The degree of functions in the Johnson and $q$-Johnson schemes

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DMV Meeting 2023
MS 2 – Combinatorial aspects of finite fields
September 25, 2023
Technische Universität Ilmenau, Germany

joint work with Jonathan Mannaert and Alfred Wassermann
Cameron-Liebler line classes

- Cameron, Liebler 1982:
  “Special” set $\mathcal{L}$ of lines in $\text{PG}(3, q)$.
- Defined by the following equivalent properties:
  - Algebraic property:
    $\chi_{\mathcal{L}} \in \mathbb{R}$-row space of the line-point incidence matrix.
  - Geometric property:
    Constant intersection with any line spread of $\text{PG}(3, q)$.

Various directions of generalization

- Ambient space $\text{PG}(n, q)$.
- Lines $\rightarrow$ $k$-spaces.
- Allow $q = 1$ (set case).
- Points $\rightarrow$ spaces of degree $d$.

Goal
Coherent theory of all generalizations.
Subset and subspace lattices

- Fix $q = 1$ (set case) or prime power $q \geq 2$ ($q$-analog case).
- Fix $n$ non-negative integer.
- Let $V$ be a $\mathbb{F}_q$-vector space of dimension $n$.
- Let $L(V)$ be the lattice of all $\mathbb{F}_q$-subspaces of $V$.
- For $U \in L(V)$ let $\text{rk}(U) = \begin{cases} \#U \\ \dim(U) \end{cases}$.
- Let $[V]_k = \{ U \in L(V) \mid \text{rk}(U) = k \}$.
  Set case: $\# [V]_k = \binom{n}{k} = \binom{n}{k}_1$ Binomial coefficient.
  $q$-analog case: $\# [V]_k = \binom{n}{k}_q$ Gaussian coefficient.
Association Schemes

- Let $X$ finite set, $\mathcal{R} = \{ R_0, \ldots, R_d \}$ partition of $X \times X$.
- $(X, \mathcal{R})$ association scheme if
  - $R_0$ identity relation
  - All relations $R_i$ are symmetric
  - There exist constants (called intersection numbers) $p_{ij}^\ell$ such that for all $x, y \in X$ with $(x, y) \in R_\ell$
    \[ \# \{ z \in X \mid (x, z) \in R_i \text{ and } (z, y) \in R_j \} = p_{ij}^\ell \]

- By definition: Set of adjacency matrices $B^{(i)}$ of $R_i$ pairwise commutable
  \[ \implies \text{are simultaneously diagonalizable} \]
  \[ \implies \mathbb{R}^X = V_0 \perp \ldots \perp V_d \text{ orthogonal sum of maximal common eigenspaces} \]
Johnson and Grassmann scheme

- Let $k \leq \frac{n}{2}$ and $X = \binom{V}{k}$.
- For $i \in \{0, \ldots, k\}$ define the relation $U_1 \ R_i \ U_2 \iff \operatorname{rk}(U_1 \cap U_2) = k - i$.
- Then $(X, (R_0, \ldots, R_k))$ is a $k$-class association scheme.

Set case: Johnson scheme

$q$-analog case: Grassmann scheme or $q$-Johnson scheme.

- Maximal common eigenspaces $V_i$ can be ordered s.t.
  \[
  \overline{V}_i := V_0 \perp \ldots \perp V_i = \mathbb{R}\text{-row space of } W^{(ki)},
  \]
  where $W^{(ki)}$ is $\binom{V}{k}$-vs-$\binom{V}{i}$ incidence matrix.
The Degree

- Let $f : \binom{V}{k} \rightarrow \mathbb{R}$.
- Definition (via algebraic property):
  Degree $\text{deg}(f) := \text{smallest } d \text{ such that } f \in \bar{V}_d$.
- Let $x_U$ be characteristic function of
  $\{K \in \binom{V}{k} \mid U \leq K\}$ (rk($U$)-pencil)
- Dually: Let $\bar{x}_U$ be characteristic function of
  $\{K \in \binom{V}{k} \mid U \geq K\}$ (dual rk($U$)-pencil).
- Alternative characterization of degree:
  $\text{deg}(f)$ is smallest $d$
  such that $f$ is a linear combination of $d$-pencils.
  The (unique) coefficients are called weights $\text{wt}_f(D)$ of $f$:
  \[
  f = \sum_{D \in \binom{V}{d}} \text{wt}_f(D) x_D
  \]
- $\Rightarrow \text{deg}(f) = 0 \iff f \text{ constant.}$
Lemma

- \( \deg(\lambda f) = \deg(f) \) for all \( \lambda \in \mathbb{R} \setminus \{0\} \).
- \( \deg(f + g) \leq \max(\deg(f), \deg(g)) \)
- \( \deg(fg) \leq \deg(f) + \deg(g) \)

Theorem

Let \( \text{rk} \, I \leq k \) and \( n - \text{rk} \, J \leq k \).

