Enumeration of generalized polyominoes

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Abstract

As a generalization of polyominoes we consider edge-to-edge connected nonoverlapping unions of regular k-gons. For $n \leq 4$ we determine formulas for the number $a_k(n)$ of generalized polyominoes consisting of n regular k-gons. Additionally give a table of the numbers $a_k(n)$ for small k and n obtained by computer enumeration. We finish with a survey of known problems for polyominoes.

Key words:

1 Introduction

A polyomino, in its original definition, is a connected interior-disjoint union of axis-aligned unit squares joined edge-to-edge. In other words, it is an edge-connected union of cells in the planar square lattice. For the origin of polyominoes we quote

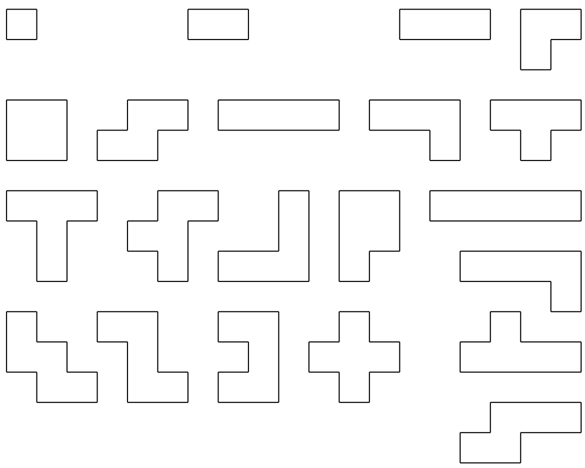

Figure 1. Polyominoes with at most 5 squares.

Klarner [10]: “Polyominoes have a long history, going back to the start of the 20th century, but they were popularized in the present era initially by Solomon Golomb i.e. [3,4,5] , then by Martin Gardner in his *Scientific American* columns.”
There at are least three ways to define two polyominoes as equivalent, namely factoring out just translations (fixed polyominoes), rotations and translations (chiral polyominoes), or reflections, rotations and translations (free polyominoes). Here we consider only free polyominoes. To give an illustration of polyominoes Figure 1 depicts the free polyominoes consisting of at most 5 unit squares.

n	A0001055(n)	n	A0001055(n)	n	A0001055(n)	n	A0001055(n)
1	1	8	369	15	3426576	22	43191857688
2	1	9	1285	16	13079255	23	168047007728
3	2	10	4655	17	50107909	24	654999700403
4	5	11	17073	18	192622052	25	2557227044764
5	12	12	63600	19	742624232	26	9999088822075
6	35	13	238591	20	2870671950	27	39153010938487
7	108	14	901971	21	11123060678	28	153511100594603

Table 1
Polyominoes or square animals.

One of the first problems for polyominoes was the determination of their number. Altough there has been some progress, a solution to this problem remains outstanding. In the literature one sometimes speaks also of the cell-growth problem and uses the term animal instead of polyomino. In Table 1 we give the known numbers of polyominoes, this is sequence A0001055 in the “Online Encyclopedia of Integer Sequences” [14].

n	A000577(n)	n	A000577(n)	n	A000577(n)	n	A000577(n)
1	1	8	66	15	73983	22	121419260
2	1	9	160	16	211297	23	353045291
3	1	10	448	17	604107	24	1028452717
4	3	11	1186	18	1736328	25	3000800627
5	4	12	3334	19	5000593	26	8769216722
6	12	13	9235	20	14448984	27	25661961260
7	24	14	26166	21	41835738	28	75195166667

Table 2
Triangular polyominoes (or polyiamonds).

Polyominoes were soon generalized to the two other tessellations of the plane. For the trigonal lattice they are called triangular polyominoes and for the hexagonal lattice they are called hexagonal polyominoes. Their know numbers are given in Table 2 and Table 3, respectively. Polyominoes were also considered on the eight Archimedean tessellations [2] and as unions of d-dimensional hypercubes instead of squares. In this article we consider polyominoes as unions of regular nonoverlapping edge-to-edge connected k-gons. For short we call them k-polyominoes. Edge-to-edge connected unions of regular k-gons which may overlap were counted by Harary [7].

n	A000228(n)	n	A000228(n)	n	A000228(n)	n	A000228(n)
1	1	6	82	11	143552	16	372868101
2	1	7	333	12	683101	17	1822236628
3	3	8	1448	13	3274826	18	8934910362
4	7	9	6572	14	15796897	19	43939164263
5	22	10	30490	15	76581875	20	216651036012

Table 3
Hexagonal polyominoes.

