Polynomials
for 'same game'

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24. Oktober 2001
In N.J.A. Sloane’s On-Line Encyclopedia of Integer Sequences the sequence A035615 deals with the winning n-digit binary strings in ‘same game’. Erich Friedman describes this game as: Strings that can be reduced to null string by repeatedly removing an entire run of two or more consecutive digits. Example: 11011001 is a winning string since 110(11)001 → (111) → null.

The first values of this sequence are: 0, 2, 2, 6, 12, 26, 58, 126, 278, 602, 1300, 2774, 5878, 12350, 25778, 53470, 110332, 226610, 463602, 945214, 1921550, 3896642, 7885092, 15927086, 32121582, 64697726.

Algorithm:

\[ \text{win}(\text{string } s, \text{int links, int rechts}) \]

if rechts-links==0 return \text{false}
if all digits in s a equal return \text{true}
for all entire runs of two or more consecutive digits r do
  if \text{win}(s-r,links',rechts')==\text{true} return \text{true}
return \text{false}

main()

solutions=0;
for all binary strings s with length n do
  if \text{win}(s,1,r) = \text{true} solutions++

In A035617 the ‘same game’ is treated for ternary strings. More general let \( a(n,b) \) denote the number of winning n-digit b-ary strings in ‘same game’. The above algorithm can obviously used for the general problem. Because of the fact, that a winning n-digit b-ary string can only have \( \left\lfloor \frac{n}{2} \right\rfloor \) different digits there exist for \( a(n,b) \) a polynomial with maximal degree \( \left\lfloor \frac{n}{2} \right\rfloor \).

The first few polynomials are:

\[
\begin{align*}
a(1, b) &= 0 \\
a(2, b) &= b \\
a(3, b) &= b \\
a(4, b) &= 2b^2 - b \\
a(5, b) &= 5b^2 - 4b \\
a(6, b) &= 5b^3 - 3b^2 - b \\
a(7, b) &= 21b^3 - 35b^2 + 15b \\
a(8, b) &= 14b^4 - 36b^2 + 23b \\
a(9, b) &= 84b^4 - 204b^3 + 162b^2 - 41b \\
a(10, b) &= 42b^5 + 60b^4 - 405b^3 + 465b^2 - 161b \\
a(11, b) &= 330b^5 - 990b^4 + 990b^3 - 341b^2 + 12b 
\end{align*}
\]

Literatur: