THE GENERALISED FERMAT EQUATION $x^2 + y^3 = z^{15}$

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ABSTRACT. We determine the set of primitive integral solutions to the generalised Fermat equation $x^2 + y^3 = z^{15}$. As expected, the only solutions are the trivial ones with $xyz = 0$ and the non-trivial pair $(x, y, z) = (\pm 3, -2, 1)$.

1. INTRODUCTION

Let p, q, $r \in \mathbb{Z}_{\geq 2}$. The equation

$$(1) \qquad x^p + y^q = z^r$$

is known as the *generalised Fermat equation* (or the *Fermat-Catalan* equation) with signature (p, q, r). As in Fermat's Last Theorem, one is interested in non-trivial primitive integer solutions. An integer solution (x, y, z) is said to be *non-trivial* if $xyz \neq 0$, and *primitive* if x, y, z are coprime. Let $\chi = p^{-1} + q^{-1} + r^{-1}$. The parametrisation of non-trivial primitive integer solutions for (p, q, r) with $\chi \geq 1$ has been completed [10]. The generalised Fermat Conjecture [8, 9] is concerned with the case $\chi < 1$. It states that the only non-trivial primitive solutions to (1) with $\chi < 1$ are (up to sign and permutation)

$$1 + 2^3 = 3^2, \quad 2^5 + 7^2 = 3^4, \quad 7^3 + 13^2 = 2^9, \quad 2^7 + 17^3 = 71^2,$$

$$3^5 + 11^4 = 122^2, \quad 17^7 + 76271^3 = 21063928^2, \quad 1414^3 + 2213459^2 = 65^7,$$

$$9262^3 + 15312283^2 = 113^7, \quad 43^8 + 96222^3 = 30042907^2, \quad 33^8 + 1549034^2 = 15613^3.$$

The generalised Fermat Conjecture has been established for many signatures (p, q, r), including for several infinite families of signatures, see for example [17] for a short overview, and [7, Chapter 17] for a relatively recent survey.

There is an abundance of solutions for the generalised Fermat equation with signatures $(2, 3, n)$, and so this subfamily is particularly interesting. The condition $\chi < 1$ within this family coincides with $n \geq 7$. The cases $n = 7$, 8, 9 are solved respectively in [16], [3], [4]. The case $n = 10$ is solved independently in [2] and [18]. Every $n > 5$ is divisible by 6, 8, 9, 10, 15, 25 or an odd prime $p \geq 7$. Given the results for $6 \leq n \leq 10$, it would be sufficient for a complete resolution of this subfamily to deal with exponents $n = 15$, 25 and prime exponents $n \geq 11$. In this note, we deal with the case $n = 15$. Our result is as follows.

Theorem 1.1. *The only primitive integer solutions to the equation*

$$(2) \qquad x^2 + y^3 = z^{15}$$

are the trivial solutions $(\pm 1, -1, 0)$, $(\pm 1, 0, 1)$, $(0, 1, 1)$, $(0, -1, -1)$ and the non-trivial solutions $(\pm 3, -2, 1)$.

There are two fairly obvious avenues for attacking equation (2). One is to use Edwards' parametrisations [10] of the solutions to $x^2 + y^3 = z^5$. In these parametrisations, z is given by a binary form of degree 12, so the requirement that z be a cube results in

a number of superelliptic curves of genus 10 of the form $w^3 = f(u, v)$ with $\deg f = 12$. To these curves (and if necessary, to their Jacobians), one can apply descent techniques analogous to those discussed in [14], [15], [17].

However, we decided to follow the other route, which is based on the parametrisations of the solutions to $x^2 + y^3 = z^3$. In this case, the equations obtained are of the form $w^5 = f(u, v)$ with a binary quartic form f. A further descent step reduces the problem to that of finding rational points on a number of hyperelliptic curves of genus 2 or 4.

2. Parametrisation of Solutions to $x^2 = y^3 + z^3$

We recall the following result [7, Theorem 14.3.1].

Theorem 2.1. *Let (x, y, z) be a triple of coprime integers such that $x^2 = y^3 + z^3$. Then, up to possibly interchanging y and z, there are coprime integers s and t such that one of the following sets of relations hold.*

(P1) $\begin{cases} x = \pm(s^2 - 2st - 2t^2)(s^4 + 2s^3t + 6s^2t^2 - 4st^3 + 4t^4) \\ y = s(s + 2t)(s^2 - 2st + 4t^2) \\ z = -4t(s - t)(s^2 + st + t^2) \end{cases}$

    with $s \not\equiv 0 \bmod 2$ and $s \not\equiv t \bmod 3$.

(P2) $\begin{cases} x = 3(s - t)(s + t)(s^4 + 2s^3t + 6s^2t^2 + 2st^3 + t^4) \\ y = s^4 - 4s^3t - 6s^2t^2 - 4st^3 + t^4 \\ z = 2(s^4 + 2s^3t + 2st^3 + t^4) \end{cases}$

    with $s \not\equiv t \bmod 2$ and $s \not\equiv t \bmod 3$.

(P3) $\begin{cases} x = = 6st(3s^4 + t^4) \\ y = = -3s^4 + 6s^2t^2 + t^4 \\ z = = 3s^4 + 6s^2t^2 - t^4 \end{cases}$

    with $s \not\equiv t \bmod 2$ and $t \not\equiv 0 \bmod 3$.

As far as we know, this parametrisation was first obtained by Mordell [13, page 235], but see the discussion in [10, Appendix B.1].

As a corollary, we see that each primitive integral solution to $x^2 + y^3 = z^{15}$ gives rise to a solution of an equation

(3) $$u^5 = f_i(s, t)$$

for some $i = 1, 2, \ldots, 6$, where $f_1, \ldots, f_6$ are the six binary quartic forms giving the values of y and z in Theorem 2.1, with s, t coprime and satisfying the relevant conditions modulo 2 and 3. For future reference, we define the f_i explicitly as follows.

$$f_1(s, t) = s(s + 2t)(s^2 - 2st + 4t^2)$$
$$f_2(s, t) = -4t(s - t)(s^2 + st + t^2)$$
$$f_3(s, t) = s^4 - 4s^3t - 6s^2t^2 - 4st^3 + t^4$$
$$f_4(s, t) = 2(s^4 + 2s^3t + 2st^3 + t^4)$$
$$f_5(s, t) = -3s^4 + 6s^2t^2 + t^4$$
$$f_6(s, t) = 3s^4 + 6s^2t^2 - t^4$$

We note that we can eliminate f_4 immediately: here s and t must be of different parity, which implies that $s^4 + 2s^3t + 2st^3 + t^4$ is odd, so $f_4(s, t)$ is divisible by 2, but not by 4 and hence cannot be a fifth power. The known solutions of equation (2) give rise to solutions of $u^5 = f_i(s, t)$ for $i \in \{1, 2, 3, 5, 6\}$, which means that it is not possible to rule out any of the other quartic forms on the basis of local considerations. In what follows we shall deal separately with equation (3) with $i = 1, 2, 3, 5, 6$ and s, t satisfying

the relevant conditions. We shall refer to these as Cases 1, 2, 3, 5 and 6, which are respectively resolved in Sections 3, 4, 5, 6, 7. All computations mentioned below were carried out using the computer algebra package `MAGMA` [1].

3. Case 1

We want to solve the equation

$$u^5 = s(s + 2t)(s^2 - 2st + 4t^2)$$

in integers s, t, u with s and t coprime, $s \not\equiv 0$ mod 2 and $s \not\equiv t$ mod 3. These congruence and coprimality conditions imply that the three factors on the right hand side are coprime in pairs. This in turn implies that there are integers w_1, w_2, w_3, coprime in pairs, such that

(4) $$s = w_1^5, \quad s + 2t = w_2^5, \quad s^2 - 2st + 4t^2 = w_3^5.$$

If we substitute the first of these equations into the last, we obtain

$$(2t - w_1^5/2)^2 + \tfrac{3}{4}w_1^{10} = w_1^{10} - 2tw_1^5 + 4t^2 = w_3^5,$$

which, upon setting $X = w_3/w_1^2$ and $Y = 4t/w_1^5 - 1$, becomes

$$Y^2 = 4X^5 - 3.$$

This is a curve of genus 2, to which by now standard methods can be applied. A 2-descent on its Jacobian as described in [20] gives an upper bound of 1 for the Mordell-Weil rank. An argument using the explicit theory of heights as developed in [19, 21] shows that the point on the Jacobian given by the class of the divisor

$$(i, 2i + 1) + (-i, -2i + 1) - 2\infty$$

(where $i^2 = -1$) generates the Mordell-Weil group (it is easy to check that the torsion subgroup is trivial). Finally, a computation using Chabauty's method and the Mordell-Weil sieve as explained in [6] shows that the rational points on the curve are

$$\infty, \quad (1, 1), \quad (1, -1).$$

The first of these leads to $s = 0$, which is excluded (s must be odd). The second leads to the contradiction $2w_1^5 = w_2^5$, whereas the last gives the trivial solutions $(x, y, z) = (\pm 1, 0, 1)$. We have shown the following.