2 Formulas for the number of nonisomorphic k-polyominoes

We denote the number of nonisomorphic k-polyominoes consisting of n regular k-gons by $a_k(n)$. Because the definition of a k-polyomino makes sense only for $k \geq 3$ we set $a_k(n) = 0$ for $k < 3$. For a k-polyomino consisting of a single cell we clearly have $a_k(1) = 1$. Because there is only one possibility to connect two cells and this union is nonoverlapping we have $a_k(2) = 1$. To handle k-polyominoes consisting of 3 cells we consider the cells $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ which are connected via the edges $\overline{P_2P_5}$ and $\overline{P_3P_6}$, see Figure 2. We call the length of the shortest path beetween P_2 and P_3 the distance $d(P_2, P_3)$. If the cells $\mathcal{C}_1$ and $\mathcal{C}_3$ are connected to the cell $\mathcal{C}_2$ as in Figure 2, we denote the minimum $\min(d(P_2, P_3), d(P_5, P_6))$ by $d(\mathcal{C}_1, \mathcal{C}_3)$. Here

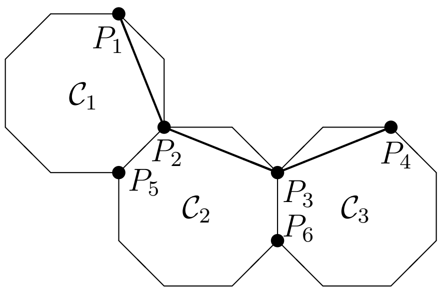

Figure 2. Distance between two k-gons neighboring a common k-gon.

we have $d(\mathcal{C}_1, \mathcal{C}_3) = 2$. The next lemma characterizes the distances of $\mathcal{C}_1$ and $\mathcal{C}_3$ where the cells do not intersect.

Lemma 1 *Two k-gons $\mathcal{C}_1$ and $\mathcal{C}_3$ joined via an edge to a k-gon $\mathcal{C}_2$ are nonoverlapping iff $d(\mathcal{C}_1, \mathcal{C}_3) \geq \left\lfloor \frac{k-1}{6} \right\rfloor$.*

Proof. We consider Figure 3 and assume $k \geq 12$. For the angles α of a regular k-gon we have $\alpha = \frac{k-2}{k}\pi$. Because the $d(\mathcal{C}_1, \mathcal{C}_3)+1$-gon has $(d(\mathcal{C}_1, \mathcal{C}_3)-1)\pi$ as sum of angles $\angle(P_6, P_2, P_3) = \angle(P_7, P_3, P_2) = \frac{d(\mathcal{C}_1,\mathcal{C}_3)-1}{k}\pi$. With $2\alpha+\angle(P_5, P_2, P_6) = 2\pi$ we get $\beta := \angle(P_1, P_2, P_3) = \frac{2d(\mathcal{C}_1,\mathcal{C}_3)+2}{k}\pi$. Because the lengths of the lines $\overline{P_1P_2}$, $\overline{P_2P_3}$, and $\overline{P_3P_4}$ are the same, the points P_1 and P_3 are equal iff $\beta = \frac{\pi}{3}$. For $\beta < \frac{\pi}{3}$ the lines $\overline{P_1P_2}$ and $\overline{P_3, P_4}$ intersect, so $\beta \geq \frac{\pi}{3}$ is a necessary condition. Inserting $\beta = \frac{2d(\mathcal{C}_1,\mathcal{C}_3)+2}{k}\pi$ yields $d(\mathcal{C}_1, \mathcal{C}_3) \geq \frac{k-6}{6}$. Because $d(\mathcal{C}_1, \mathcal{C}_3)$ and k are integers we have $d(\mathcal{C}_1, \mathcal{C}_3) \geq \left\lceil \frac{k-6}{6} \right\rceil = \left\lfloor \frac{k-1}{6} \right\rfloor$. For $k \equiv 0 \bmod 6$ and $d(\mathcal{C}_1, \mathcal{C}_3) = \frac{k-6}{6}$ also the cells $\mathcal{C}_1$ and $\mathcal{C}_3$ have an edge in common. It is not difficult to see that in this configuration the k-gons do not overlap. If we consider such a $k+6$-gon with $d(\mathcal{C}_1, \mathcal{C}_3) = \frac{k}{6}$ we can deduce that for $d(\mathcal{C}_1, \mathcal{C}_3) \geq \left\lfloor \frac{k-1}{6} \right\rfloor$ the three cells do not intersect. The proof is finished by checking the cases $k < 12$. $\square$

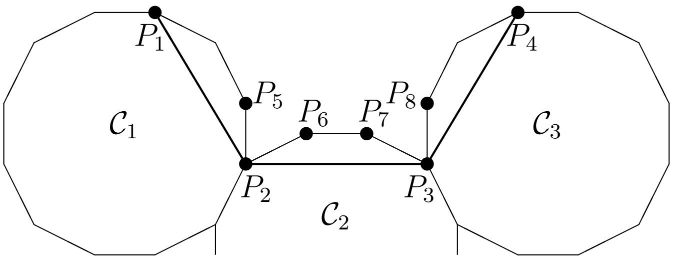

Figure 3. Nonoverlapping 16-gons.

From this we can deduce the following corolla.

Corolla 2 *The number of neighbors of a cell in a k-polyomino is at most*

$$\min\left(k, \frac{k}{\left\lfloor \frac{k+5}{6} \right\rfloor}\right) \leq 6\,.$$

For $k = \infty$ or more precisely circles we have that a circle can have at most 6 nonoverlapping circles of equal radius sharing at least a single point. The maximum number of neighbors is also called Newton number of the geometric object.

With the aid of Lemma 1 we are able to determine the number $a_k(3)$ of k-polyominoes consisting of 3 cells.

Theorem 3

$$a_k(3) = \left\lfloor \frac{k-2}{2} \right\rfloor - \left\lfloor \frac{k-1}{6} \right\rfloor + 1 \quad \textit{for } k \geq 3\,.$$

Proof. It suffices to determine the possible values for $d(\mathcal{C}_1, \mathcal{C}_3)$. Due to Lemma 1 we have $d(\mathcal{C}_1, \mathcal{C}_3) \geq \left\lfloor \frac{k-1}{6} \right\rfloor$ and due to the definition of the distance or symmetry considerations we have $d(\mathcal{C}_1, \mathcal{C}_3) \leq \left\lfloor \frac{k-2}{2} \right\rfloor$. □

In order to determine the number of k-polyominoes with more than 3 cells we describe the classes of k-polyominoes by graphs. We represent each k-gon by a node and join two nodes exactly if the corresponding k-gons are connected via an edge.

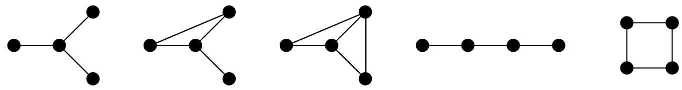

Figure 4. The possible graphs of k-polyominoes with 4 nodes.

Lemma 4 *The number of k-polyominoes with a graph isomorph to one of first three ones in Figure 4 is given by*

$$\left\lfloor \frac{\left(k - 3\left\lfloor \frac{k+5}{6} \right\rfloor\right)^2 + 6\left(k - 3\left\lfloor \frac{k+5}{6} \right\rfloor\right) + 12}{12} \right\rfloor$$

Proof. We denote on of the at most two cells corresponding to a node of degree 3 in the graph by $\mathcal{C}_0$ and the three other cells by $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$. With $d_1 := d(\mathcal{C}_1, \mathcal{C}_0, \mathcal{C}_2) - \left\lfloor \frac{k-1}{6} \right\rfloor$, $d_2 := d(\mathcal{C}_2, \mathcal{C}_0, \mathcal{C}_3) - \left\lfloor \frac{k-1}{6} \right\rfloor$, and $d_3 := d(\mathcal{C}_3, \mathcal{C}_0, \mathcal{C}_1) - \left\lfloor \frac{k-1}{6} \right\rfloor$ we have $m := d_1 + d_2 + d_3 = k - 3 - 3\left\lfloor \frac{k-1}{6} \right\rfloor = k - 3\left\lfloor \frac{k+5}{6} \right\rfloor$. Because the k-polyominoes with a graph isomorphic to one of the first three ones in Figure 4 are uniquely described by d_1, d_2, d_3, due to Lemma 1 and due to symmetry their number equals the number of partitions of m into at most three parts. This number is the coefficient of x^m in the Taylor series of $\frac{1}{(1-x)(1-x^2)(1-x^3)}$ in $x = 0$ and can be expressed as $\left\lfloor \frac{m^2+6m+12}{12} \right\rfloor$. □

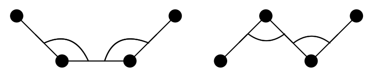

Figure 5. Paths of length 4.