- \( \deg(x_I) = \text{rk} \, I \)
- \( \deg(\bar{x}_J) = n - \text{rk} \, J \)

Proof.

First part: Use that the \( \text{Aut}(\mathcal{L}(V)) \)-orbit of \( x_U \) spans \( V_{\text{rk} \, U} \).

Second part:

- Set up linear equation system for the weights, assuming that \( \text{wt}(I) \) only depends on \( \text{rk}(I \cap J) \).
- Equation system matrix is an invertible triangular matrix.
What are the weights of $\bar{x}_U$?

**Theorem**

Let $i \in \{0, \ldots, k\}$, $J \in \binom{V}{n-i}$, $I \in \binom{V}{i}$ and $z = \text{rk}(I \cap J)$. Then

$$\text{wt}_{\bar{x}_J}(I) = \begin{cases} 
\delta_{z,k} & \text{if } i = k, \\
(-1)^{i-z} \frac{1}{q^{(k-i)(i-z)} + \binom{i-z}{2}} \binom{k-i}{1} \binom{k-z}{z} & \text{otherwise}.
\end{cases}$$

**Proof.**

Compute the solutions of the above equation system. Use negation formula and $q$-Vandermonde formula for Gaussian coefficients.
Boolean functions

- Identify sets $\mathcal{F} \subseteq \binom{V}{k}$ with their characteristic function $\chi_{\mathcal{F}}$, commonly called Boolean function in this context.
- In this way: Define $\deg(\mathcal{F}) = \deg(\chi_{\mathcal{F}})$.
- Is there a geometric characterization of $\deg(\mathcal{F})$? Suitable generalization of “spread”?

Definition: Design

A set $\mathcal{D} \subseteq \binom{V}{k}$ is called a $t-(n, k, \lambda)_q$ design, if every $T \in \binom{V}{t}$ is contained in exactly $\lambda$ elements of $\mathcal{D}$.

Fact (Delsarte)

$\mathcal{D}$ is a $t-(n, k, \lambda)_q$-design if and only if $\chi_{\mathcal{D}} \in V_0 \perp V_{t+1} \perp V_{t+2} \perp \ldots \perp V_k$. 
Combined with Delsarte’s concept of pairwise orthogonality, this leads to:

**Fact (Geometric property of the degree)**

Let $\mathcal{F} \subseteq \binom{V}{k}$. If $d = \deg \mathcal{F}$, then for each $d$-$(n, k, \lambda)_q$ design $\mathcal{D}$,

$$\#(\mathcal{F} \cap \mathcal{D}) = \frac{\#\mathcal{F} \cdot \#\mathcal{D}}{\binom{n}{k}}$$

**Remark**

- Important open question: Is the reverse implication true?
- Would follow if the characteristic functions of $d$-designs span $V_0 + V_{d+1} + V_{d+2} + \ldots + V_k$.
  (Richness statement about existence of designs)
- Hard question: This would imply Hartman’s conjecture from 1987.
Boolean degree 1 functions

- Set case: (Filmus, Ihringer 2019)
  Only the trivial examples $x_P$ and $\overline{x}_H$ ($P \in [V]_1$, $H \in [V]_{n-1}$).

- $q$-analog case:
  Boolean degree 1 function = Cameron-Liebler set of $(k-1)$-spaces in $\text{PG}(n-1, q)$.
  Non-trivial examples do exist.
  Complete classification probably out of reach.
Change of ambient space

Implication of change of ambient space

- $V \rightarrow H \quad (H \in \binom{V}{n-1} \text{ hyperplane})$
- $V \rightarrow V/P \quad (P \in \binom{V}{1} \text{ point})$

on the degree?
Theorem
Let $P \in [V]_1$ and
\[ \mathcal{A} = \{ g : [V]_k \rightarrow \mathbb{R} \mid g(K) = 0 \text{ for all } K \in [V]_k \text{ with } P \not\subseteq K \} . \]
Then
\[ \Phi : \mathbb{R}^{[V/P]}_{k-1} \rightarrow \mathcal{A}, \quad \Phi(f) : K \mapsto \begin{cases} f(K/P) & \text{if } P \subseteq K, \\ 0 & \text{if } P \not\subseteq K \end{cases} \]
is an isomorphism of $\mathbb{R}$-vector spaces and \[ \deg_V \Phi(f) = \deg_{V/P}(f) + 1 \text{ (except certain border cases).} \]

Proof.

