# Counting polyominoes with minimum perimeter

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**Abstract**

The number of essentially different square polyominoes of order $n$ and minimum perimeter $p(n)$ is enumerated.

*Key words:* polyominoes, enumeration
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## 1 Introduction

Suppose we are given $n$ unit squares. What is the best way to arrange them side by side to gain the minimum perimeter $p(n)$? In [5] F. Harary and H. Harborth proved that $p(n) = 2\left\lceil 2\sqrt{n}\right\rceil$. They constructed an example where the cells grow up cell by cell like spirals for these extremal polyominoes (see Figure 1). In general, this is not

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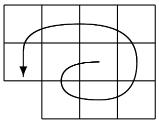

Figure 1. Spiral construction.

the only possibility to reach the minimum perimeter. Thus the question arises to determine the number $e(n)$ of different square polyominoes of order $n$ and with minimum perimeter $p(n)$ where we regard two polyominoes as equal if they can be mapped onto each other by translations, rotations, and reflections.
We will show that these extremal polyominoes can be obtained by deleting squares at the corners of rectangular polyominoes with the minimum perimeter $p(n)$ and with at least $n$ squares. The process of deletion of squares ends if $n$ squares remain forming a desired extremal polyomino. This process leads to an enumeration of the polyominoes with minimum perimeter $p(n)$.

**Theorem 1.** The number $e(n)$ of polyominoes with $n$ squares and minimum perimeter $p(n)$ is given by

$$e(n) = \begin{cases} 1 & if\ n = s^2, \\ \sum_{c=0}^{\lfloor -\frac{1}{2}+\frac{1}{2}\sqrt{1+4s-4t} \rfloor} r_{s-c-c^2-t} & if\ n = s^2 + t, \\ & \quad 0 < t < s, \\ 1 & if\ n = s^2 + s, \\ q_{s+1-t} + \sum_{c=1}^{\lfloor \sqrt{s+1-t} \rfloor} r_{s+1-c^2-t} & if\ n = s^2 + s + t, \\ & \quad 0 < t \leq s, \end{cases}$$

with $s = \lfloor \sqrt{n} \rfloor$, $r_k$, $q_k$ being the coefficient of $x^k$ in the following generating functions $r(x)$ and $q(x)$, respectively. The two generating functions

$$s(x) = 1 + \sum_{k=1}^{\infty} x^{k^2} \prod_{j=1}^{k} \frac{1}{1 - x^{2j}}$$

and

$$a(x) = \prod_{j=1}^{\infty} \frac{1}{1 - x^j}$$

are used in the definition of

$$r(x) = \frac{1}{4} \left( a(x)^4 + 3a(x^2)^2 \right)$$

and

$$q(x) = \tfrac{1}{8} \left( a(x)^4 + 3a(x^2)^2 + 2s(x)^2 a(x^2) + 2a(x^4) \right) \ .$$

The behavior of $e(n)$ is illustrated in Figure 2. It has a local maximum at $n = s^2 + 1$ and $n = s^2 + s + 1$ for $s \geq 1$. Then $e(n)$ decreases to $e(n) = 1$ at $n = s^2$ and $s = s^2 + s$. In the following we give lists of the values of $e(n)$ for $n \leq 143$ and of the two maximum cases $e(s^2 + 1)$ and $e(s^2 + s + 1)$ for $s \leq 50$,

$e(n) = 1, 1, 2, 1, 1, 1, 4, 2, 1, 6, 1, 1, 11, 4, 2, 1, 11, 6, 1, 1, 28, 11, 4, 2, 1, 35, 11, 6, 1, 1, 65, 28, 11, 4, 2, 1, 73, 35, 11, 6, 1, 1, 147, 65, 28, 11, 4, 2, 1, 182, 73, 35, 11, 6, 1, 1, 321, 147, 65, 28, 11, 4, 2, 1, 374, 182, 73, 35, 11, 6, 1, 1, 678, 321, 147, 65, 28, 11, 4, 2, 1, 816, 374, 182, 73, 35, 11, 6, 1, 1, 1382, 678, 321, 147, 65, 28, 11, 4, 2, 1, 1615, 816, 374, 182, 73, 35, 11, 6, 1, 1, 2738, 1382, 678, 321, 147, 65, 28, 11, 4, 2, 1, 3244, 1615, 816, 374, 182, 73, 35, 11, 6, 1,$

Figure 2. $e(n)$ for $n \leq 100$.

