LATTICE CONTEXTS – A GENERALIZATION IN FORMAL CONCEPT ANALYSIS

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Abstract. We generalize formal contexts to lattice contexts using a galois connection between two lattices. The galois connection defines derivation operators, yielding the concept lattice. The connection to the classical one-valued contexts is revealed. In particular, a method is presented to obtain a one-valued context with isomorphic concept lattice.

As an application, we define fuzzy contexts as well as many-valued contexts as subclasses of lattice contexts. In particular, for many-valued contexts, this construction allows to define derivation operators directly on the context (i.e. without using scales).

1. Introduction

The original definition of formal contexts (also known as one-valued contexts), on which the formal concept analysis is founded according to [GaWi99], is only partly applicable to real world problems. This led to several extensions, like the many-valued contexts ([GaWi99, ch.1.3]), triadic contexts ([LeWi95]), fuzzy contexts and fuzzy-valued contexts ([Um94]).

The strategies to handle these extensions were two folded: On one hand, one tried to find one-valued contexts corresponding to the generalization in question, like using scales in the case of many-valued contexts. On the other hand, one introduced equivalent structures (derivation operators, concept lattice, ...) directly, as in the case of fuzzy contexts. But the solutions found were always specific for only one extension, either for the many-valued contexts, or for the fuzzy contexts.

The intent of this article is to provide one general structure, the *lattice context*, which contains each of the above mentioned extension as a special subclass. Even the one-valued context itself – in this article henceforth termed *classical context* – is just a special case of this lattice context.

This general definition of a context will provide derivation operators, yielding a concept lattice. Furthermore, we will study the relation between lattice contexts and classical contexts. In particular, for each lattice context, we will give a classical context having an isomorphic concept lattice.

Using proper subclasses of lattice contexts, one can introduce other extensions of formal contexts. We will clarify this strategy with fuzzy contexts and many-valued contexts: Both can be defined as special lattice contexts. (Therewith, we specify in particular a derivation operator for many-valued contexts.)

This article presents part of the results of the diplomarbeit [Gu97]. Especially the applications of the lattice context to fuzzy contexts and to many-valued contexts are described there in more detail.

2. Generalized Contexts

Definition 2.1. A tripel $\mathbb{K} = (X, Y, \varphi)$ is called *generalized context*, or *order context*, iff $(X, \leq)$ and $(Y, \leq)$ are ordered sets, and the mapping $\varphi \colon X \to Y$ has a dual adjoint $\varphi^{\mathrm{d}} \colon Y \to X$, i.e. $(\varphi, \varphi^{\mathrm{d}})$ is a galois connection. Further, $\mathbb{K}$ is called *lattice context*, iff X and Y are complete lattices and $\mathbb{K}$ is called *power set context*, iff $X = \mathfrak{P}(G)$ and $Y = \mathfrak{P}(M)$ are power sets.
We call $x' := \varphi(x)$ the derivation of $x \in X$ and $y' := \varphi^{\mathrm{d}}(y)$ the derivation of $y \in Y$.

Lemma 2.2. *In the generalized context $\mathbb{K} = (X, Y, \varphi)$ with $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$, the following holds:*

(1) $x_1 \leq x_2 \Rightarrow x_2' \leq x_1'$ 1'. $y_1 \leq y_2 \Rightarrow y_2' \leq y_1'$
(2) $x \leq x''$ 2'. $y \leq y''$
(3) $x' = x'''$ 3'. $y' = y'''$
(4) $x \leq y' \Leftrightarrow y \leq x'$

Proof. These are basic properties of galois connections. □

In a generalized context, we can define formal concepts as follows:

Definition 2.3. Let $\mathbb{K} = (X, Y, \varphi)$ be a generalized context. A pair of elements $(x, y) \in X \times Y$ is called *concept* of $\mathbb{K}$, iff $x' = y$ and $y' = x$. Here, x is the *extent* and y the *intent* of the concept (x, y). $\mathfrak{B}(\mathbb{K})$ denotes the set of all concepts of the context $\mathbb{K}$.

Lemma 2.2 implies, that $\mathfrak{B}(\mathbb{K}) = \{(x'', x') | x \in X\} = \{(y', y'') | y \in Y\} \subseteq \varphi^{\mathrm{d}}(Y) \times \varphi(X)$. So we have natural isomorphisms $(x, y) \mapsto x$ and $(x, y) \mapsto y$ from $\mathfrak{B}(\mathbb{K})$ to $\varphi^{\mathrm{d}}(Y)$ and to $\varphi(X)$, respectively.

As a consequence, we can define an order on $\mathfrak{B}(\mathbb{K})$ via

$$(x_1, y_1) \leq (x_2, y_2) :\Leftrightarrow x_1 \leq x_2 \, (\Leftrightarrow y_1 \geq y_2).$$

In particular, the mapping $(x, y) \mapsto x$ defines an isomorphism between the ordered sets $\mathfrak{B}(\mathbb{K})$ and $\varphi^{\mathrm{d}}(Y)$, as well as $(x, y) \mapsto y$ defines an antiisomorphism between the ordered sets $\mathfrak{B}(\mathbb{K})$ and $\varphi(X)$.

In the case of $(x_1, y_1) \leq (x_2, y_2)$, we call (x_1, y_1) a *subconcept* of (x_2, y_2), and (x_2, y_2) a *superconcept* of (x_1, y_1).

Example 2.4. The (classical) formal context according to [GaWi99] is defined as a tripel $\mathbb{K} = (G, M, I)$ with two sets G and M and an incidence relation $I \subseteq G \times M$. The derivations for subsets $A \in \mathfrak{P}(G)$ and $B \in \mathfrak{P}(M)$ are given as

$$\begin{aligned} A' &:= \{m \in M | (g, m) \in I, \forall g \in A\}, \\ B' &:= \{g \in G | (g, m) \in I, \forall m \in B\}. \end{aligned}$$

These derivation operators define a galois connection between the power sets of G and M:

$$\begin{aligned} \varphi &\colon \mathfrak{P}(G) \to \mathfrak{P}(M), A \mapsto A', \\ \psi &\colon \mathfrak{P}(M) \to \mathfrak{P}(G), B \mapsto B'. \end{aligned}$$

Hence, we can view $\mathbb{K}$ as the power set context $(\mathfrak{P}(G), \mathfrak{P}(M), \varphi)$.

Vice versa, we can identify every power set context $\mathbb{K} = (\mathfrak{P}(G), \mathfrak{P}(M), \varphi)$ with a classical context (G, M, I) with

$$I \subseteq G \times M, (g, m) \in I :\Leftrightarrow m \in \varphi(\{g\}) \, .$$

Thus, the classical contexts are exactly the power set contexts.