Proposition 3.1. *The only solutions in integers (s, t, u) with s and t coprime, $s \not\equiv 0$ mod 2 and $s \not\equiv t$ mod 3 to the equation*

$$u^5 = s(s + 2t)(s^2 - 2st + 4t^2)$$

are $(s, t, u) = (\pm 1, 0, 1)$. This yields the trivial solutions $(x, y, z) = (\pm 1, 0, 1)$ to (2).

4. Case 2

Now we consider f_2, so we want to solve

$$u^5 = -4t(s - t)(s^2 + st + t^2)$$

in integers with s and t coprime, $s \not\equiv 0$ mod 2, $s \not\equiv t$ mod 3. As before, these conditions imply that the three non-constant factors on the right-hand side are coprime in pairs, and the last factor is odd. Therefore there are integers w_1, w_2, w_3, coprime in pairs, such that

(5) $$t = w_1^5, \quad s - t = 8w_2^5, \quad s^2 + st + t^2 = w_3^5,$$

or

(6) $$t = 8w_1^5, \quad s - t = w_2^5, \quad s^2 + st + t^2 = w_3^5.$$

We deal with the possibilities (5) and (6) as we have dealt with (4). If (5) holds then
$$(2s + t)^2 + 3t^2 = 4(s^2 + st + t^2) = 4w_3^5$$
which shows that $(w_3/w_1^2, (2s + w_1^5)/w_1^5)$ is a point on the curve $Y^2 = 4X^5 - 3$ whose points we have already determined; we obtain $(s, t, u) = (\pm 1, 0, 0)$, and $(x, y, z) = (\pm 1, 1, 0)$.

If (6) holds then
$$(s + 4w_1^5)^2 + 48w_1^{10} = (s + t/2)^2 + 3t^2/4 = w_3^5,$$
leading to the genus 2 curve
$$Y^2 = X^5 - 48$$
(with $X = w_3/w_1^2$ and $Y = s/w_1^5 + 4$). As before, the Mordell-Weil group is infinite cyclic, this time generated by the class of
$$(6 + 2\sqrt{2}, 124 + 76\sqrt{2}) + (6 - 2\sqrt{2}, 124 - 76\sqrt{2}) - 2\infty,$$
and a 'Chabauty plus Mordell-Weil sieve' computation shows that this curve has the point at infinity as its only rational point. The solutions corresponding to this have $w_1 = 0$, so $t = 0$ and $s = \pm 1$, $u = 0$. This gives the trivial solutions $(x, y, z) = (\pm 1, 1, 0)$ of the original equation (2). We have shown:

Proposition 4.1. *The only solutions in integers (s, t, u) with s and t coprime, $s \not\equiv 0 \bmod 2$ and $s \not\equiv t \bmod 3$ to the equation*
$$u^5 = -4t(s - t)(s^2 + st + t^2)$$
are $(s, t, u) = (\pm 1, 0, 0)$. This yields the trivial solutions $(x, y, z) = (\pm 1, 1, 0)$ to (2).

5. Case 3

The remaining three forms f_3, f_5 and f_6 are all irreducible. However, they all split over $\mathbb{Q}(\sqrt{3})$ into two quadratic factors that can be written as linear combinations of two squares of linear forms over $\mathbb{Q}$. The two factors must be again coprime (in $\mathbb{Z}[\sqrt{3}]$) and are therefore fifth powers up to a power of the fundamental unit $\varepsilon = \sqrt{3} - 2$. We then express two squares of linear forms as binary quintic forms; taking their product results in a hyperelliptic curve of genus 4. In this section we carry this out for f_3. We have
$$\begin{aligned} f_3(s, t) &= s^4 - 4s^3t - 6s^2t^2 - 4st^3 + t^4 \\ &= (s^2 - 2(1 + \sqrt{3})st + t^2)(s^2 - 2(1 - \sqrt{3})st + t^2). \end{aligned}$$
The conditions are that s and t are coprime with $s \not\equiv t \bmod 2$ and $s \not\equiv t \bmod 3$, which imply that each factor is coprime to 6. Since their resultant is 48, it follows that they are coprime. So there is $j \in \{-2, -1, 0, 1, 2\}$ and there are integers v, w such that
$$s^2 - 2(1 - \sqrt{3})st + t^2 = \varepsilon^j(v + w\sqrt{3})^5 = g_j(v, w) + h_j(v, w)\sqrt{3}$$
with binary quintic forms g_j, h_j with rational integral coefficients. Thus
$$(7) \qquad s^2 - 2st + t^2 = g_j(v, w), \qquad 2st = h_j(v, w).$$
Explicitly, we have
$$\begin{aligned} g_{-2}(v, w) &= 7v^5 + 60v^4w + 210v^3w^2 + 360v^2w^3 + 315vw^4 + 108w^5 \\ g_{-1}(v, w) &= -(2v^5 + 15v^4w + 60v^3w^2 + 90v^2w^3 + 90vw^4 + 27w^5) \\ g_0(v, w) &= v(v^4 + 30v^2w^2 + 45w^4) \\ g_1(v, w) &= -2v^5 + 15v^4w - 60v^3w^2 + 90v^2w^3 - 90vw^4 + 27w^5 = g_{-1}(v, -w) \\ g_2(v, w) &= 7v^5 - 60v^4w + 210v^3w^2 - 360v^2w^3 + 315vw^4 - 108w^5 = g_{-2}(v, -w) \end{aligned}$$

and

$$h_{-2}(v, w) = 4v^5 + 35v^4 w + 120v^3 w^2 + 210v^2 w^3 + 180vw^4 + 63w^5$$
$$h_{-1}(v, w) = -v^5 - 10v^4 w - 30v^3 w^2 - 60v^2 w^3 - 45vw^4 - 18w^5$$
$$h_0(v, w) = (5v^4 + 30v^2 w^2 + 9w^4)w$$
$$h_1(v, w) = v^5 - 10v^4 w + 30v^3 w^2 - 60v^2 w^3 + 45vw^4 - 18w^5 = h_{-1}(-v, w)$$
$$h_2(v, w) = -4v^5 + 35v^4 w - 120v^3 w^2 + 210v^2 w^3 - 180vw^4 + 63w^5 = h_{-2}(-v, w).$$

From (7),
$$(s - t)^2 = g_j(v, w) \quad \text{and} \quad (s + t)^2 = g_j(v, w) + 2h_j(v, w)$$
so, setting $Y = (s^2 - t^2)/w^5$, $X = v/w$, this gives the hyperelliptic curve
$$C_{3,j} : Y^2 = g_j(X, 1)\big(g_j(X, 1) + 2h_j(X, 1)\big)$$
of genus 4. The irreducible quintic factors all have a root in the same number field $L = \mathbb{Q}(\theta)$ where

$$(8) \qquad\qquad \theta^5 - 5\theta^3 + 5\theta - 4 = 0.$$

A 2-cover descent as described in [5] on the first two and the last curves ($j = -2, -1, 2$) proves that they do not possess any rational points. On the other hand, the two remaining curves do have some rational points, so we need to put in some more work. We consider $C_{3,1}$ first. A partial 2-descent as in [17] using the factorisation over $\mathbb{Q}$ shows that if (X, Y) is a rational point on $C_{3,1}$, then there is a rational point $(X, \tilde{Y})$ (with the same X-coordinate!) on either

$$\tilde{Y}^2 = 2X^5 - 15X^4 + 60X^3 - 90X^2 + 90X - 27$$

or

$$5\tilde{Y}^2 = 2X^5 - 15X^4 + 60X^3 - 90X^2 + 90X - 27.$$

These are curves of genus 2 again. For the first curve, we find in a similar way as we did when considering f_1 and f_2 that its Jacobian has Mordell-Weil rank 1, and a 'Chabauty plus MWS' computation shows that the rational points are the point at infinity and two points with $X = 3$; the two points with $X = 3$ do not give rise to points on $C_{3,1}$. For the second curve, the rank is again 1, and its rational points are the point at infinity and two points with $X = 1$; again the points with $X = 1$ do not lead to points on $C_{3,1}$. Thus

$$C_{3,1}(\mathbb{Q}) = \{\infty\}.$$

As $X = v/w$, we see that $w = 0$ and so $v = \pm 1$. This does not lead to any solutions of (2).