1, 5289, 2738, 1382, 678, 321, 147, 65, 28, 11, 4, 2, 1,

$e(s^2 + 1)$ = 1, 1, 6, 11, 35, 73, 182, 374, 816, 1615, 3244, 6160, 11678, 21353, 38742, 68541, 120082, 206448, 351386, 589237, 978626, 1605582, 2610694, 4201319, 6705559, 10607058, 16652362, 25937765, 40122446, 61629301, 94066442, 142668403, 215124896, 322514429, 480921808, 713356789, 1052884464, 1546475040, 2261006940, 3290837242, 4769203920, 6882855246, 9893497078, 14165630358, 20206501603, 28718344953, 40672085930, 57404156326, 80751193346,

$e(s^2 + s + 1)$ = 2, 4, 11, 28, 65, 147, 321, 678, 1382, 2738, 5289, 9985, 18452, 33455, 59616, 104556, 180690, 308058, 518648, 863037, 1420480, 2314170, 3734063, 5970888, 9466452, 14887746, 23235296, 36000876, 55395893, 84680624, 128636339, 194239572, 291620864, 435422540, 646713658, 955680734, 1405394420, 2057063947, 2997341230, 4348440733, 6282115350, 9038897722, 12954509822, 18496005656, 26311093101, 37295254695, 52682844248, 74170401088, 104083151128.

## 2 Proof of Theorem 1

The perimeter cannot be a minimum if the polyomino is disconnected or if it has holes. For connected polyominoes without holes the property of having the minimum perimeter is equivalent to the property of having the maximum number of common edges since an edge which does not belong to two squares is part of the perimeter. The maximum number of common edges $B(n)$ is determined in [5] to be

$$B(n) = 2n - \left\lceil 2\sqrt{n} \right\rceil . \qquad (\star)$$

Denote the degree of a square by the number of its edge-to-edge neighbors. There is a closed walk trough all edge-to-edge neighboring squares of the perimeter. Now we use the terms of graph theory [4] and consider the squares as vertices. So we can define $H$ to be the cycle $x_1 x_2 \ldots x_k x_1$ where the $x_i$ are the squares of the above defined closed walk. For short we will set $|H| = k$ in the following

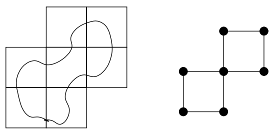

Figure 3.

lemmas. We would like to mention that $x_i = x_j$ with $i \neq j$ is possible in this definition. An example is depicted in Figure 3 together with the corresponding graph of $H$. Let furthermore $h_i$ denote the number of squares $x_j$ in $H$ having degree $i$ in the given polyomino. So

$$|H| = h_1 + h_2 + h_3 + h_4 \ .$$

If a polyomino with minimum perimeter $p(n)$ contains a square of degree 1 (i.e. $h_1 > 0$) then $B(n) - B(n-1) = 1$. Considering the formula $(\star)$ for $B(n)$, this is equivalent to $n = s^2 + 1$ or $n = s^2 + s + 1$ so that we can assume $h_1 = 0$ in general. In the following two lemmas we prove a connection between the number of common edges of a polyomino and $|H|$.

**Lemma 1.** If $h_1 = 0$ then $h_2 = h_4 + 4$.

***Proof.*** Consider the polygon connecting the centers of the squares of $H$. For $2 \leq i \leq 4$ there is an inner angle of $\frac{(i-1)\pi}{2}$ in a square of degree $i$. The sum of the angles of an $|H|$-gon is $(|H| - 2)\pi$. Thus

$$(h_2 + h_3 + h_4 - 2)\pi = h_2\frac{\pi}{2} + h_3\pi + h_4\frac{3\pi}{2}$$

implies the desired equation. □

**Lemma 2.** If $h_1 = 0$ then the number $m$ of common edges of squares of the polyomino is

$$m = 2n - \frac{|H|}{2} - 2.$$

***Proof.*** Every inner square of the polyomino has 4 neighbors. Counting the common edges twice yields

$$2m = 4(n - |H|) + 2h_2 + 3h_3 + 4h_4.$$

From Lemma 1 we obtain

$$2m = 4n - 4|H| + 3(h_2 + h_3 + h_4) - 4 = 4n - |H| - 4.$$ □

In the next lemma we use the knowledge of $|H|$ to bound the number of squares $n$ of a polyomino.

**Lemma 3.** For the maximum area $A(|H|)$ of a polyomino with boundary $H$ and $h_1 = 0$ we have

$$A(|H|) = \begin{cases} \left(\frac{|H|+4}{4}\right)^2 & if \ \ |H| \equiv 0 \ (mod\, 4), \\ \left(\frac{|H|+4}{4}\right)^2 - \frac{1}{4} & if \ \ |H| \equiv 2 \ (mod\, 4). \end{cases}$$

***Proof.*** Because of Lemma 2 the integer $|H|$ has to be an even number. Consider the smallest rectangle surrounding a polyomino and denote the side lengths by $a$ and $b$. Using the fact that the cardinality of the boundary $H$ of a polyomino is at least the cardinality of the boundary of its smallest surrounding rectangle we conclude $|H| \geq 2a + 2b - 4$. The maximum area of the rectangle with given perimeter is obtained if the integers $a$ and $b$ are as equal as possible. Thus $a = \left\lceil \frac{|H|+4}{4} \right\rceil$ and $b = \left\lfloor \frac{|H|+4}{4} \right\rfloor$. The product yields the asserted formula. $\square$

Now we use the fact that we deal with polyominoes with minimum perimeter $p(n)$ and compute $|H|$ as a function of $n$.

**Lemma 4.** For a polyomino with $h_1 = 0$ and with minimum perimeter $p(n)$ we have $|H| = 2\left\lceil 2\sqrt{n} \right\rceil - 4$.

***Proof.*** Since for connected polyominoes without holes the property of having minimum perimeter $p(n)$ is equivalent to the property of having the maximum number $B(n)$ of common edges, we can use $B(n) = 2n - \left\lceil 2\sqrt{n} \right\rceil$ and Lemma 2. $\square$

After providing those technical lemmas we give a strategy to construct all polyominoes with minimum perimeter.

**Lemma 5.** Each polyomino with $h_1 = 0$ and minimum perimeter $p(n)$ can be obtained by deleting squares of a rectangular polyomino with perimeter $p(n)$ consisting of at least $n$ squares.

***Proof.*** Consider a polyomino $P$ with boundary $H$ and minimum perimeter $p(n)$. Denote its smallest surrounding rectangle by $R$. If the cardinality of the boundary of $R$ is less than $|H|$ then $P$ does not have the minimum perimeter due to Lemma 2 and due to the fact that $m = B(n)$ is increasing. Thus $|H|$ equals the cardinality of the boundary of $R$ and $P$ can be obtained by deleting squares from a rectangular polyomino with perimeter $p(n)$ and with an area at least $n$. Only squares of degree 2 can be deleted successively if the perimeter does not change. $\square$

For the following classes of $n$ with $s = \lfloor \sqrt{n} \rfloor$ we now characterize all rectangles being appropriate for a deletion process to obtain $P$ with minimum perimeter $p(n)$.

**(i)** $n = s^2$.
From Lemmas 3 and 4 we know that the unique polyomino with minimum perimeter $p(n)$ is indeed the $s \times s$ square.

**(ii)** $n = s^2 + t$, $0 < t < s$.
Since

$$s^2 < n < \left(s + \frac{1}{2}\right)^2 = s^2 + s + \frac{1}{4}$$

Lemma 4 yields $|H| = 4s - 2$. Denote the side lengths of the surrounding rectangle by $a$ and $b$. With $2a + 2b - 4 = |H| = 4s - 2$ we let $a = s + 1 + c$ and $b = s - c$ with an integer $c \geq 0$. Since at least $n$ squares are needed for the delation process we have $ab \geq n$, yielding

$$0 \leq c \leq \Big\lfloor -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4s - 4t}\Big\rfloor.$$

**(iii)** $n = s^2 + s$.
The $s \times (s + 1)$ rectangle is the unique polyomino with minimum perimeter $p(n)$ due to Lemmas 3 and 4.

**(iv)** $n = s^2 + s + t$, $0 < t \leq s$.
Since

$$\left(s + \frac{1}{2}\right)^2 = s^2 + s + \frac{1}{4} < n < (s+1)^2 = s^2 + 2s + 1$$

Lemma 4 yields $|H| = 4s$. Again $a$ and $b$ denote the side lengths of the surrounding rectangle and we let $a = s+1+c$ and $b = s+1-c$ with an integer $c \geq 0$. The condition $ab \geq n$ now yields

$$0 \leq c \leq \left\lfloor \sqrt{1+s-t} \right\rfloor.$$

We remark that the deletion process does not change the smallest surrounding rectangle since $ab - n < b$, that is the number of deleted squares is less than the number of squares of the smallest side of this rectangle.

In Lemmas 1, 2, 4, and 5 we have required $h_1 = 0$. We now argue that all polyominoes with $h_1 > 0$ and with minimum perimeter $p(n)$ are covered by the deletion process described above ((i)-(iv)).

**Lemma 6.** The construction of Lemma 5 also yields all polyominoes with minimum perimeter $p(n)$ when $h_1 > 0$.
***Proof.*** Any square of degree 1 determines two cases, $n = s^2 + 1$ or $n = s^2 + s + 1$. (See the remark preceeding Lemma 1.) The deletion of this square leaves a polyomino $P$ with minimum perimeter $p(n-1)$.

In the first case $P$ has the shape of the $s \times s$ square as in (i). Thus we get the original polyomino by deleting $s-1$ squares from the $s \times (s+1)$ rectangle and this is covered in (ii).

In the second case $P$ has the shape of the $s \times (s+1)$ rectangle as in (iii). Thus we get the original polyomino by deleting $s-1$ squares from the $s \times (s+2)$ rectangle or by deleting $s$ squares from the $(s+1) \times (s+1)$ square, and this is covered in (iv). $\square$

So far we have described those rectangles from which squares of degree 2 are removed. Now we examine the process of deleting squares from a rectangular polyomino. Squares of degree 2 can only be located in the corners of the polyomino. What shape has the set of deleted squares at a corner? There is a maximum square of squares

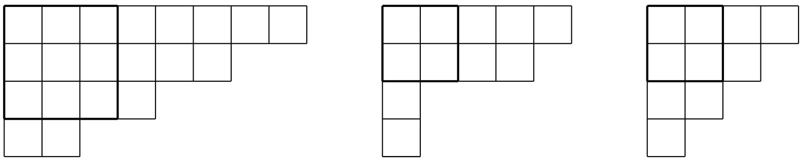

Figure 4. Shape of the deleted squares at the corners.

at the corner, the so called “Durfee square”, together with squares in rows and columns of decreasing length from outside to the interior part of the polyomino. To count the different possibilities of the sets of deleted squares with respect to the number of the deleted squares we use the concept of a generating function $f(x) = \sum_{i=0}^{\infty} f_i x^i$. Here the coefficient $f_i$ gives the number of different ways to use $i$ squares. Since the rows and columns are ordered by their lengths they form Ferrer’s diagrams with generating function $\prod_{j=1}^{\infty} \frac{1}{1-x^j}$ each [2]. So the generating function for the sets of deleted squares in a single corner is given by

$$a(x) = \prod_{j=1}^{\infty} \frac{1}{1-x^j} \, .$$

Later we will also need the generating function $s(x)$ for the sets of deleted squares being symmetric with respect to the diagonal of the corner square. Since such a symmetric set of deleted squares consists of a square of $k^2$ squares and the two mirror images of a Ferrer’s diagrams with height or width at most $k$ we get

$$s(x) = 1 + \sum_{k=1}^{\infty} x^{k^2} \prod_{j=1}^{k} \frac{1}{1-x^{2j}} \, .$$

We now consider the whole rectangle. Because of different sets of symmetry axes we distinguish between squares and rectangles. We define generating functions $q(x)$ and $r(x)$ so that the coefficient of $x^k$ in $q(x)$ and $r(x)$ is the number of ways to remove $k$ squares from all four corners of a square or a rectangle, respectively. We mention that the coefficient of $x^k$ gives the desired number only if $k$ is smaller than the small side of the rectangle.

Since we want to count polyominoes with minimum perimeter up to translation, rotation, and reflection, we have to factor out these symmetries. Here the general tool is the lemma of Cauchy-Frobenius, see e.g. [6]. We remark that we do not have to consider translations because we describe the polyominoes without coordinates.

**Lemma (Cauchy-Frobenius, weighted form).** Given a group action of a finite group $G$ on a set $S$ and a map $w : S \longrightarrow R$ from $S$ into a commutative ring $R$ containing $\mathbb{Q}$ as a subring. If $w$ is constant on the orbits of $G$ on $S$, then we have, for any transversal $\mathcal{T}$ of the orbits:

$$\sum_{t\in\mathcal{T}} w(t) = \frac{1}{|G|} \sum_{g\in G} \sum_{s\in S_g} w(s)$$

where $S_g$ denotes the elements of $S$ being fixed by $g$, i.e.

$$S_g = \{s \in S | s = gs\} .$$

For $G$ we take the symmetry group of a square or a rectangle, respectively, for $S$ we take the sets of deleted squares on all 4 corners, and for the weight $w(s)$ we take $x^k$, where $k$ is the number of squares in $s$. Here we will only describe in detail the application of this lemma for a determination of $q(x)$. We label the 4 corners of the square by $1$, $2$, $3$, and $4$, see Figure 5. In Table 1 we list the 8