In Lemma 1 we have given a condition for a path of 3 cells avoiding an overlapping. For paths of length 4 we have to consider two cases. We depict the position of the distance of three neighboring cells by an arc, see Figure 5. In the second case the two nodes of degree two are not able to overlap so we need a lemma in the spirit of Lemma 1 only for the first case.

Lemma 5 *Four k-gons $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ arranged as in the first case of Figure 5 are nonoverlapping iff Lemma 1 is fulfilled for the two subpaths of length 3 and*

$$d(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)+d(\mathcal{C}_2,\mathcal{C}_3,\mathcal{C}_4)\geq\left\lfloor\frac{k-3}{2}\right\rfloor .$$

The path is indeed a 4-cycle iff

$$d(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)+d(\mathcal{C}_2,\mathcal{C}_3,\mathcal{C}_4)=\frac{k-4}{2} .$$

Proof. We start with the second statement and consider the quadrangle of the centers of the 4 cells. Because the angle sum of a quadrangle is 2π we get

$$d(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)+d(\mathcal{C}_2,\mathcal{C}_3,\mathcal{C}_4)+d(\mathcal{C}_3,\mathcal{C}_4,\mathcal{C}_1)+d(\mathcal{C}_4,\mathcal{C}_1,\mathcal{C}_2)=k-4 .$$

Due to the fact that the side lengths of the quadrangle are equal we have

$$d(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)+d(\mathcal{C}_2,\mathcal{C}_3,\mathcal{C}_4)=d(\mathcal{C}_3,\mathcal{C}_4,\mathcal{C}_1)+d(\mathcal{C}_4,\mathcal{C}_1,\mathcal{C}_2)$$

which is equivalent to the statement.
Thus we have that $d(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)+d(\mathcal{C}_2,\mathcal{C}_3,\mathcal{C}_4)\geq\left\lfloor\frac{k-3}{2}\right\rfloor$ is a necessary condition. For $k\equiv 0 \mod 2$ it is clear that this condition is also sufficient and for $k\equiv 1 \mod 2$ we consider the corresponding configurations of $(k-1)$-polyominoes and of $(k+1)$-polyominoes. □

With this lemma we are able to count the k-polyominoes having one of the two remaining graphs as their graphs.

Lemma 6 *The number of k-polyominoes with a graph isomorph to one of last two ones in Figure 4 is given by*

$$\begin{aligned}
&\frac{5k^2+4k}{48} \text{ for } k\equiv 0 \mathit{mod}\, 12, && \frac{5k^2+6k-11}{48} \text{ for } k\equiv 1 \mathit{mod}\, 12,\\
&\frac{5k^2+12k+4}{48} \text{ for } k\equiv 2 \mathit{mod}\, 12, && \frac{5k^2+14k+9}{48} \text{ for } k\equiv 3 \mathit{mod}\, 12,\\
&\frac{5k^2+20k+32}{48} \text{ for } k\equiv 4 \mathit{mod}\, 12, && \frac{5k^2+22k+5}{48} \text{ for } k\equiv 5 \mathit{mod}\, 12,\\
&\frac{5k^2+4k-12}{48} \text{ for } k\equiv 6 \mathit{mod}\, 12, && \frac{5k^2+6k+1}{48} \text{ for } k\equiv 7 \mathit{mod}\, 12,\\
&\frac{5k^2+12k+16}{48} \text{ for } k\equiv 8 \mathit{mod}\, 12, && \frac{5k^2+14k-3}{48} \text{ for } k\equiv 9 \mathit{mod}\, 12,\\
&\frac{5k^2+20k+20}{48} \text{ for } k\equiv 10 \mathit{mod}\, 12, && \frac{5k^2+22k+17}{48} \text{ for } k\equiv 11 \mathit{mod}\, 12 .
\end{aligned}$$