In order to have some classification of contexts, we need an idea of isomorphy on contexts. We call two contexts isomorphic iff their sets of concepts are isomorphic:

Definition 2.5. Let $\mathbb{K}_i = (X_i, Y_i, \varphi_i)$ $(i = 1, 2)$ be two generalized contexts. Then $\mathbb{K}_1$ is *isomorphic* to $\mathbb{K}_2$, in short $\mathbb{K}_1 \cong \mathbb{K}_2$, iff their exists an order isomorphism between the ordered sets $\mathfrak{B}(\mathbb{K}_1)$ and $\mathfrak{B}(\mathbb{K}_2)$.

From now on, we restrict ourself to lattice contexts, i.e. contexts (X, Y, φ), where X and Y are complete lattices. As these contexts have rich structure, we can develop further propositions.

Lemma 2.6. *If $\mathbb{K} = (X, Y, \varphi)$ is a lattice context, then for $A \subseteq X$ and $B \subseteq Y$, the derivations of the suprema are as follows:*

$$\begin{aligned} (\bigvee A)' &= \bigwedge\{x' | x \in A\}, \\ (\bigvee B)' &= \bigwedge\{y' | y \in B\}. \end{aligned}$$

Proof. This, too, is a well known property of galois connections. □

Having this result, we can generalize the first part of the basic theorem on concept lattices ([GaWi99, Theorem 3, p. 20]):

Theorem 2.7. *Let $\mathbb{K} = (X, Y, \varphi)$ be a lattice concept. Then $\mathfrak{B}(\mathbb{K})$ is a complete lattice. Infimum und supremum for a subset of concepts $\{(x_i, y_i) | i \in I\} \subseteq \mathfrak{B}(\mathbb{K})$ (with an arbitrary set of indices I) is given via:*

$$\begin{aligned} \bigwedge(x_i, y_i) &= (\bigwedge x_i, (\bigwedge x_i)') = (\bigwedge x_i, (\bigvee y_i)''), \\ \bigvee(x_i, y_i) &= ((\bigwedge y_i)', \bigwedge y_i) = ((\bigvee x_i)'', \bigwedge y_i). \end{aligned}$$

Proof. $\mathfrak{B}(\mathbb{K})$ is isomorphic to $\varphi^{\mathrm{d}}(Y)$, which is, by a well known property of galois connections on complete lattices, a complete lattice. The given representations of infimum and supremum follow from Lemma 2.6. □

3. Incidence relations

The success of classical concept analysis is attributed to a large amount to the fact, that it is not necessary to store the incidence information for every element of the power sets $\mathfrak{P}(G)$ and $\mathfrak{P}(M)$. It is solely necessary to store information for the singletons (which can be identified with the elements $g \in G$ and $m \in M$, respectively). There exists an efficient algorithm to decide, if an element $A \in \mathfrak{P}(G)$ stands in relation with an element $B \in \mathfrak{P}(M)$: Just test if $(g, m) \in I$ for all $g \in A$ and for all $m \in B$.

Our next question is, how we can generalize this idea of storing only as little information as necessary without loosing any information. We will do this by considering *supremum-dense* subsets of a lattice X. A set $G \subseteq X$ is supremum-dense in X, iff one can represent each element $x \in X$ as a supremum of elements of G:

$$\forall x \in X : x = \bigvee\{g \in G | g \leq x\}.$$

In most cases relevant to practice, we will have a nice supremum-dense subset, namely the set $I^{\vee}(X)$ of all $\vee$-irreducible elements of X. (An element $x \in X$ is called *$\vee$-irreducible*, iff it cannot be represented as a supremum of strictly smaller elements.) If $I^{\vee}(X)$ is supremum-dense, it is the best possible supremum-dense subset of X in the sense, that it is the smallest one with this property. For example, if X is finite, then $I^{\vee}(X)$ is always supremum-dense.

In power sets $\mathfrak{P}(G)$, for example, we can represent each element (i.e. each subset of G) as a union of singletons (which are isomorphic to the elements of G). The singletons in turn are exactly the $\vee$-irreducible elements of the power set. We

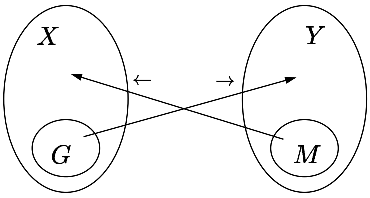
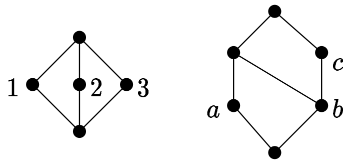


FIGURE 3.1. Illustration of the arrow operators



I:

	a	b	c
1	X	X	
2		X	X
3		X	

J:

	a	b	c
1	X	X	
2			X
3		X	

FIGURE 3.2. I is incidence relation between the lattices X and Y, but J is not.

specify a classical context, by giving a relation between the elements of G and M, i.e. between the $\vee$-irreducible elements of $\mathfrak{P}(G)$ and $\mathfrak{P}(M)$.

In the case of a lattice context (X, Y, φ), we will show, that it is sufficient, to store the incidence information between two *supremum-dense* subsets $G \subseteq X$ and $M \subseteq Y$. Thus, every lattice context (X, Y, φ) can be represented as a relation $I \subseteq G \times M$.

Hence, within this section, we will always consider two complete lattices X and Y, having supremum-dense subsets $G \subseteq X$ respective $M \subseteq Y$.

For a relation $I \subseteq G \times M$, we introduce two operators ${}^{\rightarrow}: G \to Y$ and ${}^{\leftarrow}: M \to X$ (see also Figure 3.1):

$$g^{\rightarrow} := \bigvee \{m \in M | (g, m) \in I\},$$
$$m^{\leftarrow} := \bigvee \{g \in G | (g, m) \in I\}.$$

Using these arrow operators, we define a special class of relations on $G \times M$:

Definition 3.1. Let X and Y be complete lattices, and let $G \subseteq X$ and $M \subseteq Y$ be supremum-dense subsets. The relation $I \subseteq G \times M$ is called *incidence relation* (between X and Y, resp. G and M), iff for all $g \in G$ and $m \in M$ the following conditions are satisfied:

$$m \leq g^{\rightarrow} \Rightarrow (g, m) \in I,$$
$$g \leq m^{\leftarrow} \Rightarrow (g, m) \in I.$$

Example 3.2.

(1) Figure 3.2 shows two lattices X and Y with supremum-dense subsets $G = \{1, 2, 3\}$ of X and $M = \{a, b, c\}$ of Y, as well as an incidence relation I and a further relation J between G and M. J is not an incidence relation, as we have: $b \leq c = 2^{\rightarrow}$, but $(2, b) \notin J$.

(2) If $X = \mathfrak{P}(G)$ and $Y = \mathfrak{P}(M)$ are power sets, then every relation $I \subseteq I^{\vee}(\mathfrak{P}(G)) \times I^{\vee}(\mathfrak{P}(M)) \cong G \times M$ is an incidence relation between $\mathfrak{P}(G)$ and $\mathfrak{P}(M)$ (in the sense of Definition 3.1), as for example, $\{g\} \subseteq \{m\}^{\leftarrow} = \{g \in G | (\{g\}, \{m\}) \in I\}$ implies, that $(\{g\}, \{m\}) \in I$.

There is a close connection between incidence relations and galois functions: We will show, that the incidence relations between X and Y (resp. arbitrary supremum-dense sets G and M) describe exactly the galois functions between X and Y.

Lemma 3.3. *If I is an incidence relation, then we know about the arrow operators $^{\rightarrow}\colon G \to Y$ and $^{\leftarrow}\colon M \to X$:*

(1) *For all $g \in G$ and all $m \in M$ we have:*

$$m \le g^{\rightarrow} \Leftrightarrow g \le m^{\leftarrow}.$$

(2) *The arrow operators are antitone.*

Proof.
1. For the incidence relation I are equivalent:

$$\begin{aligned} m \le g^{\rightarrow} &\Leftrightarrow (g,m) \in I, \quad \text{as well as} \\ g \le m^{\leftarrow} &\Leftrightarrow (g,m) \in I. \end{aligned}$$

("$\Leftarrow$" follows from the definition of the arrow operators, and "$\Rightarrow$" are the demanded properties of incidence relations.)
2. If $g_1, g_2 \in G$ with $g_1 \le g_2$, then, for every $m \in M$:

$$m \le g_2^{\rightarrow} \Rightarrow g_2 \le m^{\leftarrow} \Rightarrow g_1 \le m^{\leftarrow} \Rightarrow m \le g_1^{\rightarrow}.$$

Therewith, and with the supremum-density of M, it follows for arbitrary $y \in Y$:

$$\begin{aligned} y \le g_2^{\rightarrow} &\Rightarrow m \le g_2^{\rightarrow}, \forall m \le y \\ &\Rightarrow m \le g_1^{\rightarrow}, \forall m \le y \\ &\Rightarrow y = \bigvee\{m \in M | m \le y\} \le g_1^{\rightarrow} \end{aligned}$$

Thus, $g_1^{\rightarrow} \ge g_2^{\rightarrow}$. It follows, that $^{\rightarrow}$ is antitone. Analoguously we can show, that $^{\leftarrow}$ is antitone, too. □

Thus, in the case of incidence relations, the arrow operators have similar properties as a galois connection. The difference is, that the domains of the arrow operators are the sets G and M instead of the whole lattices X and Y.

We will now specify a bijection between the set of all galois functions from X to Y, and the set of all incidence relations between X and Y resp. G and M.

Definition 3.4. Let X, Y, G and M be as in 3.1.

(1) If $\varphi : X \to Y$ is a galois function, we call $\mathrm{IR}_{G,M}(\varphi) \subseteq G \times M$, defined by

$$(g,m) \in \mathrm{IR}_{G,M}(\varphi) :\Leftrightarrow m \le \varphi(g),$$

the incidence relation corresponding to φ (resp. G and M).

(2) If $I \subseteq G \times M$ is an incidence relation, we call $\mathrm{GF}_{X,Y}(I)\colon X \to Y$, defined by

$$\mathrm{GF}_{X,Y}(I)(x) := \bigwedge\{g^{\rightarrow} | g \in G, g \le x\},$$

the galois function corresponding to I (between X and Y).

Lemma 3.5. *Let X, Y, G and M be as in 3.1.*

(1) *If $\varphi : X \to Y$ is a galois function, then $I := \mathrm{IR}_{G,M}(\varphi)$ is an incidence relation, and for all $g \in G$ and $m \in M$, we have:*

$$\begin{aligned} g^{\rightarrow} &= \varphi(g), \\ m^{\leftarrow} &= \varphi^d(m). \end{aligned}$$

(2) *If $I \subseteq G \times M$ is an incidence relation, then $\varphi := \mathrm{GF}_{X,Y}(I)$ is a galois function, and for all $g \in G$ and $m \in M$, we have:*

$$\begin{aligned} \varphi(g) &= g^{\rightarrow}, \\ \varphi^{d}(m) &= m^{\leftarrow}. \end{aligned}$$

Proof.
1. Let φ be a galois connection and $I := \mathrm{IR}_{G,M}(\varphi)$. Because of the supremum-density of G and M, and because of 2.2(4) (needed for the second equation), we have:

$$\begin{aligned} g^{\rightarrow} &= & \bigvee\{m \in M | m \leq \varphi(g)\} & &= \varphi(g), \\ m^{\leftarrow} &= \bigvee\{g \in G | m \leq \varphi(g)\} &= \bigvee\{g \in G | g \leq \varphi^{\mathrm{d}}(m)\} & &= \varphi^{\mathrm{d}}(m). \end{aligned}$$

Hence, we can show, that I fulfills the conditions of incidence relations:

$$\begin{aligned} m \leq g^{\rightarrow} &\Rightarrow & m \leq \varphi(g) & &\Rightarrow (g,m) \in I. \\ g \leq m^{\leftarrow} &\Rightarrow g \leq \varphi^{d}(m) &\Rightarrow m \leq \varphi(g) & &\Rightarrow (g,m) \in I. \end{aligned}$$

2. Vice versa, let $I \subseteq G \times M$ be an incidence relation, and let $\varphi := \mathrm{GF}_{X,Y}(I)$. We will show, that φ has the dual adjoint ψ:

$$\psi : y \mapsto \bigwedge\{m^{\leftarrow} | m \in M, m \leq y\}.$$

For all $x \in X$ and $y \in Y$, the following equations are equivalent (Again, we use the supremum-density of G and M, and Lemma 3.3(1)):

$$\begin{aligned} x &\leq \psi(y), \\ \bigvee\{g \in G | g \leq x\} &\leq \bigwedge\{m^{\leftarrow} | m \in M, m \leq y\}, \\ g &\leq m^{\leftarrow} \qquad \forall g \in G : g \leq x,\ \forall m \in M : m \leq y, \\ m &\leq g^{\rightarrow} \qquad \forall g \in G : g \leq x,\ \forall m \in M : m \leq y, \\ \bigvee\{m \in M | m \leq y\} &\leq \bigwedge\{g^{\rightarrow} | g \in G : g \leq x\}, \\ y &\leq \varphi(x). \end{aligned}$$

Thus, (φ, ψ) is a galois connection. Additionally, because of the antitony of $^{\rightarrow}$ and $^{\leftarrow}$ (Lemma 3.3(2)), we see, that for all $g \in G$:

$$\begin{aligned} \varphi(g) &= \bigwedge\{\tilde{g}^{\rightarrow} | \tilde{g} \in G, \tilde{g} \leq g\} &= g^{\rightarrow}, \\ \psi(m) &= \bigwedge\{\tilde{m}^{\leftarrow} | \tilde{m} \in M, \tilde{m} \leq m\} &= m^{\leftarrow}. \end{aligned}$$