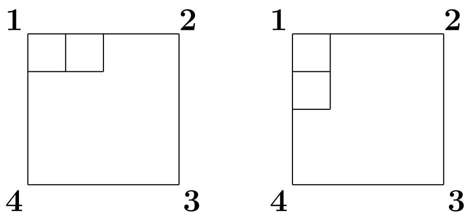

Figure 5.

permutations $g$ of the symmetry group of a square, the dihedral group on 4 points, together with the corresponding generating functions for the sets $S_g$ being fixed by $g$.

| | |
|---|---|
| $(1)(2)(3)(4)$ | $a(x)^4$ |
| $(1,2,3,4)$ | $a(x^4)$ |
| $(1,3)(2,4)$ | $a(x^2)^2$ |
| $(1,4,3,2)$ | $a(x^4)$ |
| $(1,2)(3,4)$ | $a(x^2)^2$ |
| $(1,4)(2,3)$ | $a(x^2)^2$ |
| $(1,3)(2)(4)$ | $s(x)^2a(x^2)$ |
| $(1)(2,4)(3)$ | $s(x)^2a(x^2)$ |

Table 1. Permutations of the symmetry group of a square together with the corresponding generating functions of $S_g$.

The generating function of the set of deleted squares on a corner is $a(x)$. If we consider the configurations being fixed by the identity element $(1)(2)(3)(4)$ we see that the sets of deleted squares at the 4 corners are independent and so $|S_{(1)(2)(3)(4)}| = a(x)^4$. In the case when $g = (1,2,3,4)$ the sets of deleted squares have to be the same for all 4 corners and we have $|S_{(1,2,3,4)}| = a(x^4)$. For the double transposition $(1,2)(3,4)$ the sets of deleted squares at corners

$1$ and $2$, and the sets of deleted squares at corners $3$ and $4$ have to be equal. Because the sets of deleted squares at corner points $1$ and $3$ are independent we get $|S_{(1,2)(3,4)}| = a(x^2)^2$. Next we consider $g = (1)(2,4)(3)$. The sets of deleted squares at corners $2$ and $4$ have to be equal. If we apply $g$ on the polyomino of the left hand side of Figure 5 we receive the polyomino on the right hand side and we see that in general the sets of deleted squares at corners $1$ and $3$ have to be symmetric. Thus $|S_{(1)(2,4)(3)}| = s(x)^2 a(x^2)$. The other cases are left to the reader. Summing up and a division by 8 yields

$$q(x) = \tfrac{1}{8}\left(a(x)^4 + 3a(x^2)^2 + 2s(x)^2 a(x^2) + 2a(x^4)\right) .$$

For the symmetry group of a rectangle we analogously obtain

$$r(x) = \frac{1}{4}\left(a(x)^4 + 3a(x^2)^2\right).$$

With Lemma 6, the preceeding characterization of rectangles being appropriate for a deletion process and the formulas for $a(x)$, $s(x)$, $q(x)$, and $r(x)$ we have the proof of Theorem 1 at hand.

We would like to close with the first entries of a complete list of polyominoes with minimum perimeter $p(n)$, see Figure 6.

## 3 Acknowledgments

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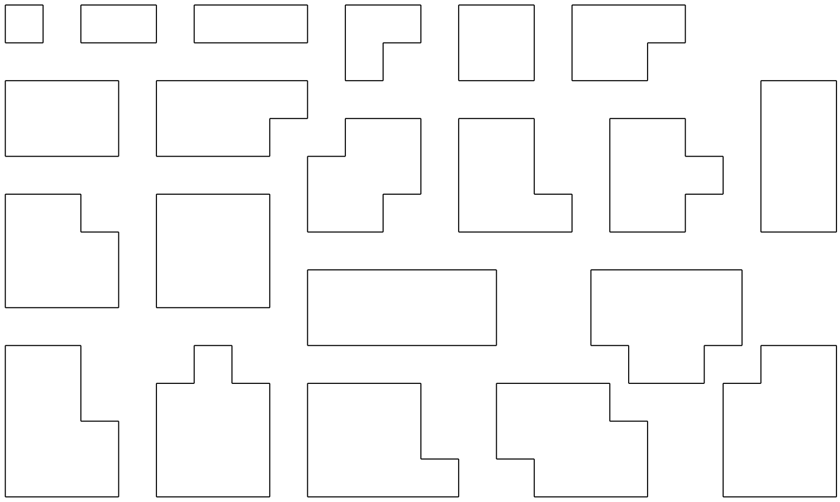

Figure 6. Polyominoes with minimum perimeter $p(n)$ for $n \leq 11$.

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