▶ Everything straightforward, except
“$\deg_V \Phi(f) \geq \deg_{V/P}(f) + 1$”.

▶ Lemma. $P \not\subseteq D \implies \operatorname{wt}_g(D) = 0$ for all $g \in \mathcal{A}, D \in [V]_{\deg(g)}$.

▶ Can be shown using a result of Guo, Li, Wang (2014) stating that the incidence matrices of certain attenuated geometries are of full rank.
Theorem
Let $H \in \mathcal{V}_{n-1}$ and

$$\mathcal{B} = \{ g : \mathcal{V}_k \to \mathbb{R} \mid g(K) = 0 \text{ for all } g \in \mathcal{V}_k \text{ with } K \not\in H \}.$$  

Then

$$\Psi: \mathbb{R}^{[H]} \to \mathcal{B}, \quad \Psi(f): K \mapsto \begin{cases} f(K) & \text{if } P \subseteq H, \\ 0 & \text{if } P \not\subseteq H \end{cases}$$

is an isomorphism of $\mathbb{R}$-vector spaces and

$$\deg_{\mathcal{V}}(\Psi(f)) = \deg_{\mathcal{H}}(f) + 1 \text{ (except certain border cases).}$$

Proof.
Follows from the previous theorem by dualization. \qed
Basic sets of degree $d$

- Let $I, J \in \mathcal{L}(V)$ with $I \leq J$ and $\text{rk} \ I + \text{cork} \ J \leq k$ where $\text{corank} \ \text{cork} \ J = n - \text{rk} \ J$.
- Let $\mathcal{F}(I, J) = \{ K \in \binom{V}{k} \mid I \leq K \leq J \}$.
- By the above theorems

$$\text{deg} \ \mathcal{F}(I, J) = \text{rk} \ I + \text{cork} \ J.$$  

- Basic sets include pencils ($\text{rk} \ I = 0$) and dual pencils ($\text{cork} \ J = 0$).
The paired construction

- Construction for the set case \( q = 1 \).
- Idea: Disjoint union of two “opposite” basic sets.
- Let \( I, J \subseteq V \) disjoint, not both empty. Let

\[
P(I, J) = \mathcal{F}(I, J^c) \cup \mathcal{F}(J, I^c)
\]

- Clear: \( \deg P(I, J) \leq \min(\#I + \#J, k) \).
- There are cases with a strict “<”!
Theorem
Let $q = 1$, $I, J \subseteq V$ disjoint, $i = \#I$, $j = \#J$, $k \leq \frac{n}{2}$, $i \leq k \leq n - i$, $j \leq k \leq n - j$.

In the cases

1. $i + j \leq k$ and $i + j$ odd;
2. $i + j \geq k$ and $k$ odd and $n = 2k$

we have $\deg P(I, J) \leq \min(i + j, k) - 1$.

Proof (Idea).
Case 1: Write $\chi_{P(I,J)}$ as an integer linear combination of basic characteristic functions of degree $i + j - 1$.

Case 2: Induction based on

- $P(X, Y) = P(X \cup \{x\}, Y) \cup P(X, Y \cup \{x\})$
  (where $X, Y, \{x\}$ are pairwise disjoint)
- $P(K, J) = P(K, \emptyset)$ for $K \in \binom{V}{k}$ and all $J$.
- Case 1
Small sets of degree $d$

- Natural question: Smallest size $m_q(d, k, n)$ of a non-empty set of degree $d$?
- From $\deg x_D = d$ we get the trivial bound

$$ m_q(d, k, n) \leq \left\lceil \frac{n - d}{k - d} \right\rceil. $$

- Trivial bound is sharp for $d = 1$.
- For $q = 1$, $n = 2k$, $d \geq 2$ even, $i = 0$ and $j = d + 1$, the paired construction beats the trivial bound!

**Corollary**

Let $d \in \{0, \ldots, k - 1\}$ be even. Then

$$ m_1(d, k, 2k) \leq 2 \cdot \binom{2k - d - 1}{k}. $$
Thank you!

Slides will be uploaded at
https://mathe2.uni-bayreuth.de/michaelk/