In principle, one could try the same approach with $C_{3,0}$, which has equation

$$(9) \qquad Y^2 = X(X^4 + 30X^2 + 45)(X^5 + 10X^4 + 30X^3 + 60X^2 + 45X + 18)$$

Here one obtains the two genus 2 curves

$$\tilde{Y}^2 = X(X^5 + 10X^4 + 30X^3 + 60X^2 + 45X + 18)$$

and

$$5\tilde{Y}^2 = X(X^5 + 10X^4 + 30X^3 + 60X^2 + 45X + 18).$$

The second of these has Jacobian with trivial Mordell-Weil group and therefore only the rational point $(0, 0)$. However, the first curve has Jacobian of Mordell-Weil rank 2; therefore Chabauty is not applicable. Note that the genus 1 curve $Y^2 = X^4 + 30X^2 + 45$ (which one could also consider) is an elliptic curve of positive rank and so is of little use for our purposes. We have also tried to work directly with the Jacobian $J_{3,0}$ of $C_{3,0}$. A 2-descent on $J_{3,0}$ shows that its 2-Selmer rank 2, so Chabauty's method is applicable in

principle. One point of infinite order in $J_{3,0}(\mathbb{Q})$ is obtained by taking the difference of the two points at infinity on $C_{3,0}$, and the conjectured finiteness of the Tate-Shafarevich group leads us to expect the existence of a second generator. Unfortunately, this second generator appears to have large height, and we have been unable to find it. Instead we shall perform a partial descent on $C_{3,0}$ over the number field K defined by the polynomial $X^4 + 30X^2 + 45$ appearing in the factorisation on the right-hand side of (9). Take $\phi = \sqrt{-15 + 6\sqrt{5}}$ (a root of this polynomial), and let $K = \mathbb{Q}(\phi)$. The field K is normal with cyclic automorphism group of order 4, generated by the automorphism

$$\sigma \colon \phi \longmapsto \frac{\phi^3 + 27\phi}{6}.$$

The result of the partial descent on $C_{3,0}$ over K is that if (X, Y) is a $\mathbb{Q}$-rational point on $C_{3,0}$ (with $X \neq 0$) then there are non-zero α, β, γ in L and a in $\mathbb{Q}$ such that

(10)
$$\begin{cases} X = a \cdot \alpha^2 \\ X - \phi = a \cdot \beta^2 \\ X^5 + 10X^4 + 30X^3 + 60X^2 + 45X + 18 = a \cdot \gamma^2, \end{cases}$$

or

(11)
$$\begin{cases} X = 2a \cdot \alpha^2 \\ X - \phi = -a \cdot \phi \cdot \beta^2 \\ X^5 + 10X^4 + 30X^3 + 60X^2 + 45X + 18 = a \cdot \frac{(-\phi^2+3)}{6} \cdot \gamma^2. \end{cases}$$

We point out that (10) and (11) are respectively consistent with the images of the points $\infty_\pm$ and $(0,0)$ in the Selmer set. We shall show that (11) is inconsistent with the condition $s \not\equiv t \pmod 3$. Let $\mathcal{O}_K$ be the ring of integers of K. The prime 3 ramifies in K as $3\mathcal{O}_K = \mathfrak{q}^2$. Observe that $\mathrm{ord}_{\mathfrak{q}}(\phi) = 1$. If r is a non-zero rational, then $\mathrm{ord}_{\mathfrak{q}}(r) = 2\,\mathrm{ord}_3(r)$ which is even. In particular $\mathrm{ord}_{\mathfrak{q}}(a)$ and $\mathrm{ord}_{\mathfrak{q}}(X)$ are both even. If $X - \phi = -a \cdot \phi \cdot \beta^2$ as in (11) then $\mathrm{ord}_3(X)$ must be positive. Since $X = v/w$, we see that $3 \mid v$. However, from (7) we have

$$(s - t)^2 = g_0(v, w) = v(v^4 + 30v^2w^2 + 45w^4)$$

which contradicts $s \not\equiv t \pmod 3$. Thus (11) does not arise and we may suppose that (10) holds. Letting

$$\tilde{Y} = a^2 \cdot \alpha \cdot \sigma(\beta) \cdot \sigma^2(\beta) \cdot \sigma^3(\beta)$$

we see that $(X, \tilde{Y}) \in D(K)$ where D is the genus 1 curve

$$D \colon \tilde{Y}^2 = X\big(X - \sigma(\phi)\big)\big(X - \sigma^2(\phi)\big)\big(X - \sigma^3(\phi)\big).$$