Proof. Because each of the two last graphs in Figure 4 contains a path of length 4 as a subgraph we consider the two cases of Figure 5. We abbreviate the two interesting distances, which describe the k-polyominoes uniquely, by d_1 and d_2. Due to symmetry reasons we may assume $d_1 \leq d_2$ and due to Lemma 1 we have $d_1, d_2 \geq \left\lfloor \frac{k}{6} \right\rfloor$ (the graphs do not contain a triangle). From the definition of the distance we have $d_1, d_2 \leq \left\lfloor \frac{k-2}{2} \right\rfloor$. To avoid double counting we assume $\left\lfloor \frac{k}{6} \right\rfloor \leq d_1, d_2 \leq \left\lfloor \frac{k-3}{2} \right\rfloor$ in the second case, so that we get a number of

$$\binom{\left\lfloor \frac{k-3}{2} \right\rfloor - \left\lfloor \frac{k}{6} \right\rfloor + 2}{2}$$

k-polyominoes. With Lemma 5 the number of k-polyominoes in the first case is given by

$$\sum_{d_1=\left\lfloor \frac{k}{6} \right\rfloor}^{\left\lfloor \frac{k-2}{2} \right\rfloor} \sum_{d_2=\max(d_1, \left\lfloor \frac{k-3}{2} \right\rfloor - d_1)}^{\left\lfloor \frac{k-2}{2} \right\rfloor} 1 \,.$$

A little calculation yields the proposed formula. □

Theorem 7

$$a_k(4) = \begin{cases} \frac{3k^2+8k+24}{24} & \textit{for} \quad k \equiv 0 \textit{ mod } 12, \\ \frac{3k^2+4k-7}{24} & \textit{for} \quad k \equiv 1 \textit{ mod } 12, \\ \frac{3k^2+8k-4}{24} & \textit{for} \quad k \equiv 2 \textit{ mod } 12, \\ \frac{3k^2+10k+15}{24} & \textit{for} \quad k \equiv 3 \textit{ mod } 12, \\ \frac{3k^2+14k+16}{24} & \textit{for} \quad k \equiv 4 \textit{ mod } 12, \\ \frac{3k^2+16k+13}{24} & \textit{for} \quad k \equiv 5 \textit{ mod } 12, \\ \frac{3k^2+8k+12}{24} & \textit{for} \quad k \equiv 6 \textit{ mod } 12, \\ \frac{3k^2+4k-7}{24} & \textit{for} \quad k \equiv 7 \textit{ mod } 12, \\ \frac{3k^2+8k+8}{24} & \textit{for} \quad k \equiv 8 \textit{ mod } 12, \\ \frac{3k^2+10k+3}{24} & \textit{for} \quad k \equiv 9 \textit{ mod } 12, \\ \frac{3k^2+14k+16}{24} & \textit{for} \quad k \equiv 10 \textit{ mod } 12, \\ \frac{3k^2+16k+13}{24} & \textit{for} \quad k \equiv 11 \textit{ mod } 12. \end{cases}$$

Proof. The graphs depicted in Figure 4 are all possible graphs of k-polyominoes, because the graphs have to be connected and the complete graph on 4 nodes k_4 is not a unit distance graph. Adding the formulas from Lemma 4 and Lemma 6 yields the theorem. □

3 Computer enumeration of k-polyominoes

By computer construction of k-polyominoes we obtained the following tables of values for $a_k(n)$.

k/n	1	2	3	4	5	6	7	8	9	10	11
3	1	1	1	3	4	12	24	66	160	448	1186
4	1	1	2	5	12	35	108	369	1285	4655	17073
5	1	1	2	7	25	118	551	2812	14445	76092	403976
6	1	1	3	7	22	82	333	1448	6572	30490	143552
7	1	1	2	7	25	118	558	2876	14982	80075	431889
8	1	1	3	11	50	269	1605	10102	65323	430302	
9	1	1	3	14	82	585	4418	34838	280014		
10	1	1	4	19	127	985	8350	73675			
11	1	1	4	23	186	1750	17507	181127			
12	1	1	5	23	168	1438	13512	131801			
13	1	1	4	23	187	1765	17775	185297			
14	1	1	5	29	263	2718	30467	352375			
15	1	1	5	35	362	4336	55264				
16	1	1	6	42	472	6040	83252				
17	1	1	6	48	614	8814	134422				
18	1	1	7	47	566	7678	112514				
19	1	1	6	48	615	8839	135175				
20	1	1	7	57	776	11876	195122				

Table 4
Number of k-polyominoes with n cells for small k and n.

