$, and we have $r' > r$. This implies $1/r' < 1/r$, hence $\phi(\zeta_{0,r'}) = \zeta_{0,1/r'} \in \mathcal{D}(0, 1/r)^-$.

   Taking everything together shows $\phi(\mathcal{D}(0, r)) = \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p} \setminus \mathcal{D}(0, 1/r)^-$.

Note that for points in $\mathscr{D}$ the order relation is preserved by ϕ. Since it is easy to see that $\rho(\phi(\zeta_{a,r}), \phi(\zeta_{a,r'})) = \rho(\zeta_{a,r}, \zeta_{a,r'})$, it follows that ρ is invariant under ϕ on $\mathscr{D}$ and therefore on all of $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p}$ by continuity.