We are only interested in points $(X, \tilde{Y})$ on $D(K)$ with $X \in \mathbb{Q}$. It is convenient to carry out a birational transformation on D that takes the Weierstrass point $(0, 0)$ to infinity, and gives a Weierstrass model. Specifically, let E be the elliptic curve

$$E \colon \eta^2 = (\mu - 1)(\mu + 2 - \sqrt{5})(\mu - 2 + \sqrt{5}).$$

Then $(\mu, \eta) = (-\phi/X, \phi\tilde{Y}/3\sqrt{5}X)$ belongs to $E(K)$, and satisfies additionally that $\mu/\phi \in \mathbb{Q}$. Let $\zeta \colon E \to \mathbb{P}^1$ be given by $\zeta(\mu, \eta) = \mu/\phi$. Then we are interested in computing the set

(12)
$$\{P \in E(K) : \zeta(P) \in \mathbb{P}^1(\mathbb{Q})\}.$$

Elliptic curve Chabauty [3, 11, 12] should succeed in determining these points provided the rank of $E(K)$ is strictly less than the degree $[K : \mathbb{Q}] = 4$. The curve E is defined over $\mathbb{Q}(\sqrt{5})$, and so a standard argument shows that the rank of $E(K)$ is the sum of the ranks of $E(\mathbb{Q}(\sqrt{5}))$ and $E_\tau(\mathbb{Q}(\sqrt{5}))$ where E_τ denotes the quadratic twist of E by

$\tau = -15 + 6\sqrt{5} = \phi^2$. A 2-descent shows that both ranks are 1 and so $E(K)$ has rank 2. In fact,

$$E(K) = \frac{\mathbb{Z}}{2\mathbb{Z}}(1,0) \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}(2 - \sqrt{5}, 0) \oplus \mathbb{Z}(0, 2 - \sqrt{5}) \oplus \mathbb{Z}\left(\frac{4 - \sqrt{5}}{3}, \frac{(2 - 2\sqrt{5})\phi}{9}\right).$$

Applying the `MAGMA` implementation of elliptic curve Chabauty due to Nils Bruin we find that the set in (12) is

$$\{\infty, (0, 2 - \sqrt{5}), (0, -2 + \sqrt{5})\}.$$

Hence $X = 0$ or ∞ and so $v = 0$ or $w = 0$. By (7), $v = 0$ gives $s = t$ which contradicts $s \not\equiv t \bmod 2$, and so does not lead to a solution of (2). However $w = 0$ implies (again from (7)) that $s = 0$ or $t = 0$, corresponding to the non-trivial solutions $(x, y, z) = (\pm 3, -2, 1)$ of equation (2). We have shown:

Proposition 5.1. *The only solutions in integers (s, t, u) with s and t coprime, $s \not\equiv t \bmod 2$ and $s \not\equiv t \bmod 3$ to the equation*

$$u^5 = s^4 - 4s^3 t - 6s^2 t^2 - 4st^3 + t^4$$

are $(s, t, u) = (\pm 1, 0, 1)$ and $(0, \pm 1, 1)$. These yield the non-trivial solutions $(x, y, z) = (\pm 3, -2, 1)$ to (2).

6. CASE 5

In this section we apply the method of the previous section to f_5. We have

$$f_5(s, t) = -3s^4 + 6s^2 t^2 + t^4 = \left((3 + 2\sqrt{3})s^2 + t^2\right)\left((3 - 2\sqrt{3})s^2 + t^2\right).$$

The conditions on s and t are coprimality and $s \not\equiv t \bmod 2$, $t \not\equiv 0 \bmod 3$. They again imply that the two factors are coprime. Writing

$$(3 + 2\sqrt{3})s^2 + t^2 = \varepsilon^j (v + w\sqrt{3})^5 = g_j(v, w) + h_j(v, w)\sqrt{3},$$

we find that

(13) $$2s^2 = h_j(v, w) \quad \text{and} \quad 2t^2 = 2g_j(v, w) - 3h_j(v, w)$$

so that

$$(2st)^2 = h_j(v, w)\left(2g_j(v, w) - 3h_j(v, w)\right).$$

This defines again five hyperelliptic curves $C_{5,j}$ of genus 4. It can be checked that (taking $j \bmod 5$) $C_{3,j} \cong C_{5,j-1}$. We know from the previous section that $C_{3,j}(\mathbb{Q}) = \emptyset$ for $j = -2, -1$ and 2. Thus $C_{5,j}(\mathbb{Q}) = \emptyset$ for $j = 2, -2$ and 1.

The models given above for $C_{5,0}$ and $C_{3,1}$ are in fact identical, so by the results of the previous section $C_{5,0}(\mathbb{Q}) = \{\infty\}$. The point at infinity on $C_{5,0}$ gives $w = 0$, which this time gives $s = 0$ and $2t^2 = 2v^5$, hence the solutions $(s, t) = (0, \pm 1)$, corresponding to the solution $(x, y, z) = (0, 1, 1)$ to (2).

We were unable to determine the rational points of $C_{3,0} \cong C_{5,-1}$. Thus we need to argue differently in this case. When $j = -1$ the second equation in (13) is

$$2t^2 = -v(v^4 + 30v^2 w^2 + 45w^4).$$

Now v, w are coprime, and $t \not\equiv 0 \bmod 3$. If v and w are both odd then

$$-v(v^4 + 30v^2 w^2 + 45w^4) \equiv 4 \pmod 8,$$

which is impossible. Hence v is even and w is odd. It follows that

$$v = -2t_1^2, \qquad v^4 + 30v^2 w^2 + 45w^4 = t_2^2$$

with $t = t_1 t_2$ or

$$v = -10t_1^2, \qquad v^4 + 30v^2 w^2 + 45w^4 = 5t_2^2$$

with $t = 5t_1t_2$. The first case gives $5w^4 \equiv t_2^2 \pmod 8$ and the second $v^4 \equiv 2t_2^2 \pmod 3$. Both are impossible. We obtain the following.

Proposition 6.1. *The only solutions in integers (s, t, u) with s and t coprime, $s \not\equiv t \bmod 2$ and $t \not\equiv 0 \bmod 3$ to the equation*

$$u^5 = -3s^4 + 6s^2t^2 + t^4$$

are $(s, t, u) = (0, \pm 1, 1)$. This yields the trivial solution $(x, y, z) = (0, 1, 1)$ to (2).

7. CASE 6

To complete the proof of Theorem 1.1 it remains to deal with f_6, which we can write as

$$f_6(s, t) = 3s^4 + 6s^2t^2 - t^4 = -\left(t^2 - 3s^2 + 2s^2\sqrt 3\right)\left(t^2 - 3s^2 - 2s^2\sqrt 3\right).$$

The conditions on s and t are coprimality, $s \not\equiv t \bmod 2$ and $t \not\equiv 0 \bmod 3$. They again imply that the two factors are coprime. Writing

$$t^2 - 3s^2 + 2s^2\sqrt 3 = \varepsilon^j(v + w\sqrt 3)^5 = g_j(v, w) + h_j(v, w)\sqrt 3,$$

we find that

(14) $$2s^2 = h_j(v, w) \quad \text{and} \quad 2t^2 = 2g_j(v, w) + 3h_j(v, w)$$

so that

$$(2st)^2 = h_j(v, w)\big(2g_j(v, w) + 3h_j(v, w)\big).$$

We therefore obtain five curves $C_{6,j}$, which are easily seen to be the quadratic twists by -1 of the curves $C_{5,-j}$. We deal with $C_{6,-2}$ last as it requires slightly more delicate arguments. We can rule out $C_{6,-1}$ via a 2-cover descent. Partial 2-descent on $C_{6,0}$ shows that rational points give rise to rational points with the same X-coordinate on the genus 2 curve

$$\tilde Y^2 = 10X^5 + 75X^4 + 300X^3 + 450X^2 + 450X + 135,$$

which has Jacobian with trivial Mordell-Weil group. This implies that the only rational point on $C_{6,0}$ is the point at infinity. This gives the solutions $(s, t) = (0, \pm 1)$, corresponding to $(x, y, z) = (0, -1, -1)$.

We consider the rational points on $C_{6,1}$. A partial 2-descent shows that rational points give rise to rational points with the same X-coordinate on the genus 1 curve

$$5\tilde Y^2 = X^4 + 30X^2 + 45.$$

This curve has Mordell-Weil rank 1 and so infinitely many rational points. However, considering the equation 3-adically, we see that 3-adic points must satisfy $\mathrm{ord}_3(X) \geq 1$. As $X = v/w$, we see that $3 \mid v$. From (14) we have

$$2t^2 = -v(v^4 + 30v^2w^2 + 45w^4)$$

which contradicts $t \not\equiv 0 \pmod 3$. Thus the rational points on $C_{6,1}$ do not give rise to solutions to (2).