□

As summary of this chapter, we formulate the following theorem:

Theorem 3.6. *Let X, Y, G and M be as in 3.1. Then $\mathrm{IR}_{G,M}$ and $\mathrm{GF}_{X,Y}$ are inverse bijections between the set of all galois functions from X to Y, and the set of all incidence relations between X and Y resp. G and M.*

Proof.
1. Let $\varphi\colon X \to Y$ be a galois function, and $I := \mathrm{IR}_{G,M}(\varphi)$. Considering $\mathrm{GF}_{X,Y}(I)$, we can show for each $x \in X$:

$$\begin{aligned} \mathrm{GF}_{X,Y}(I)(x) &= \bigwedge\{g^{\rightarrow} | g \in G, g \leq x\} \\ &= \bigwedge\{\varphi(g) | g \in G, g \leq x\} \\ &= \varphi(\bigvee\{g \in G | g \leq x\}) \\ &= \varphi(x), \end{aligned}$$

using Lemma 3.5, Lemma 2.6, and the supremum density of G.
2. Let $I \subseteq G \times M$ be an incidence relation and $\varphi := \mathrm{GF}_{X,Y}(I)$. Then, we can specify $\mathrm{IR}_{G,M}(\varphi)$ by:

$$(g,m) \in \mathrm{IR}_{G,M}(\varphi) \Leftrightarrow m \leq \varphi(g) \Leftrightarrow m \leq g^{\rightarrow} \Leftrightarrow (g,m) \in I.$$

Altogether, we have shown, that $\mathrm{IR}_{G,M}$ and $\mathrm{GF}_{X,Y}$ are inverse bijections. □

As a consequence, we can represent each galois function φ between complete lattices X and Y – and herewith each lattice context (X, Y, φ) – by a cross table, like it is common for classical contexts. As rows, we take a (preferably small) supremum-dense subset $G \subseteq X$, and as columns a supremum-dense subset $M \subseteq Y$. We enter a cross into the cell (g,m), iff $m \leq \varphi(g)$. Then, we can retrieve the derivations from this cross table: $x' = \bigwedge\{g^{\rightarrow} | g \leq x\}$ and $y' = \bigwedge\{m^{\leftarrow} | m \leq y\}$.

The close connection between the power set context $\mathbb{K} = (\mathfrak{P}(G), \mathfrak{P}(M), \varphi)$ and the classical context $(G, M, \mathrm{IR}_{G,M}(\varphi))$ was already shown in Example 2.4. (It is not difficult to see, that the relation specified there is equal to $\mathrm{IR}_{G,M}(\varphi)$, and that the operators $^{\rightarrow}$ and $^{\leftarrow}$ describe the classical derivation operators.) It follows, that the concept lattices of the power set context and the classical context are isomorphic.

Thus, what we have done in this section is, that we have generalized the former considerations, which applied only to power set contexts, to more general lattice contexts: Now, we can associate to each lattice context (X, Y, φ) a classical context $(G, M, \mathrm{IR}_{G,M}(\varphi))$, whereas $G \subseteq X$ and $M \subseteq Y$ are supremum-dense subsets.

4. INCIDENCE-ISOMORPHIC CONTEXTS

We could now assume, that the concept lattices of a lattice context and of the associated classical context are isomorphic – as it is the case for power set contexts. We will prove this assumption within this section by considering a certain relation between lattice contexts based on $\mathrm{IR}_{G,M}(\varphi)$: the *incidence-isomorphy*. In order to do this, we first introduce isomorphisms between incidence relations:

Definition 4.1. For $i = 1,2$ let X_i and Y_i be complete lattices, $G_i \subseteq X_i$, $M_i \subseteq Y_i$ respectively supremum-dense subsets, and $I_i \subseteq G_i \times M_i$ incidence relations. Then, I_1 and I_2 are called *isomorphic*, in short $I_1 \cong I_2$, iff there exist two bijections $\gamma_X : G_1 \to G_2$ and $\gamma_Y : M_1 \to M_2$ such, that

$$(g,m) \in I_1 \Leftrightarrow (\gamma_X(g), \gamma_Y(m)) \in I_2.$$

In this case, $\gamma := (\gamma_X, \gamma_Y)$ is called *incidence-isomorphism.*

Example 4.2. Let X and Y be complete lattices, $G \subseteq X$ and $M \subseteq Y$ supremum-dense, and $I \subseteq G \times M$ an incidence relation. Consider the power set lattices $\mathfrak{P}(G)$ and $\mathfrak{P}(M)$, and the incidence relation $I_* \subseteq I^{\vee}(\mathfrak{P}(G)) \times I^{\vee}(\mathfrak{P}(M))$ between the singletons of G and M, with

$$(\{g\}, \{m\}) \in I_* :\Leftrightarrow (g,m) \in I.$$

Then, both incidence relations I and I_* are isomorphic. The two bijections are given by $\gamma_x : g \mapsto \{g\}$ and $\gamma_y : m \mapsto \{m\}$.

Using the isomorphy between incidence relations, we introduce incidence-isomorphy between lattice contexts:

Definition 4.3. Let $\mathbb{K}_1 = (X_1, Y_1, \varphi_1)$ and $\mathbb{K}_2 = (X_2, Y_2, \varphi_2)$ be lattice contexts. Then $\mathbb{K}_1$ and $\mathbb{K}_2$ are called *incidence-isomorphic*, iff there exist supremum-dense subsets $G_i \subseteq X_i$ and $M_i \subseteq Y_i$ $(i = 1,2)$, such that the incidence relations corresponding to φ_i are isomorphic, i.e. iff

$$\mathrm{IR}_{G_1,M_1}(\varphi_1) \cong \mathrm{IR}_{G_2,M_2}(\varphi_2).$$

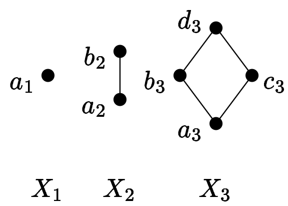


FIGURE 4.1. The lattices X_1 and X_2 are incidence-isomorphic, as well as X_2 and X_3, but X_1 and X_3 are not.

Using the corresponding incidence-isomorphism $\gamma = (\gamma_X, \gamma_Y)$, we then define two new functions $\mu_X : X_1 \to X_2$ and $\mu_Y : Y_1 \to Y_2$ as follows:

$$\mu_X(x) := \bigvee \{\gamma_X(g) | g \in G_1, g \leq x\},$$
$$\mu_Y(y) := \bigvee \{\gamma_Y(m) | m \in M_1, m \leq y\}.$$

We call $\mu := (\mu_X, \mu_Y)$ the *concept isomorphism* between $\mathbb{K}_1$ and $\mathbb{K}_2$.

