The paired construction for Boolean functions in the Johnson scheme

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joint work with Jonathan Mannaert and Alfred Wassermann

Setting

- V finite set.
- $n := \#V$.
- Fix $k \in \{0, \ldots, n\}$. Without restriction, $k \leq \frac{n}{2}$.
- Investigate functions $f : \binom{V}{k} \to \mathbb{R}$.
- Via support: Boolean functions $\leftrightarrow$ subsets of $\binom{V}{k}$.
- f has (non-unique) polynomial representation $h \in \mathbb{R}[X_a : a \in V]$.
- Degree $\deg(f) :=$ smallest possible $\deg(h)$.

Baby example

- $n = 6$, $V = \{1, 2, 3, 4, 5, 6\}$, $k = 3$.
- Define Boolean $f : \binom{V}{3} \to \{0, 1\}$ via
$$\mathrm{supp}(f) = \{\{1, 2, 3\},\ \{4, 5, 6\}\}.$$
- f represented by polynomial
$$h = X_1X_2X_3 + X_4X_5X_6.$$
- Example:
  - $f(\{1, 2, 3\}) = h(1, 1, 1, 0, 0, 0) = 1 \cdot 1 \cdot 1 + 0 \cdot 0 \cdot 0 = 1.$
  - $f(\{1, 2, 4\}) = h(1, 1, 0, 1, 0, 0) = 1 \cdot 1 \cdot 0 + 1 \cdot 0 \cdot 0 = 0.$
- $\implies \deg(f) \leq \deg(h) = 3.$
- Alternative polynomial representation:
$$\begin{aligned} h_2 &= X_1X_2X_3 + (1 - X_1)(1 - X_2)(1 - X_3) \\ &= X_1X_2 + X_1X_3 + X_2X_3 - X_1 - X_2 - X_3 + 1. \end{aligned}$$
- $\implies \deg(f) \leq \deg(h_2) = 2$ ("degree-drop").

Connections

- Application of the degree in complexity theory ($\rightsquigarrow$ Filmus, Ihringer).
- Considering f in the Johnson scheme $J(n, k)$ ...
  - $\deg(f)$ = largest index of a non-vanishing entry in the dual inner distribution of f.
  - Boolean degree $\leq t$ functions = t-antidesigns.
- In q-analog setting:
  Cameron-Liebler sets of k-spaces
  = Boolean degree 1 functions.
- Details: See
  MK, Jonathan Mannaert and Alfred Wassermann:
  The degree of functions in the Johnson and q-Johnson schemes,
  Journal of Combinatorial Theory, Series A 212 (2025), Paper No. 105979.

Properties of the degree (as usual...)

- $\deg(\lambda f) = \deg(f)$ for $\lambda \in \mathbb{R} \setminus \{0\}$.
- $\deg(f + g) \leq \max(\deg(f), \deg(g))$,
  with equality whenever $\deg(f) \neq \deg(g)$.

  Reformulation.
  Among the degrees $\deg(f)$, $\deg(g)$ and $\deg(f + g)$,
  two are equal and the third one is smaller or equal.

Natural problem

Study smallest possible size

$$m(n, k, t) = \#\,\mathrm{supp}(f) =: \#f$$

of a non-zero Boolean function f of degree t.

Candidate functions

In the following: $I, J \subseteq V$ disjoint and $i := \#I$ and $j := \#J$.

Basic function

$$f_{I,J} = \prod_{a \in I} X_a \cdot \prod_{b \in J} (1 - X_b).$$

- $\mathrm{supp}(f_{I,J}) = \{K \in \binom{V}{k} \mid I \subseteq K \text{ and } J \cap K = \emptyset\}$.
- Known:

$$\deg f_{I,J} = \begin{cases} -\infty & \text{if } i > k \text{ or } j > n - k, \\ \min(i + j, k) & \text{otherwise.} \end{cases}$$

- $\overset{i=t,\, j=0}{\Longrightarrow}$ pencil bound

$$m(n, k, t) \le \binom{n - t}{k - t}.$$

(Sharp for $t = 1$, looks strong in general.)

Candidate functions (cont.)

Paired function

$$p_{I,J} = f_{I,J} + f_{J,I}.$$

- Boolean (unless $I = J = \emptyset$).
- Symmetry: $p_{I,J} = p_{J,I}$.
- Baby example $\{\{1,2,3\},\ \{4,5,6\}\}$ is paired function $p_{\{1,2,3\},\emptyset}$ (with $V = \{1,\dots,6\}$).
  - Seen: $\deg(p_{\{1,2,3\},\emptyset}) = 2$.
    $\implies m(6,3,2) \le 2$.
  - Beats pencil bound $\binom{6-2}{3-2} = 4$!
  - "Reason": Degree-drop observed in the beginning.
- $\rightsquigarrow$ Goal: Determine degree

$$t_{i,j} := \deg(p_{I,J}).$$

of a general paired function.