Let us deal with $j = 2$. It turns out to be more convenient to work directly with (14) than with the curve $C_{6,2}$. Explicitly, (14) is the following pair of equations

$$2s^2 = -4v^5 + 35v^4w - 120v^3w^2 + 210v^2w^3 - 180vw^4 + 63w^5$$

$$2t^2 = 2v^5 - 15v^4w + 60v^3w^2 - 90v^2w^3 + 90vw^4 - 27w^5.$$

Since $s \not\equiv t \bmod 2$, we see that v, w cannot both be even. In fact, by listing all the possibilities for v, w modulo 8, we see that v must be odd and w must be even. Now we take a linear combination that eliminates the v^5 term:

$$2(s^2 + 2t^2) = w(5v^4 + 30v^2w^2 + 9w^4).$$

Recall that s, t are coprime. Any odd prime p dividing the left-hand side must satisfy $(-2/p) = 1$ and so $p \equiv 1$ or $3 \bmod 8$. Thus, $5v^4 + 30v^2w^2 + 9w^4 \equiv 1$ or $3 \bmod 8$. As v is odd and w is even we see that $5v^4 + 30v^2w^2 + 9w^4 \equiv 5 \bmod 8$, giving a contradiction.

The curve $C_{6,-2}$ has equation

$$Y^2 = (4X^5 + 35X^4 + 120X^3 + 210X^2 + 180X + 63)$$
$$\times (26X^5 + 225X^4 + 780X^3 + 1350X^2 + 1170X + 405).$$

The two irreducible quintic factors on the right-hand side each acquire a root over $L = \mathbb{Q}(\theta)$ where θ is given by (8). These roots are respectively

$$\phi_1 = \frac{\theta^4 - 5\theta^2 - 4\theta - 3}{4}, \qquad \phi_2 = \frac{7\theta^4 - 2\theta^3 - 27\theta^2 + 4\theta - 33}{26}.$$

Let

$$\mu_1 = 2\theta^4 + 2\theta^3 - 3\theta^2 + \theta + 1, \qquad \mu_2 = \frac{18\theta^4 + 19\theta^3 - 10\theta^2 - 12\theta + 21}{26}.$$

We perform a partial 2-cover descent on $C_{6,-2}$ over L. The outcome of this is that if (X, Y) is a rational point then there are non-zero $a \in \mathbb{Q}$ and $\alpha, \beta \in L$ such that

$$X - \phi_1 = \mu_1 \cdot a \cdot \alpha^2, \qquad X - \phi_2 = \mu_2 \cdot a \cdot \beta^2.$$

We note in passing that the only points on the curve $C_{6,-2}$ appear to be $\{(-1, \pm 4)\}$, and that

$$-1 - \phi_1 = 3\mu_1 \frac{(3\theta^4 + 2\theta^3 - 15\theta^2 - 10\theta + 13)^2}{6^2}, \quad -1 - \phi_2 = 3\mu_2 \frac{(\theta^3 + 2\theta^2 - 4\theta - 9)^2}{3^2},$$

which provides a useful check on the correctness of our partial descent implementation.

The ring of integers of L is $\mathbb{Z}[\theta]$. The ideal $2 \cdot \mathbb{Z}[\theta]$ factors as $\mathfrak{p}\mathfrak{q}^2$ where $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals. If $r \in \mathbb{Q}$ is non-zero then $\mathrm{ord}_\mathfrak{q}(r) = 2\,\mathrm{ord}_2(r)$; in particular $\mathrm{ord}_\mathfrak{q}(X)$ and $\mathrm{ord}_\mathfrak{q}(a)$ are even. We note that $\mathrm{ord}_\mathfrak{q}(\mu_1) = 1$. Thus $\mathrm{ord}_\mathfrak{q}(X - \phi_1)$ is odd. However, $\mathrm{ord}_\mathfrak{q}(\phi_1) = 0$. This forces $\mathrm{ord}_2(X) = 0$. As $X = v/w$ and v, w are coprime, we see that v and w are both odd. Now (14) forces $2s^2 \equiv 2t^2 \equiv 0 \pmod{4}$, contradicting the coprimality of s and t.

Proposition 7.1. *The only solutions in integers (s, t, u) with s and t coprime, $s \not\equiv t \bmod 2$ and $t \not\equiv 0 \bmod 3$ to the equation*

$$u^5 = 3s^4 + 6s^2t^2 - t^4$$

are $(s, t, u) = (0, \pm 1, -1)$. This yields the trivial solution $(x, y, z) = (0, -1, -1)$ to (2).

This completes the proof of Theorem 1.1.

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