COARSE LENGTH CAN BE UNBOUNDED IN 3-STEP NILPOTENT LIE GROUPS

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Abstract. In [1], we remarked that, contrary to 2-step nilpotent simply connected Lie groups, in 3-step nilpotent simply connected Lie groups it is possible that '$\mathbb{R}$-words' in the given generators cannot be replaced by an equally long $\mathbb{R}$-word representing the same group element and having a bounded number of direction changes. a given set of generators. In this note, we present an example.

1. Introduction

Let G be a (connected and) simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Via the exponential map, we can identify $\mathfrak{g}$ with G. We fix a set $S = \{x_1, \ldots, x_n\}$ of generators of $\mathfrak{g}$. Let $S_{\mathbb{R}} = \{x^t : x \in S, t \in \mathbb{R}\}$. We consider words of the form $w = x_{i_1}^{t_1} x_{i_2}^{t_2} \cdots x_{i_k}^{i_k} \in S_{\mathbb{R}}^*$, i.e., words over the alphabet $S_{\mathbb{R}}$. We will call such words $\mathbb{R}$-*words over* S. We define the *length* of w to be

$$\ell(w) = \sum_{j=1}^{k} |t_j|$$

and the *coarse length* of w to be

$$\ell'(w) = k\,.$$

We also let $|w|$ denote the result of evaluating w in G. In this context, we set $|x^t| = tx$ for $t \in \mathbb{R}$ and $x \in G = \mathfrak{g}$, where the product with t is taken in the Lie algebra. Then we can define the *length* and *coarse length* of an element g of G by

$$\ell(g) = \inf\{\ell(w) : w \in S_{\mathbb{R}}^*, |w| = g\} \in \mathbb{R}_{\geq 0}$$

and

$$\ell'(g) = \inf\{\ell'(w) : w \in S_{\mathbb{R}}^*, |w| = g, \ell(w) = \ell(g)\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}\,.$$

In our paper [1, Lemma 3.3], we show that in the case that G is 2-step nilpotent, there is a number N such that for every $\mathbb{R}$-word $w \in S_{\mathbb{R}}^*$, there is another $\mathbb{R}$-word $w' \in S_{\mathbb{R}}^*$ satisfying $|w'| = |w|$, $\ell(w') \leq \ell(w)$, and $\ell'(w') \leq N$. This implies that $\ell'(g) \leq N$ for all $g \in G$.

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We then remark that this assertion is not true in general for 3-step nilpotent simply connected Lie groups. Since various people have asked for more precise information, we give an explicit example in this note.

2. The Example

Consider the free 3-step nilpotent Lie algebra $\mathfrak{g}$ on two generators X and Y. Then

$$X, Y, \tfrac{1}{2}[X,Y], \tfrac{1}{12}[X,[X,Y]], \tfrac{1}{12}[Y,[Y,X]]$$

is a basis of $\mathfrak{g}$, which we use to identify $\mathfrak{g}$ with $\mathbb{R}^5$. We can also identify $\mathfrak{g}$ with the corresponding Lie group G (i.e., we take the exponential and logarithm maps to be the identity on the underlying set); then the group law in G is given by

$$a \cdot b = a + b + \tfrac{1}{2}[a,b] + \tfrac{1}{12}[a,[a,b]] + \tfrac{1}{12}[b,[b,a]]$$

in terms of the Lie bracket on $\mathfrak{g}$.

Now we consider elements of G that can be written in the form

$$X^{s_1} Y^{t_1} X^{s_2} Y^{t_2} \cdots X^{s_N} Y^{t_N}$$

with s_i, t_j nonnegative and $\sum_i s_i = \sum_j t_j = 1$. In terms of the basis above, the result will be of the form $(1,1,u,v,w)$. Let M be the subset of $\mathbb{R}^3$ consisting of all triples (u,v,w) that can be obtained in this way.

Lemma 1. *The point $(0,0,0)$ is in the closure of M, but not in M itself.*

Proof. Let Σ be the set of $\mathbb{R}$-words

$$X^{s_1} Y^{t_1} X^{s_2} Y^{t_2} \cdots X^{s_N} Y^{t_N}$$

with s_i, t_j nonnegative and $\sum_i s_i = \sum_j t_j = 1$ as above, and let Σ' be its image in the group G (which corresponds to $\{(1,1)\} \times M$). For $0 \le t \le 1$, define maps $a_t : \Sigma \to \Sigma$ and $b_t : \Sigma \to \Sigma$ by

$$a_t(w) = w(X^{1-t}, Y)X^t \qquad \text{and} \qquad b_t(w) = w(X, Y^{1-t})Y^t$$

(i.e., for a_t, we replace each X^s in w by $X^{s(1-t)}$ and multiply the result from the right by X^t; similarly for b_t). Since each element of Σ can be obtained by a succession of applications of these maps to one of the words XY or YX, Σ' is the smallest subset of G containing XY and YX that is invariant under these maps. Therefore M is the smallest subset of $\mathbb{R}^3$ containing $(\pm 1, 1, 1)$ that is invariant under the maps induced by a_t and b_t ($0 \le t \le 1$) on $\mathbb{R}^3$, which are

$$\begin{aligned} a_t(u,v,w) &= ((1-t)u - t, (1-t)^2 v - 3t(1-t)u + t(2t-1), (1-t)w + t) \\ b_t(u,v,w) &= ((1-t)u + t, (1-t)v + t, (1-t)^2 w + 3t(1-t)u + t(2t-1))\,. \end{aligned}$$

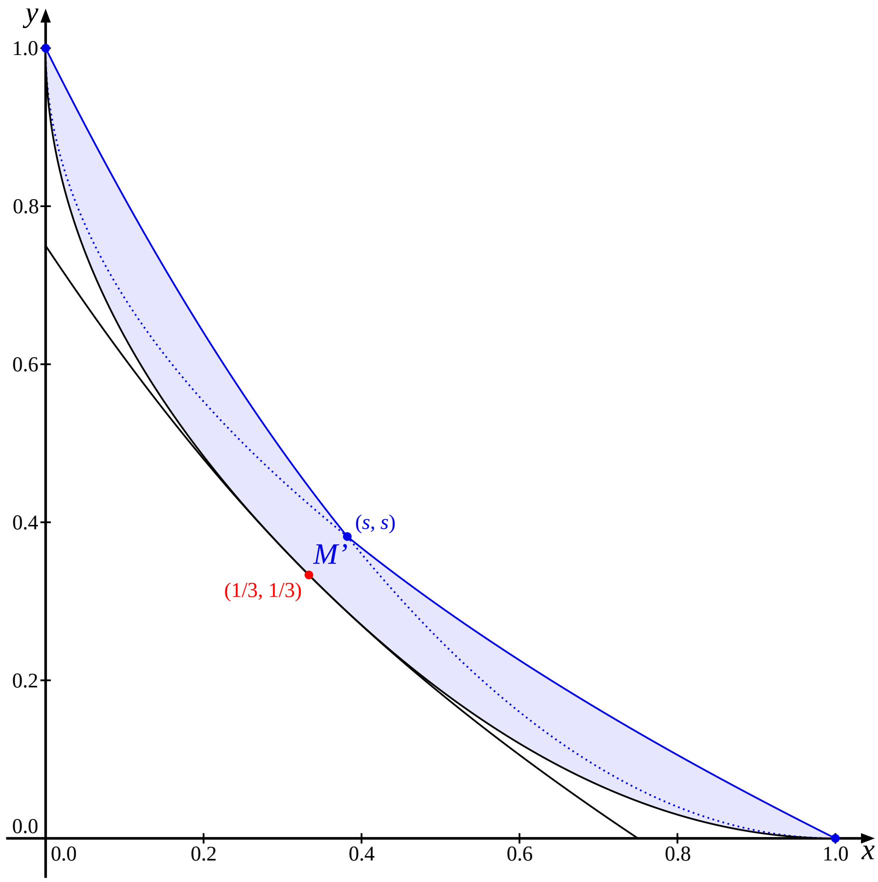

FIGURE 1. The set M'. Here $s = (3 - \sqrt{5})/2$.