We would like to describe in short how to efficiently generate k-polyominoes. The

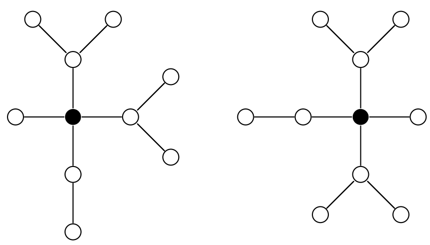

Figure 6. Two nonisomorphic geometric trees.

first fact we observe is that the graphs of k-polyominoes are in most cases trees. If we enhance the trees by distances for each path of length 3 we have a description for a unique k-polyomino. To avoid redundancy in assigning the values for the distances we consider drawings of the trees in the plane and call them geometric trees. The two graphs in Figure 6 are nonisomorphic in a graph theoretic sense, but we would like to regard them as different. Therefore we consider the neighbors of a node as being ordered. So we can generate the k-polyominoes as follows. At first we generate all trees with n nodes and a maximum degree given by Corolla 2. Then for each tree we assign the possible values for the distances with restrictions from Lemma 1 and Lemma 5. Because the center of a tree, depicted by a filled circle

k/n	1	2	3	4	5	6
21	1	1	7	64	972	16410
22	1	1	8	74	1179	20970
23	1	1	8	82	1437	27720
24	1	1	9	81	1347	24998
25	1	1	8	82	1439	27787
26	1	1	9	93	1711	34763
27	1	1	9	103	2045	44687
28	1	1	10	115	2376	54133
29	1	1	10	125	2786	67601
30	1	1	11	123	2641	62252
31	1	1	10	125	2790	67777
32	1	1	11	139	3204	81066
33	1	1	11	150	3707	99420
34	1	1	12	165	4193	116465
35	1	1	12	177	4790	140075
36	1	1	13	175	4575	130711
37	1	1	12	177	4796	140434
38	1	1	13	193	5380	163027
39	1	1	13	207	6089	193587
40	1	1	14	224	6760	221521

Table 5
Number of k-polyominoes with n cells for small k and n.

in Figure 6, is unique and due to the fact that the maximum degree is at most 6 according to Corolla 2 each isomorphic assignment of distances to the geometric trees can occur at most 12 times. To avoid the construction of isomorphic k-polyominoes we consider the assigned geometric tree under the rotations and mirrorings which fix the center. If the constructed assigned geometric tree is minimal with respect to lexicographical ordering in this set of at most 12 structures then we take it else we dismiss it. Because the conditions of Lemma 1 and Lemma 5 are only necessary and not sufficient we have to check if the cells of the corresponding k-polyomino are nonoverlapping. If two cells have an edge in common we add the corresponding edge to the geometric tree which is then a geometric graph. If the cells are nonoverlapping and the graph remains a tree then we have constructed a k-polyomino which we have not constructed before. In the case were the graph is not a tree we define a unique spanning tree for the graph and check if the original tree is this unique spanning tree. The big advantage of this construction strategy is, that there is no need to store the constructed k-polyominoes in the memory.

4 Problems for k-polyominoes

For 4-polyominoes the maximum area of the convex hull was considered in [1]. If the area of a cell is normalized to 1 then the maximum area of a 4-polyomino consisting of n squares is given by $n + \frac{1}{2}\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor$. One of the present authors proved an analogous result for the maximum content of the convex hull of a union of d-dimensional units hypercubes [11] and for the area of the convex hull of 6-polyominoes [12]. For n hypercubes the maximum content of the convex hull is given by

$$\sum_{I\subseteq\{1,\ldots,d\}} \frac{1}{|I|!}\prod_{i\in I}\left\lfloor\frac{n-2+i}{d}\right\rfloor .$$

The maximum area of the convex hull of 6 polyominoes is given by $\frac{1}{6}\lfloor n^2+\frac{14}{3}n+1\rfloor$. For other values of k the question for the maximum area of the convex hull of k-polyominoes is still open. Beside from [9] no results are known for the question of the minimum area of the convex hull, which is non trivial for $k \neq 3, 4$.

Another class of problems is the question for the minimum and the maximum number of edges of k-polyominoes. The following sharp inequalities for the number q of edges of k-polyominoes consisting of n cells were found in [6] and are also given in [8].

$$k = 3: \quad n + \left\lceil\frac{1}{2}\left(n+\sqrt{6n}\right)\right\rceil \leq q \leq 2n+1$$

$$k = 4: \quad 2n + \left\lceil 2\sqrt{n}\right\rceil \leq q \leq 3n+1$$

$$k = 6: \quad 3n - \left\lceil\sqrt{12n-3}\right\rceil \leq q \leq 5n+1$$

In general the maximum number of edges is given by $(k-1)n+1$. We would like to mention that the numbers of 4-polyominoes with a minimum number of edges were enumerated in [13].

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