Warning 4.4.

(1) There may exist elements $g, h \in G_1$ with $g \geq h$, but $\gamma_X(g) \not\geq \gamma_X(h)$. Then $\mu_X(g) \geq \gamma_X(g) \vee \gamma_X(h) > \gamma_X(g)$. So, we then have in particular an element $g \in G_1$ with: $\mu_X(g) \neq \gamma_X(g)$.

(2) Incidence-isomorphy is not transitive:
Consider for example the lattices X_1, X_2 and X_3 in Figure 4.1, together with a convenient complete lattice Y, a supremum-dense subset $M \subseteq Y$ and convenient galois functions φ_i $(i = 1, 2, 3)$. We therewith have three lattice contexts $\mathbb{K}_i := (X_i, Y, \varphi_i)$.
There are isomorphic, supremum dense subsets $G_1 := \{a_1\} \subseteq X_1$ and $G_2 := \{b_2\} \subseteq X_2$, thus the two lattice contexts $\mathbb{K}_1$ and $\mathbb{K}_2$ are incidence-isomorphic, if for all $m \in M$ holds: $m \leq \varphi_1(a_1) \Leftrightarrow m \leq \varphi_2(b_2)$. Furthermore, there are isomorphic supremum-dense subsets $G_2' := \{a_2, b_2\} \subseteq X_2$ and $G_3 := \{b_3, c_3\} \subseteq X_3$. Thus $\mathbb{K}_2$ and $\mathbb{K}_3$ are incidence-isomorphic, too, if for all $m \in M$ holds: $m \leq \varphi_2(a_2) \Leftrightarrow m \leq \varphi_3(b_3)$ and $m \leq \varphi_2(b_2) \Leftrightarrow m \leq \varphi_3(c_3)$. But $\mathbb{K}_1$ and $\mathbb{K}_3$ can't be incidence-isomorphic, as there does not exist any supremum-dense subset of X_3, which is isomorphic to a supremum-dense subset of X_1.

We still need to show, that the concept isomorphism μ is really an isomorphism between the concept lattices $\mathfrak{B}(\mathbb{K}_1)$ and $\mathfrak{B}(\mathbb{K}_2)$ of two lattice contexts.

As we will need the following inequalities several times, we formulate them in a separate lemma:

Lemma 4.5. *Let $\mathbb{K}_1 = (X_1, Y_1, \varphi_1)$ and $\mathbb{K}_2 = (X_2, Y_2, \varphi_2)$ be incidence-isomorphic lattice contexts. Let further be $\gamma = (\gamma_X, \gamma_Y)$ the incidence-isomorphism between the corresponding incidence relations $\mathrm{IR}_{G_1,M_1}(\varphi_1)$ and $\mathrm{IR}_{G_2,M_2}(\varphi_2)$, as well as $\mu = (\mu_X, \mu_Y)$ the corresponding concept isomorphism.*
Then for all $g \in G_1$, $m \in M_1$ and for all $x \in X_1$, $y \in Y_1$ we have:

$$g \leq y' \Leftrightarrow \gamma_X(g) \leq \mu_Y(y)'$$
$$m \leq x' \Leftrightarrow \gamma_Y(m) \leq \mu_X(x)'$$

Proof.
1. For all $g \in G_1$ and $m \in M_1$ holds $g \leq m' \Leftrightarrow \gamma_X(g) \leq \gamma_Y(m)'$:
$g \leq m'$ is equivalent to $(g, m) \in \mathrm{IR}_{G_1,M_1}(\mathbb{K}_1)$ and thus, because of the definition of the incidence-isomorphism γ, it is in turn equivalent to $(\gamma_X(g), \gamma_Y(m)) \in$

$\mathrm{IR}_{G_2,M_2}(\mathbb{K}_2)$, thus to $\gamma_X(g) \le \gamma_Y(m)'$.
2. For all $g \in G_1$ and $y \in Y_1$ are equivalent:

$$\begin{aligned} g &\le y' \\ g &\le (\bigvee\{m \in M_1 | m \le y\})' \\ g &\le \bigwedge\{m' | m \in M_1, m \le y\} \\ g &\le m' \qquad \forall m \in M_1 \text{ mit } m \le y \\ \gamma_X(g) &\le \gamma_Y(m)' \qquad \forall m \in M_1 \text{ mit } m \le y \\ \gamma_X(g) &\le \bigwedge\{\gamma_Y(m)' | m \in M_1, m \le y\} \\ \gamma_X(g) &\le (\bigvee\{\gamma_Y(m) | m \in M_1, m \le y\})' \\ \gamma_X(g) &\le \mu_Y(y)' \end{aligned}$$

One can show the second inequation analoguously. □

Theorem 4.6. *Let $\mathbb{K}_i = (X_i, Y_i, \varphi_i)$, G_i, M_i $(i = 1, 2)$, as well as γ and μ like in 4.5. Then the derivation operators of the contexts are compatible with μ, i.e. for $x \in X_1$ and $y \in Y_1$ holds:*

$$\begin{aligned} \mu_X(x)' &= \mu_Y(x') \\ \mu_Y(y)' &= \mu_X(y') \end{aligned}$$

Proof. We consider $\mu_X(x)'$:
Because of the supremum-density of M_2 this is equal to $\bigvee\{m_2 \in M_2 | m_2 \le \mu_X(x)'\}$. As $\gamma_Y : M_1 \to M_2$ is bijective, there exists to each $m_2 \in M_2$ exactly one $m \in M_1$ with $\gamma_Y(m) = m_2$. Further, we have, according to 4.5, $\gamma_Y(m) \le \mu_X(x)' \Leftrightarrow m \le x'$. Altogether, we thus have: $\mu_X(x)' = \bigvee\{\gamma_Y(m) | m \in M_1, m \le x'\} = \mu_Y(x')$. □

Corollary 4.7. *$\mu : X_1 \times Y_1 \to X_2 \times Y_2$ maps concepts onto concepts.*

Proof. Let $(x, y) \in \mathfrak{B}(\mathbb{K}_1)$ be a concept. Then $x' = y$ and $y' = x$. Thus for $\mu(x, y) = (\mu_X(x), \mu_Y(y))$ holds: $\mu_X(x)' = \mu_Y(x') = \mu_Y(y)$ and $\mu_Y(y)' = \mu_X(y') = \mu_X(x)$. □

Theorem 4.8. *Let $\mathbb{K}_i = (X_i, Y_i, \varphi_i)$, G_i, M_i $(i = 1, 2)$, as well as γ and μ like in 4.5. Then μ is an order-isomorphism between $\mathfrak{B}(\mathbb{K}_1)$ and $\mathfrak{B}(\mathbb{K}_2)$, and so the contexts $\mathbb{K}_1$ and $\mathbb{K}_2$ are isomorphic.*