We now project to the new coordinates

$$x = \frac{v + 3u + 2}{6} \qquad \text{and} \qquad y = \frac{w - 3u + 2}{6}.$$

In terms of x and y, we get (again abusing notation slightly)

$$a_t(x, y) = ((1-t)^2 x, (1-t)y + t)$$
$$b_t(x, y) = ((1-t)x + t, (1-t)^2 y).$$

Let M' be the image of M under this projection. Then M' is the smallest subset of $\mathbb{R}^2$ containing $(1, 0)$ and $(0, 1)$ that is invariant under a_t and b_t as above. It is then not hard to show that $M' \setminus \{(1, 0), (0, 1)\}$ is given by the following inequalities:

$$x < 1 \quad \text{and} \quad y < 1 \quad \text{and} \quad 4x > 3(1-y)^2 \quad \text{and} \quad 4y > 3(1-x)^2$$
$$\text{and} \quad (x \leq (1-y)^2 \quad \text{or} \quad y \leq (1-x)^2).$$

(One shows that the inequalities are satisfied for $a_t(1,0)$ when $t < 1$, similarly for $b_t(0,1)$ when $t < 1$, and that they are preserved under a_t and b_t when $t < 1$. This shows the inclusion relevant for us. The other one follows from the fact that all points on the diagonal $x = y$ that satisfy the inequalities can be reached; all other points in the domain can be reached from there.) See Figure 2, where $s = (3 - \sqrt{5})/2$.

In particular, $(1/3, 1/3)$ is not in M', so $(0,0,0)$ is not in M.

On the other hand,

$$X + Y = \lim_{n\to\infty} (X^{1/n} Y^{1/n})^n ,$$

so $(0,0,0)$ is in the closure of M. In other words, $X + Y$ is in the closure of Σ', but not in Σ'. □

This implies that the coarse length is unbounded on the set of elements of G in the form given above. (Since otherwise, M would have to be compact, compare [1].) Note that the length of a word as above cannot be shortened to be less than 2, since otherwise we could not reach the projection $(1,1)$ in G/G'.

Corollary 2. *Let $g_0 \in G$ correspond to $(1,1,0,0,0)$ with respect to the basis above, i.e., $g_0 = X + Y$ as an element of $\mathfrak{g}$. Then $\ell'(g_0) = \infty$. (I.e., there is no geodesic joining the origin to g_0 consisting of finitely many steps in directions given by the generators.)*

Proof. It is clear that $\ell(g_0) \geq 2$, since for the element $g \in G$ corresponding to (s,t,u,v,w), we always have $\ell(g) \geq |s| + |t|$. We claim that $\ell(g_0) = 2$. This follows from the facts that ℓ is continuous on G, that ℓ takes the value 2 on Σ' and that $g_0 \in \bar{\Sigma}'$. Now, since g_0 is not in Σ', by definition of Σ' there is no $\mathbb{R}$-word w of length 2 that represents g_0, hence the set in the definition of $\ell'(g_0)$ is empty. □

Corollary 3. *For every N, there exists an $\mathbb{R}$-word $w \in \Sigma$ such that there is no $\mathbb{R}$-word w' satisfying $\ell(w') \leq 2 = \ell(w)$, $|w'| = |w|$ and $\ell'(w') \leq N$.*

Proof. If $g \in \Sigma'$, then any $\mathbb{R}$-word representing g must have length ≥ 2, and any $\mathbb{R}$-word of length 2 representing g must be in Σ. The previous corollary implies that as we let g approach g_0 within Σ', the minimal coarse length of an $\mathbb{R}$-word in Σ representing g tends to infinity. □

References

[1] M. Stoll: *Asymptotics of the growth of 2–step nilpotent groups,* J. London Math. Soc. **58**, 38–48 (1998).

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