\[ x^2 + y^3 = z^7 \]

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The Result

The following is now a 100% theorem.
(Last year in Oberwolfach, it was only 90%.)

**Theorem.**
The complete list of primitive integral solutions of

\[ x^2 + y^3 = z^7 \]

is given by

\[
(±1,−1,0), \quad (±1,0,1), \quad ±(0,1,1)
\]
\[
(±3,−2,1), \quad (±71,−17,2), \quad (±2213459,1414,65)
\]
\[
(±15312283,9262,113), \quad (±21063928,−76271,17)
\]
Some General Theory

We consider the Generalized Fermat Equation

\[ x^p + y^q = z^r. \]

Let

\[ \chi_{p,q,r} = \chi = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1. \]

Let \( P \subset \mathbb{P}^2 \) be the line \( x + y = z \).

There is a branched Galois covering \( X \xrightarrow{\pi} P \), defined over \( \mathbb{Q} \), ramified above \((0 : 1 : 1), (1 : 0 : 1), (1 : -1 : 0)\) with ramification index \( p, q, r \), respectively.

- \( \chi > 0 \implies X \cong \mathbb{P}^1 \)
- \( \chi = 0 \implies X \) is an elliptic curve
- \( \chi < 0 \implies \text{genus}(X) \geq 2 \)

Note that \( \chi_{2,3,7} = -1/42 \) is closest to zero from below.
A General Theorem

Let \( S_{p,q,r} = \{(a^p : b^q : c^r) : a, b, c \in \mathbb{Z}, \gcd(a, b, c) = 1, a^p + b^q = c^r\} \subset P(\mathbb{Q}) \).

**Theorem** (Darmon-Granville).
There is a number field \( K \) such that \( \pi(X(K)) \supset S_{p,q,r} \).

**Corollary.**
If \( \chi < 0 \), then \( S_{p,q,r} \) is finite.

**Theorem (Variant).**
There are finitely many twists \( X_j \xrightarrow{\pi_j} P \) of \( X \xrightarrow{\pi} P \) such that \( \bigcup_j \pi_j(X_j(\mathbb{Q})) \supset S_{p,q,r} \).
(These twists are all unramified outside \( pqr \).)

More precisely:
Let \( Y_j(\mathbb{Q}) \subset X_j(\mathbb{Q}) \) be the points satisfying certain conditions mod powers of the primes dividing \( pqr \). Then \( S_{p,q,r} = \bigsqcup_j \pi_j(Y_j(\mathbb{Q})) \).
A Side Remark

Instead of $P$, one should consider $x^p + y^q = z^r$ as a curve $X_{p,q,r}$ in a weighted $\mathbb{P}^2$.

Note that $S_{p,q,r} \cong X_{p,q,r}(\mathbb{Z})$ (modulo signs).

$X_{p,q,r}$ is $P$ with the points $(0 : 1 : 1)$, $(1 : 0 : 1)$, $(1 : -1 : 0)$ replaced with $1/p$, $1/q$, $1/r$ times a point, respectively. (Assuming $p, q, r$ coprime in pairs.)

Then $X \overset{\pi}{\longrightarrow} X_{p,q,r}$ is unramified
and $\chi$ is the Euler characteristic of $X_{p,q,r}$.

This explains why one can do descent using $\pi$.

We also find $2 - 2 \text{genus}(X) = \deg(\pi) \chi$. 
Overall Strategy for (2,3,7)

In our special case, $X$ can be taken to be the *Klein Quartic*

$$X : x^3 y + y^3 z + z^3 x = 0$$

Note that $X \cong X(7)$ (the modular curve).

1. Find the relevant twists $X_j$ of $X$ explicitly.
2. Find $X_j(\mathbb{Q})$ (or at least $Y_j(\mathbb{Q})$).

Step 1 uses modular arguments (plus a separate consideration for reducible 7-torsion).

Step 2 uses descent on the Jacobians of the $X_j$ and Chabauty, plus a Brauer-Manin obstruction argument for one curve where Chabauty fails.
Step 1: Quick Overview

Given a solution \( a^2 + b^3 = c^7 \), consider

\[
E_{(a,b,c)} : y^2 = x^3 + 3bx - 2a.
\]

The usual arguments show that \( E_{(a,b,c)} \) is semistable outside \( \{2, 3\} \) and that, if \( E_{(a,b,c)}[7] \) is irreducible, \( E_{(a,b,c)}[7] \cong E[7] \) (up to quadratic twist) for \( E \) out of a list of 13 elliptic curves.

By work of Kraus and Halberstadt, we can write down explicit equations of twists \( X_E(7) \) and \( X_{\overline{E}}(7) \) classifying \( E' \) such that \( E'[7] \cong E[7] \).

A more direct argument produces several hundreds of twists arising from the case when \( E_{(a,b,c)}[7] \) is reducible.

For many of these curves, local considerations show \( Y_j(\mathbb{Q}) = \emptyset \). We are left with just 10 twists \( X_j \).
The Twists

\[ X_1 : 6x^3y + y^3z + z^3x = 0 \]
\[ X_2 : 3x^3y + y^3z + 2z^3x = 0 \]
\[ X_3 : 3x^3y + 2y^3z + z^3x = 0 \]
\[ X_4 : 7x^3z + 3x^2y^2 - 3xyz^2 + y^3z - z^4 = 0 \]
\[ X_5 : -2x^3y - 2x^3z + 6x^2yz + 3xy^3 - 9xy^2z + 3xyy^2z - xz^3 + 3y^3z - yz^3 = 0 \]
\[ X_6 : x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + 18xyz^2 + 9y^2z^2 - 9z^4 = 0 \]
\[ X_7 : -3x^4 - 6x^3z + 6x^2y^2 - 6x^2yz + 15x^2z^2 - 4xy^3 \\
- 6xyz^2 - 4xz^3 + 6y^2z^2 - 6yz^3 = 0 \]
\[ X_8 : 2x^4 - x^3y - 12x^2y^2 + 3x^2z^2 - 5xy^3 - 6xy^2z \\
+ 2xz^3 - 2y^4 + 6y^3z + 3y^2z^2 + 2yz^3 = 0 \]
\[ X_9 : 2x^4 + 4x^3y - 4x^3z - 3x^2y^2 - 6x^2yz + 6x^2z^2 \\
- xy^3 - 6xyz^2 - 2y^4 + 2y^3z - 3y^2z^2 + 6yz^3 = 0 \]
\[ X_{10} : x^3y - x^3z + 3x^2z^2 + 3xy^2z + 3xy^2z + 3xz^3 - y^4 \\
+ y^3z + 3y^2z^2 - 12yz^3 + 3z^4 = 0 \]
The Points

We find the following rational points on these curves.

\[ X_1 : (1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : -1 : 2) \]
\[ X_2 : (1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : -1), (1 : -2 : -1) \]
\[ X_3 : (1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : -1) \]
\[ X_4 : (1 : 0 : 0), (0 : 1 : 0), (0 : 1 : 1) \]
\[ X_5 : (1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1) \]
\[ X_6 : (0 : 1 : 0), (1 : -1 : 0), (0 : 1 : 1), (0 : 1 : -1) \]
\[ X_7 : (0 : 1 : 0), (0 : 0 : 1), (0 : 1 : 1) \]
\[ X_8 : (0 : 0 : 1), (2 : -1 : 0) \]
\[ X_9 : (0 : 0 : 1), (1 : 1 : 0) \]
\[ X_{10} : (1 : 0 : 0), (1 : 1 : 0) \]

The boxed points lead to nontrivial primitive solutions.
Step 2: Overview

It remains to prove that there are no other points in $X_j(\mathbb{Q})$, or at least in $Y_j(\mathbb{Q})$.

The only good and widely applicable way of doing this is Chabauty’s method.

Chabauty’s method works when the rank of $J_j(\mathbb{Q}) = \text{Jac}(X_j)(\mathbb{Q})$ is less than 3.

So we first have to determine this rank and find generators of a finite index subgroup of $J_j(\mathbb{Q})$. 
2-Descent on $J_j$

For general plane quartics, descent is infeasible.