Proof.
1. First, we show, that for $x_1, x_2 \in \varphi_1^d(Y_1) \subseteq X_1$ holds

$$x_1 \le x_2 \Leftrightarrow \mu_X(x_1) \le \mu_X(x_2) :$$

x_1 can be represented as $\bigvee\{g \in G_1 | g \le x_1\}$, and for x_2 holds: $x_2'' = x_2$. Thus, $x_1 \le x_2$ is equivalent to $g \le x_2''$ for all $g \in G_1$ with $g \le x_1$. Here, we can apply 4.5, determining $\gamma_X(g) \le \mu_Y(x_2')'$ for the same g's. Forming the supremum on the left sides, as well as 4.6 leads to $\mu_X(x_1) \le \mu_Y(x_2')' = \mu_X(x_2'') = \mu_X(x_2)$. The steps of this proof are reversible, and so the equivalence holds.
2. The bijectivity of $\mu_X : \varphi_1^d(Y_1) \to \varphi_2^d(Y_2)$ follows from the symmetry of the definition:
If $\mathbb{K}_1 \cong \mathbb{K}_2$, then as well $\mathbb{K}_2 \cong \mathbb{K}_1$, thus we have besides $\mu_X : X_1 \to X_2$ a further function $\nu_X : X_2 \to X_1$:

$$\nu_X(x) := \bigvee\{\gamma_X^{-1}(g) | g \in M_2, g \le x\}.$$

According to 4.7, $\mu_X(\varphi_1^d(Y_1)) \subseteq \varphi_2^d(Y_2)$ and $\nu_X(\varphi_2^d(Y_2)) \subseteq \varphi_1^d(Y_1)$. We show for $x \in \varphi_1^d(Y_1)$, that $\nu_X(\mu_X(x)) = x$:
At first, for every extent $x \in \varphi_1^d(Y_1)$ of a concept holds $x'' = x$. Thus, $\nu_X(\mu_X(x)) = \nu_X(\mu_X(x'')) = \nu_X(\mu_Y(x')')$ because of 4.6. According to the definition of ν_X, the

latter is equal to $\bigvee\{\gamma_X^{-1}(g)|g \in G_2, g \leq \mu_Y(x')'\} = \bigvee\{g \in G_1|\gamma_X(g) \leq \mu_Y(x')'\}$. If we apply 4.5 to the terms of this set ($\gamma_X(g) \leq \mu_Y(x')' \Leftrightarrow g \leq x'' = x$), then we get $\nu_X(\mu_X(x)) = \bigvee\{g \in G_1|g \leq x\} = x$.
Because of symmetry, $\mu_X(\nu_X(x)) = x$ for $x \in \varphi_2^d(Y_2)$, as well.
3. Due to (1) and (2), $\mu_X : \varphi_1^d(Y_1) \to \varphi_2^d(Y_2)$ is an order-isomorphism. Furthermore, we have order-isomorphisms $\mathfrak{B}(\mathbb{K}_1) \to \varphi_1^d(Y_1), (x,y) \mapsto x$ and $\varphi_2^d(Y_2) \to \mathfrak{B}(\mathbb{K}_2), x \mapsto (x, x')$. Thus, if the following diagram commutates, then $\mu = (\mu_X, \mu_Y) : \mathfrak{B}(\mathbb{K}_1) \to \mathfrak{B}(\mathbb{K}_2), (x,y) \mapsto (\mu_X(x), \mu_Y(y))$ is an order-isomorphismus, too:

$$\begin{tikzcd}
\mathfrak{B}(\mathbb{K}_1) \arrow[r, "(x,y)\mapsto\mu(x,y)"] \arrow[d, "(x,y)\mapsto x"'] & \mathfrak{B}(\mathbb{K}_2) \\
\varphi_1^d(Y_1) \arrow[r, "x\mapsto\mu_X(x)"'] & \varphi_2^d(Y_2) \arrow[u, "x\mapsto(x,x')"']
\end{tikzcd}$$

But this commutes because of 4.6:

$$(\mu_X(x), \mu_X(x)') = (\mu_X(x), \mu_Y(x')) = (\mu_X(x), \mu_Y(y)) = \mu(x,y). \qquad \square$$

A direct consequence of this theorem is, that the information, needed for the concept lattice, is totally contained in the incidence relation $\mathrm{IR}_{G,M}(\varphi)$. In particular, if we have a relation $I \subseteq G \times M$, which is incidence relation both, between the lattices $X_1 \supseteq G$ and $Y_1 \supseteq M$, as well as between the lattices $X_2 \supseteq G$ and $Y_2 \supseteq M$, then the lattice contexts $(X_1, Y_1, \mathrm{GF}_{X_1,Y_1}(I))$ and $(X_2, Y_2, \mathrm{GF}_{X_2,Y_2}(I))$ are isomorphic.

Corollary 4.9. *Let $\mathbb{K} = (X, Y, \varphi)$ be a lattice context, and let $G \subseteq X$ and $M \subseteq Y$ be respectively supremum-dense. Then, $\mathbb{K}$ is isomorph to the classical context*

$$\begin{aligned} \mathbb{K}_* &:= (G, M, \mathrm{IR}_{G,M}(\varphi)) \\ &= (\mathfrak{P}(G), \mathfrak{P}(M), \mathrm{GF}_{\mathfrak{P}(G),\mathfrak{P}(M)}(\mathrm{IR}_{G,M}(\varphi)). \end{aligned}$$

Thus, we can use the algorithms already invented for classical contexts, in order to generate a concept lattice for a lattice context.

In the remainder of this article, we will display two examples, where we will apply this generalization of the classical context in order to understand two already known special cases of formal contexts: The fuzzy context and the multi-valued context. We will see, that one can consider these two types of contexts as special cases of the lattice context.

5. Applications

5.1. **Fuzzy contexts.**

One possible generalization of contexts are fuzzy contexts, which were introduced in [Um94]. Because of lack of space, we can't introduce the underlying theory of fuzzy sets here. We will just briefly show in short, that lattice concepts are a natural and useful basic structure for defining fuzzy contexts. In [Gu97] this aspect is worked out in more detail.

In order to be able to define fuzzy contexts, we need the L-fuzzy algebra, a mathematical structure invented in [We78]. Having a L-fuzzy algebra L, one can introduce L-fuzzy power sets $\mathfrak{P}_L(M)$ over some universe M. Such L-fuzzy power sets together with some proper definition of an inclusion operator build complete lattices. Additionally, one can introduce (binary) L-fuzzy relations (between two crisp sets G and M), which are nothing else than L-fuzzy subsets of $G \times M$.

Then, the L-fuzzy context is defined as a triple $(G, M, \mathcal{R})$, consisting of the sets G (of objects) and M (of attributes), as well as of an L-fuzzy relation $\mathcal{R}$ between G and M. As derivation operators, a galois connection (φ, ψ) between the L-fuzzy power sets of G and M is introduced, associating each L-fuzzy subset of G with an L-fuzzy subset of M, and vice versa.

Comparing this definition with our inventions in this article, we can easily see, that we have two complete lattices $\mathfrak{P}_L(G)$ and $\mathfrak{P}_L(M)$ together with a galois connection in between – and thus a lattice context:

Definition 5.1. Given an L-fuzzy algebra L, two sets G (of objects) and M (of attributes), and an L-fuzzy relation $\mathcal{R}$ between G and L, a fuzzy context $(G, M, \mathcal{R})$ is defined to be the lattice context $(\mathfrak{P}_L(G), \mathfrak{P}_L(M), \varphi)$.

Using this definition, we can specify the concept lattice and, by studying the $\vee$-irreducible elements in L-fuzzy power sets, we have a natural way of getting a classical context with an isomorphic concept lattice.

5.2. **Many-valued contexts.**
We now want to introduce many-valued context (G, M, W, I) as lattice contexts. In order to keep the complexity small, we restrict here to complete many-valued contexts, i.e. in every cell of the cross-table should be an entry. The general case with empty cells is handled in [Gu97].

In order to define a lattice context, we first need two complete lattices X and Y. As X, we can take the power set $\mathfrak{P}(G)$ of all objects. The following choice of Y is inspired by the way we describe objects in everyday life: We give all possible values, a set of objects can have.

For example, if we want to describe teenagers within the context of all human beings, using the attributes age and sex, we could say, that their age lies in between 15 and 20 years (or something like that). As the sex does not play any role in order to classify someone as teenager or not, we could mention, that the teenagers could be both male and female. So, what we do is, that we assigns to each attribute a subset of values.

Definition 5.2. Let $\mathbb{K} = (G, M, W, I)$ be a many-valued context. A *description* in $\mathbb{K}$ is a mapping $B\colon M \to \mathfrak{P}(W)$ from the set M of attributes to the set W of values.

A description is more precise, i.e. applies to less objects, iff one restricts the amount of values, which an object can have. Thus we have a natural order on the set of all descriptions:

Remark 5.3. The set $\mathfrak{P}(W)^M$ of all descriptions in $\mathbb{K}$, together with the order

$$B_1 \leq B_2 :\Leftrightarrow B_1(m) \supseteq B_2(m), \forall m \in M,$$

forms a complete lattice.

Proof. For an arbitrary index set T, let B_t, $t \in T$ be descriptions in $\mathbb{K}$. Infimum und supremum of all B_t, $t \in T$ are given by:

$$\bigwedge_{t\in T} B_t\colon m \mapsto \bigcup_{t\in T} B_t(m) \quad \text{and} \quad \bigvee_{t\in T} B_t\colon m \mapsto \bigcap_{t\in T} B_t(m). \qquad \square$$

In a classical context, the derivation of a set A of objects is the greatest set of attributes, which is in relation with all objects $g \in A$ ($A' = \bigvee\{B \subseteq M | gIm, \forall m \in B, \forall g \in A\}$). Analoguously, we take now the most precise description, which applies to all $g \in A$: $A' = \bigvee\{B | m(g) \in B(m), \forall m \in M, \forall g \in A\}$.

Definition 5.4. Let $\mathbb{K} = (G, M, W, I)$ be a complete many-valued context. We define the mappings $\varphi : \mathfrak{P}(G) \to \mathfrak{P}(W)^M$ and $\psi : \mathfrak{P}(W)^M \to \mathfrak{P}(G)$, by specifying for $A \in \mathfrak{P}(G)$ and $B \in \mathfrak{P}(W)^M$:

$$\begin{aligned} \varphi(A) &:= M \to \mathfrak{P}(W), m \mapsto m(A), \\ \psi(B) &:= \{g \in G | m(g) \in B(m),\ \forall m \in M\}. \end{aligned}$$

As φ and ψ are intented to be the derivation operators of our lattice context, we write in short A' instead of $\varphi(A)$ and B' instead of $\psi(B)$. But we still have to show, that these two mappings form a galois connection:
Remark 5.5. The mappings φ and ψ form a galois connection between the lattices $\mathfrak{P}(G)$ and $\mathfrak{P}(W)^M$.

Proof. We show this using the criterion $A \subseteq B' \Leftrightarrow A' \geq B$:
„$\Rightarrow$". Let $A \subseteq B' \;\Rightarrow A'(m) = m(A) \subseteq m(B') \subseteq B(m), \forall m \in M \Rightarrow A' \geq B$
„$\Leftarrow$". Let $A' \geq B \;\Rightarrow m(A) = A'(m) \subseteq B(m), \forall m \in M \Rightarrow A \subseteq B'$ □

As a result of our reflectings, we now can interpret the many-valued context as lattice context:
Theorem 5.6. *Let $\mathbb{K} = (G, M, W, I)$ be a complete many-valued context. We identify $\mathbb{K}$ with the lattice context*

$$\mathbb{K} = (\mathfrak{P}(G), \mathfrak{P}(W)^M, \varphi),$$

where φ is defined by 5.4.
In this way, we get derivation operators on many-valued contexts, and thus concepts in $\mathbb{K}$. The set of all concepts $\mathfrak{B}(\mathbb{K})$ is a complete lattice.

In order to get an incidence-isomorphic classical context, we need supremum-dense subsets $\tilde{G} \subseteq \mathfrak{P}(G)$ and $\tilde{M} \subseteq \mathfrak{P}(W)^M$. Then, $\mathbb{K} = (G, M, W, I)$ is isomorphic zu

$$\widetilde{\mathbb{K}} = (\tilde{G}, \tilde{M}, \mathrm{IR}_{\tilde{G},\tilde{M}}(\varphi)).$$

The $\vee$-irreducible elements of $\mathfrak{P}(G)$, i.e. the singletons $\{g\}$ with $g \in G$, are supremum-dense in $\mathfrak{P}(G)$:

$$\tilde{G} := I^{\vee}(\mathfrak{P}(G)) \cong G$$

In order to be able to prescribe the $\vee$-irreducible elements of $\mathfrak{P}(W)^M$, we consider the following definition:
Definition 5.7. For $m \in M$ and $w \in W$, the description $m \neq w$ is defined by:

$$(m \neq w)(n) := \begin{cases} W \setminus \{w\} & \text{if } n = m \\ W & \text{else} \end{cases}$$

Remark 5.8. The $\vee$-irreducible elements of $\mathfrak{P}(W)^M$ are given by:

$$I^{\vee}(\mathfrak{P}(W)^M) = \{m \neq w | m \in M, w \in W\}.$$

This set is supremum-dense in $\mathfrak{P}(W)^M$.