Elementary degree bound

$$t_{i,j} = \deg(f_{I,J} + f_{J,I}) \ \leq \ \max(\deg(f_{I,J}), \deg(f_{J,I})) = \min(i+j, k).$$

Theorem (K, Mannaert, Wassermann 2025)

$$t_{i,j} = \begin{cases} i+j-1 & \text{if } i+j \text{ odd and } i+j \leq k, \\ k-1 & \text{if } k \text{ odd and } n = 2k \text{ and } i+j \geq k, \\ \min(i+j, k) & \text{otherwise.} \end{cases}$$

Remark

- Previously (2024): Only upper bound $t_{i,j} \leq \ \dots$
- New in this talk: Equality $t_{i,j} = \ \dots$

Proof (sketch)

- Reduce situation $n > 2k$ to $n = 2k$ via derived and residual functions.
- $\rightsquigarrow$ Consider critical case $n = 2k$.

Proof (sketch) cont.

Observation:

$$
\begin{aligned}
p_{I,J} &= f_{I,J} + f_{J,I} \\
&= \prod_{i \in I} X_i \prod_{j \in J} (1 - X_j) + \prod_{i \in I} (1 - X_i) \prod_{j \in J} X_j \\
&= \underbrace{((-1)^j + (-1)^i)}_{=0 \Leftrightarrow i+j \text{ odd}} \underbrace{\prod_{i \in I \cup J} X_i}_{=f_{I \cup J, \emptyset}} + (\text{terms of deg } < i + j).
\end{aligned}
$$

Consequence:

Lemma

Let $i + j \leq k$.

- For $i + j$ odd, $t_{i,j} \leq i + j - 1$.
- For $i + j$ even, $t_{i,j} = i + j$.

Proof (sketch) cont.

Lemma: Monotonicity

$t_{i,j}$ is monotone in i and j (on $\{0,\ldots,k\} \times \{0,\ldots,k\}$).

Proof.

- Symmetry: Enough to consider i.
- Double counting: For $i < k$,

$$(k-i)f_{I,J} = \sum_{\substack{L \in \binom{V \setminus J}{i+1} \\ I \subseteq L \subseteq J^{\complement}}} f_{L,J}. \qquad (*)$$

- Dually (using $n = 2k$):

$$(k-i)f_{J,I} = \sum_{\substack{L \in \binom{V \setminus J}{i+1} \\ I \subseteq L \subseteq J^{\complement}}} f_{J,L}. \qquad (**)$$

- $(*) + (**) \rightsquigarrow p_{I,J}$ linear combination of $p_{L,J}$'s $(\#L = i+1)$.
- $\implies t_{i,j} \le t_{i+1,j}$.

Proof (sketch) cont.

Lemma

For $i + j \leq k$ and $i + j$ odd, $\quad t_{i+j} = i + j - 1$.

Proof.

- $t_{i,j} \leq i + j + 1 \quad$ (as $i + j$ odd).
- Monotonicity: $t_{i,j} \geq t_{i-1,j} = i + j - 1 \quad$ (as $(i - 1) + j$ even).

□

Lemma of triads

Among the degrees $t_{i,j}$, $t_{i+1,j}$ and $t_{i,j+1}$,
two are equal and the third is smaller or equal.

Proof.

Use Pascal-type decomposition

$$p_{I,J} = p_{I \cup \{a\},J} + p_{I,J \cup \{a\}}$$

where $a \in V \setminus (I \cup J)$.

□

Proof (sketch) cont.

Lemma: Constant on diagonals

The value $t_{i,j}$ only depends on $s = i + j$.

Proof.

Combine Lemma of triads with monotonicity. □

Lemma

Let $i + j \geq k$. Then

$$t_{i,j} = \begin{cases} k & \text{if } k \text{ even,} \\ k - 1 & \text{if } k \text{ odd.} \end{cases}$$

Proof.

- Enough to consider end of diagonal $t_{k,j}$ with $j \in \{1, \ldots, k\}$.
- $\mathrm{supp}(p_{K,J}) = \{K, K^{\complement}\} = \mathrm{supp}(p_{K,\emptyset})$.
- Hence $t_{k,j} = t_{k,0}$, which is already known.

Proof of Theorem completed! □

Corollary

$$m(n,k,t) \leq \begin{cases} 2 \cdot \binom{2k-t-1}{k} & \text{if } n = 2k \text{ and } t \text{ even and } t \neq k, \\ \binom{n-t}{k-t} & \text{otherwise.} \end{cases}$$

Proof.

For the first case, consider $p_{I,J}$ with $i = t + 1$ and $j = 0$. □

Conjecture

Always equality.

Corollary

For k odd,

$$m(2k, k, k-1) = 2.$$

Thank you!

Slides will be uploaded at
https://mathe2.uni-bayreuth.de/michaelk/