*However,* our curves are very special: They are twists of $X$, so they have a large automorphism group:

$$\text{Aut}(X_j) \cong \text{Aut}(X) \cong \text{PSL}(2, \mathbb{F}_7)$$

In fact, $X_{2,3,7} \cong X_j/\text{Aut}(X_j)$ via $\pi_j$.

This results in a special geometry.
A Feature of the Klein Quartic

The Klein Quartic
A Feature of the Klein Quartic

A flex point
A Feature of the Klein Quartic

A flex point with its tangent
A Feature of the Klein Quartic

We get another flex point!
A Feature of the Klein Quartic

In the end, we have a “flex triangle”
2-Descent: Structure of $J_j[2]$

Let $T_i$ (1 ≤ $i$ ≤ 8) be the eight degree 3 divisors corresponding to the flex triangles on $J_j$.

**Lemma.**
Let $V = \mathbb{F}_2 \cdot T_1 \oplus \cdots \oplus \mathbb{F}_2 \cdot T_8$; this is a Galois module. Consider

$$\mathbb{F}_2 \xrightarrow{\alpha} V \xrightarrow{\beta} \mathbb{F}_2$$

where $\alpha(1) = T_1 + \cdots + T_8$ and $\beta(a_1T_1 + \cdots + a_8T_8) = a_1 + \cdots + a_8$. Then $J_j[2] \cong \ker(\beta)/\im(\alpha)$ as a Galois module.

This is completely analogous to hyperelliptic genus 3 curves (where the $T_i$ are replaced by the Weierstrass points).

Hence we can transfer the 2-descent method from hyperelliptic curves to our Klein Quartic twists.
2-Descent: More detail

To carry out the 2-descent, we need a suitable function $F$ on $X_j$.

Fix a basepoint $P_0 \in X_j(\mathbb{Q})$.
Fit a cubic $F_i$ through $3P_0 + 2T_i$.
Then $\text{div}(F_i) = 2T_i + (3P_0 + R)$, where $R$ is rational and independent of $i$.

Usually, all the $T_i$ are conjugate. Assume this.
Set $T = T_1$ and $K = \mathbb{Q}(T)$ the corresponding octic number field.
Then $F = F_1/z^3$ is defined over $K$ and induces a homomorphism

$$F : J_j(\mathbb{Q})/2J_j(\mathbb{Q}) \longrightarrow \ker\left(N_{K/\mathbb{Q}} : K^\times/(K^\times)^2 \mathbb{Q}^\times \to \mathbb{Q}^\times/(\mathbb{Q}^\times)^2\right)$$

with kernel of order 2 generated by $[R - 3P_0]$ and image contained in the subgroup unramified outside \{2, 3, 7\}.

We compute the 2-Selmer group in the usual way and get a bound on the rank of $J_j(\mathbb{Q})$. 
2-Descent and Chabauty: Results

Our 2-descent, applied to $X_j$, gives the following.

**Proposition.**
For all $1 \leq j \leq 10$, the subgroup of $J_j(\mathbb{Q})$ generated by divisors supported in the known rational points has finite index.

We have (with $r_j = \text{rank } J_j(\mathbb{Q})$)

\[
\begin{align*}
    r_1 &= r_2 = r_3 = 1 \\
    r_4 &= r_6 = r_7 = r_8 = r_9 = r_{10} = 2 \\
    r_5 &= 3
\end{align*}
\]

Knowing this, we can use Chabauty on nine of the curves:

**Proposition.** For $1 \leq j \leq 10$, $j \neq 5$, the listed points exhaust $X_j(\mathbb{Q})$. 
The Last Curve

It remains to deal with the last curve:

\[ X_5 : -2x^3y - 2x^3z + 6x^2yz + 3xy^3 - 9xy^2z + 3xyz^2 - xz^3 + 3y^3z - yz^3 = 0 \]

Here, we have

\[ Y_5(\mathbb{Q}) = \{ P \in X_5(\mathbb{Q}) : P \equiv (0 : 1 : 0) \mod 3, \]
\[ P \equiv (1 : 0 : 0) \text{ or } (1 : 1 : 1) \mod 2 \} \]

The known points in \( X_5(\mathbb{Q}) \),

\( (1 : 0 : 0), \quad (0 : 1 : 0), \quad (0 : 0 : 1), \quad (1 : 1 : 1), \)

all violate one of the conditions.

