Uniform bounds on the number of rational points on hyperelliptic curves of low Mordell-Weil rank

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- stronger than M-Z: effective and explicit.
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- proof uses 2-adic properties of division polynomials.

Motivation (1)

Theorem. (‘Mordell’s Conjecture’, Faltings 1983)
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A geometric variant:

Theorem. ('Mordell-Lang Conjecture', Faltings 1994)
Let C be a curve of genus $g \geq 2$ over $\mathbb{C}$,
with an embedding $i \colon C \to J$ into its Jacobian.
Let $\Gamma \subset J(\mathbb{C})$ be a subgroup of finite rank r.
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Theorem.
The Zilber-Pink Conjecture for families of abelian varieties
implies a positive answer.

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For example, we can take

$$B(\mathbb{Q}_3, g, r) = 8(r + 4)(g - 1) + \max\{1, 4r\} \cdot g\,.$$

Bound for the number of rational points

Taking $K = \mathbb{Q}_3$, C defined over $\mathbb{Q}$ and $\Gamma = J(\mathbb{Q})$, we obtain:

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More generally, there is a bound $N(d, g, r)$ (for $r \leq g - 3$) such that
for all number fields K with $[K : \mathbb{Q}] \leq d$ and
all hyperelliptic curves of genus g over K with $J(K)$ of rank r, we have

$$\#C(K) \leq N(d, g, r)\,.$$

Chabauty-Coleman

A method pioneered by Chabauty and developed further by Coleman gives:

Theorem. (Coleman 1985, Stoll 2006, Katz & Zureick-Brown 2013)
Let C be a curve of genus g over $\mathbb{Q}_p$ with $p > 2$,
with (minimal) proper regular model $\mathcal{C}$ over $\mathbb{Z}_p$ (with special fiber $\mathcal{C}_s$).
Let $i \colon C \to J$ be an embedding given by a base-point $P_0 \in C(\mathbb{Q}_p)$
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Problem: $\#\mathcal{C}_s^{\mathrm{smooth}}(\mathbb{F}_p)$ cannot be bounded in terms of g and p!

Sketch of proof (1)

We have canonical isomorphisms $\Omega^1_C(\mathbb{Q}_p) \cong \Omega^1_J(\mathbb{Q}_p) \cong T_0 J(\mathbb{Q}_p)^*$
and the p-adic abelian logarithm $\log_J \colon J(\mathbb{Q}_p) \to T_0 J(\mathbb{Q}_p)$.

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We obtain a pairing $\Omega^1_C(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \to \mathbb{Q}_p, \quad (\omega, P) \mapsto \langle \omega, P \rangle$
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Note that $i^{-1}(\Gamma) \subset \{P \in C(\mathbb{Q}_p) : \forall \omega \in V_\Gamma \colon \langle \omega, i(P) \rangle = 0\}$.

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Let $\rho\colon C(\mathbb{Q}_p) \to \mathcal{C}_S^{\text{smooth}}(\mathbb{F}_p)$ be the reduction map.
Fix a residue disk $D = \rho^{-1}(\bar{P})$.
There is an analytic isomorphism $\varphi\colon \{\xi \in \mathbb{Q}_p : |\xi|_p < 1\} \to D$.

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Then

$$\langle \omega, i(\varphi(\tau)) \rangle = \int_{P_0}^{\varphi(\tau)} \omega = \int_{P_0}^{\varphi(0)} \omega + \int_0^{\tau} w(t)\,dt = c + \ell(\tau)\,.$$

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Considering the Newton Polygon of w and ℓ, one shows that

$$\#\{P \in D : \langle \omega, i(P) \rangle = 0\} \le 1 + \nu(\omega, D) + \left\lfloor \frac{\nu(\omega, D)}{p-2} \right\rfloor,$$

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Picking an 'optimal' ω for each D and summing gives the result.

How to fix the problem

The only source for the unboundedness of $\#\mathcal{C}_s^{\mathrm{smooth}}(\mathbb{F}_p)$ is arbitrarily long chains of $\mathbb{P}^1$'s in $\mathcal{C}_s$ (Artin & Winters 1971).

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So we can cover $C(\mathbb{Q}_p)$ by $\ll pg$ disks D
coming from points in $\mathcal{C}_s^{\mathrm{smooth}}(\mathbb{F}_p)$ outside chains
and by $\ll g$ annuli A coming from the chains.

The key result

Proposition.

For each annulus A there is a subspace $V_A \subset \Omega^1_C(\mathbb{Q}_p)$ with $\operatorname{codim} V_A \leq 2$ such that for $0 \neq \omega \in V_A$, if C is hyperelliptic,

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The pull-back of ω is $\quad \varphi^*\omega = w(t)\,dt = d\ell(t) + c(\omega)\dfrac{dt}{t}$

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Set $V_A = \{\omega \in \Omega^1_C(\mathbb{Q}_p) : a(\omega) = c(\omega) = 0\}$.

End of proof

Since $r \leq g - 3$, we have $V_\Gamma \cap V_A \neq \{0\}$ for all A,
so we can always pick a suitable $\omega \neq 0$ to get a bound

$$\#\left(i^{-1}(\Gamma) \cap A\right) \leq \nu(\omega, A) + \left\lfloor \frac{\nu(\omega, A)}{p-2} \right\rfloor .$$

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Taking the 'optimal' ω for each annulus and for each disk and summing, we obtain the desired bound, which is of the shape

$$\ll (p + r + 1)g .$$

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Since $r \le g - 3$, we have $V_\Gamma \cap V_A \neq \{0\}$ for all A,
so we can always pick a suitable $\omega \neq 0$ to get a bound

$$\#\bigl(i^{-1}(\Gamma) \cap A\bigr) \le \nu(\omega, A) + \left\lfloor \frac{\nu(\omega, A)}{p - 2} \right\rfloor .$$

Taking the 'optimal' ω for each annulus and for each disk and summing,
we obtain the desired bound, which is of the shape

$$\ll (p + r + 1)g .$$

For a general p-adic field with ramification index $e < p - 1$
and residue field of size q, the bound takes the shape

$$\ll \bigl(q + e(r + 1)\bigr)g .$$