Proof. The smallest element of the lattice $\mathfrak{P}(W)^M$ is

$$0_{\mathfrak{P}(W)^M} : M \to \mathfrak{P}(W), m \mapsto W.$$

1. $I^{\vee}(\mathfrak{P}(W)^M) = \{m \neq w | m \in M, w \in W\}$:
„$\subseteq$". For each description $B \neq 0_{\mathfrak{P}(W)^M}$, which is not of the form $m \neq w$, we give two strictly smaller descriptions, whose supremum is equal to B:
As $B \neq 0_{\mathfrak{P}(W)^M}$, there exists a $m_1 \in M$ with $B(m_1) \subsetneq W$. Hence, there is a $w_1 \in W$ with $w_1 \notin B(m_1)$. As $B \neq (m_1 \neq w_1)$, there must be another pair $m_2 \in M$, $w_2 \in W$ mit $w_2 \notin B(m_2)$. Thus, the following descriptions are strictly smaller than B:

$$B_1 : m \mapsto \begin{cases} B(m_1)\dot{\cup}\{w_1\} & \text{if } m = m_1 \\ B(m) & \text{else} \end{cases}$$

$$B_2 : m \mapsto \begin{cases} B(m_2)\dot{\cup}\{w_2\} & \text{if } m = m_2 \\ B(m) & \text{else} \end{cases}$$

Considering the supremum thereof, we easily see, that $B = B_1 \vee B_2 : m \mapsto B_1(m) \cap B_2(m)$, and so, B is not $\vee$-irreducible. Of course, the $0_{\mathfrak{P}(W)^M}$ is not $\vee$-irreducible, as $0_{\mathfrak{P}(W)^M} = \bigvee \emptyset$.
„$\supseteq$". The only description, which is strictly smaller than $m \neq w$, is the null $0_{\mathfrak{P}(W)^M}$. Thus, $m \neq w$ has exactly one lower neighbour, and is therefore $\vee$-irreducible.
2. It is still necessary to show the supremum-density of $I^{\vee}(\mathfrak{P}(W)^M)$:
This follows from the definition of the descriptions $m \neq w$: Every $B \in \mathfrak{P}(W)^M$ is representable as

$$B = \bigvee \{m \neq w | m \in M, w \notin B(m)\}. \qquad \square$$

We now know supremum-dense subsets $G \subseteq \mathfrak{P}(G)$ and $\tilde{M} := I^{\vee}(\mathfrak{P}(W)^M) \subseteq \mathfrak{P}(W)^M$. In order to get the incidence-isomorphic classical context corresponding to $\mathbb{K} = (\mathfrak{P}(G), \mathfrak{P}(W)^M, \varphi)$, we still need to analyse the incidence relation $\tilde{I} := \mathrm{IR}_{G,\tilde{M}}(\varphi)$.

A pair $(g, m \neq w)$ is per definition contained in the incidence relation $\tilde{I}$, iff $(m \neq w) \leq \varphi(g)$, thus $(m \neq w)(n) \supseteq n(g)$ for all $n \in M$. This is a real restriction only for $n = m$, thus one can prescribe $\tilde{I}$ as follows:

$$(g, m \neq w) \in \tilde{I} \Leftrightarrow m(g) \neq w.$$

Herewith, we have described an incidence-isomorphic classical context $\widetilde{\mathbb{K}} = (G, \tilde{M}, \tilde{I})$:

Theorem 5.9. *Let $\mathbb{K} = (G, M, W, I)$ be a complete many-valued context. Then, the concept lattice of $\mathbb{K}$ is isomorphic to the concept lattice of the classical context $\widetilde{\mathbb{K}} = (G, \tilde{M}, \tilde{I})$, whereas*

$$\tilde{M} := \{m \neq w | m \in M, w \in W\} \quad \textit{and}$$

$$(g, m \neq w) \in \tilde{I} :\Leftrightarrow m(g) \neq w.$$

Example 5.10. As an example, let us consider a complete lattice, which represents some real numbers together with the attributes irrational, algebraic and transcendent. Thereby, the attributes irrational and transcendent can have the values *yes* or *no*, whereas algebraic can be *no*, or a number $n \in \mathbb{N}$ signifying, that this number is algebraic of degree n. Thus, we have a complete many-valued context $\mathbb{K} = (G, M, W, I)$ with

$$\begin{aligned} G &= \{2, \sqrt{2}, \sqrt[3]{2}, \pi\} \\ M &= \{\mathrm{i}, \mathrm{a}, \mathrm{t}\} \\ W &= \{\mathrm{y}, \mathrm{n}\} \cup \mathbb{N} \end{aligned}$$

I	i	a	t
2	n	1	n
$\sqrt{2}$	y	2	n
$\sqrt[3]{2}$	y	3	n
π	y	n	y

The classical context $\widetilde{\mathbb{K}}$ has an infinite number of attributes, as W is infinite. But we can ignore attributes, which hold for all objects, because they are reducible. $m \neq w$ does hold for all objects, if there is no object $g \in G$ wich has for the attribute m the value w. Thus, the context $\widetilde{\mathbb{K}}$ – and hence $\mathbb{K}$ – is isomorphic to:

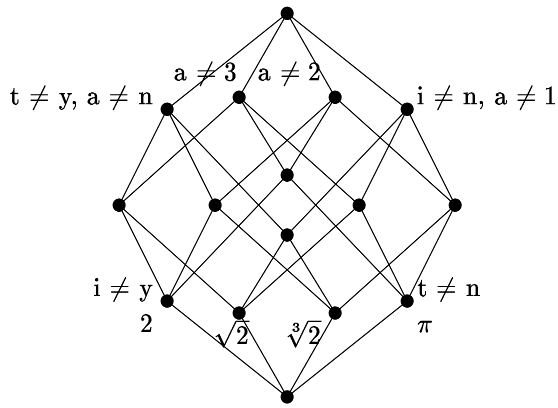


FIGURE 5.1. The concept lattice of the many-valued context $\mathbb{K}$.

	$i \neq n$	$i \neq y$	$a \neq n$	$a \neq 1$	$a \neq 2$	$a \neq 3$	$t \neq n$	$t \neq y$
2		×	×		×	×		×
$\sqrt{2}$	×		×	×		×		×
$\sqrt[3]{2}$	×		×	×	×			×
π	×			×	×	×	×	

The concept lattice is drawn in Figure 5.1.

CONCLUSIONS

In this article, we introduced a new general idea of contexts, the lattice contexts, and we briefly showed, how to apply the results to already known extensions of formal concept analysis like fuzzy contexts or complete many valued contexts.

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