**Can we show that** \( Y_5(\mathbb{Q}) \) **is empty?**
The Idea

We have $J_5(\mathbb{Q}) \cong \mathbb{Z}^3$. Let $P_1, P_2, P_3$ be generators.

We have a map

$$\iota : X_5(\mathbb{Q}) \ni P \mapsto [P - P_0] \in J_5(\mathbb{Q}) \xrightarrow{\cong} \mathbb{Z} \cdot P_1 \oplus \mathbb{Z} \cdot P_2 \oplus \mathbb{Z} \cdot P_3.$$

So $\iota(P) = \iota_1(P)P_1 + \iota_2(P)P_2 + \iota_3(P)P_3$.

Let us find conditions on these coefficients!

For a prime $p$, we can find finite groups $G_p$ such that

$$\phi_p : J_5(\mathbb{Q}) \hookrightarrow J_5(\mathbb{Q}_p) \longrightarrow G_p$$

On the other hand, $Y_5(\mathbb{Q}) \hookrightarrow Y_5(\mathbb{Q}_p) \ (= X_5(\mathbb{Q}_p) \text{ if } p \neq 2, 3)$ maps to a certain subset $S_p \subset G_p$.

Now $\iota(Y_5(\mathbb{Q})) \subset \bigcap_p \phi_p^{-1}(S_p)$,

and $\phi_p(n_1P_1 + n_2P_2 + n_3P_3) \in S_p$ gives congruence conditions on $n_1, n_2, n_3$. 
Choosing $G_p$

We need to bring in the conditions at 2 and 3, so we need to consider suitable $G_2$ and $G_3$.

Note that 2 and 3 are primes of bad reduction, so some work is required.

For additional $p$ of good reduction, we can simply take $G_p = J_5(\mathbb{F}_p)$ or a quotient of it; then $S_p = \nu(X_5(\mathbb{F}_p))$ is the image of the points mod $p$. 
How to get $G_3$

We need to find the structure of $J_5(\mathbb{Q}_3)$.

To do this, we compute a minimal regular model of $X_5$ over $\mathbb{Z}_3$. The special fiber looks like this:

From this, we find $\Phi_3 \cong \mathbb{Z}/7\mathbb{Z}$; we set $G_3 = \Phi_3$ and get $n_1 + 3n_3 \equiv 1$ mod 7.
How to get $G_2$

The minimal regular model of $X_5$ over $\mathbb{Z}_2$ has special fiber

From this, we find $\Phi_2 \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$; we set $G_2 = \Phi_2$ and get $n_1 + n_2 \equiv 0 \text{ mod } 4$ and $n_3 \equiv 0$ or $1 \text{ mod } 4$.

We need some more primes to “connect” this mod 7 and mod 4 information.
Auxiliary Primes

Besides

\[ J_5(\mathbb{Q}_2) \rightarrow \Phi_2 \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \]
\[ J_5(\mathbb{Q}_3) \rightarrow \Phi_3 \cong \mathbb{Z}/7\mathbb{Z} \]

we find

\[ J_5(\mathbb{Q}_{13}) \rightarrow J_5(\mathbb{F}_{13}) \rightarrow \mathbb{Z}/14\mathbb{Z} \]
\[ J_5(\mathbb{Q}_{23}) \rightarrow J_5(\mathbb{F}_{23}) \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \]
\[ J_5(\mathbb{Q}_{97}) \rightarrow J_5(\mathbb{F}_{97}) \rightarrow \mathbb{Z}/14\mathbb{Z} \times \mathbb{Z}/14\mathbb{Z} \times \mathbb{Z}/14\mathbb{Z} \]

and computing the image of \( X_5(\mathbb{F}_p) \) in the group on the right, we finally obtain contradictory conditions on \( n_1, n_2, n_3 \) mod 14.

**Proposition.**

\( Y_5(\mathbb{Q}) \) is empty.