Derived designs of q-Fano planes and q-analogs of group divisible designs

Michael Kiermaier

Institut für Mathematik
Universität Bayreuth

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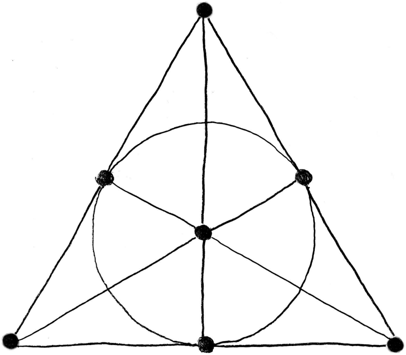
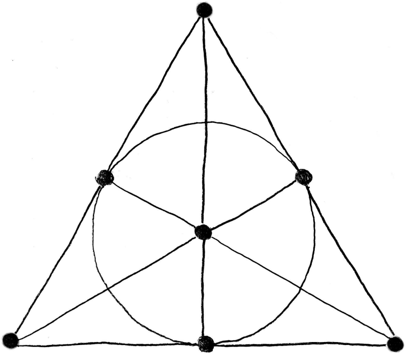
What is this?



- Most frequent image in discrete math.
- Fano plane.

What is special about it?



- Smallest projective plane.
- Smallest non-trivial Steiner triple system.

Outline

Block designs and their q-analogs

Derived q-Fano planes and α-points

q-analogs of group divisible designs

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Block designs and their q-analogs

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Subset lattice

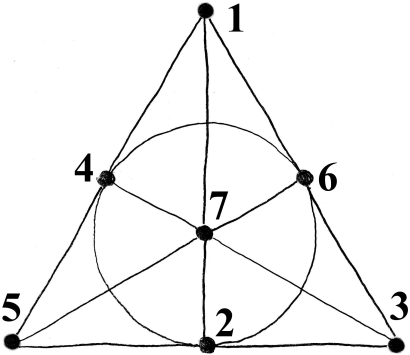
- Let V be a v-element set.
- $\binom{V}{k} :=$ Set of all k-subsets of V.
- $\#\binom{V}{k} = \binom{v}{k}$.
- Subsets of V form a distributive lattice (wrt. $\subseteq$).

Definition

$D \subseteq \binom{V}{k}$ is a t-(v, k, λ) (block) design
if
each $T \in \binom{V}{t}$ is contained in exactly λ blocks (elements of D).

- If $\lambda = 1$: D Steiner system
- If $\lambda = 1$, $t = 2$ and $k = 3$: D Steiner triple system STS(v)

Example



$$V = \{1, 2, 3, 4, 5, 6, 7\}$$

$$D = \{\{1, 2, 7\}, \{1, 3, 6\}, \{1, 4, 5\}, \{2, 3, 5\},$$
$$\{2, 4, 6\}, \{3, 4, 7\}, \{5, 6, 7\}\}$$

Fano plane D is a 2-$(7, 3, 1)$ design, i.e an $\mathrm{STS}(7)$.

Lemma

Let D be a t-(v, k, λ) design and $i, j \in \{0, \ldots, t\}$ with $i + j \leq t$. Then for all $I \in \binom{V}{i}$ and $J \in \binom{V}{v-j}$ with $I \subseteq J$

$$\lambda_{i,j} := \#\{B \in D \mid I \subseteq B \subseteq J\} = \lambda \cdot \frac{\binom{v-i-j}{k-i}}{\binom{v-t}{k-t}}.$$

In particular, $\#D = \lambda_{0,0}$.

Example

Fano plane STS(7) ($v = 7$, $k = 3$, $t = 2$, $\lambda = 1$):

$$\lambda_{0,0} = 7$$
$$\lambda_{1,0} = 3 \qquad \lambda_{0,1} = 4$$
$$\lambda_{2,0} = 1 \qquad \lambda_{1,1} = 2 \qquad \lambda_{0,2} = 2$$

Corollary: Integrality conditions

If a t-(v, k, λ) design exists, then all $\lambda_{i,j} \in \mathbb{Z}$.

Sufficient to check: $\lambda_i := \lambda_{i,0} \in \mathbb{Z}$ (Parameters admissible)

Lemma

STS(v) admissible $\iff$ $v \equiv 1, 3 \pmod 6$.

STS(v) for small v

- STS(3) $= \{V\}$ exists trivially.
- Smallest non-trivial Steiner triple system: Fano plane STS(7).
- Next admissible case: STS(9) exists (affine plane of order 3).

Theorem (Kirkman 1847)

All admissible STS(v) do exist.

Subspace lattice

- Let V be a v-dimensional $\mathbb{F}_q$ vector space.
- Grassmannian $\begin{bmatrix} V \\ k \end{bmatrix}_q :=$ Set of all k-dim. subspaces of V.
- Gaussian Binomial coefficient

$$\#\begin{bmatrix} V \\ k \end{bmatrix}_q = \begin{bmatrix} v \\ k \end{bmatrix}_q = \frac{(q^v - 1)(q^{v-1} - 1) \cdot \ldots \cdot (q^{v-k+1} - 1)}{(q - 1)(q^2 - 1) \cdot \ldots \cdot (q^k - 1)}$$

- Subspaces of V form a modular lattice (wrt. $\subseteq$).
- Subspace lattice of V = projective geometry $\mathrm{PG}(v-1, q)$
  - Elements of $\begin{bmatrix} V \\ 1 \end{bmatrix}_q$ are points.
  - Elements of $\begin{bmatrix} V \\ 2 \end{bmatrix}_q$ are lines.
  - Elements of $\begin{bmatrix} V \\ 3 \end{bmatrix}_q$ are planes.
  - Elements of $\begin{bmatrix} V \\ v-1 \end{bmatrix}_q$ are hyperplanes.
- Fano plane is the projective geometry $\mathrm{PG}(2, 2)$.

q-analogs in combinatorics

Replace subset lattice by subspace lattice!

orig.	q-analog
v-element set V	v-dim. $\mathbb{F}_q$ vector space V
$\binom{V}{k}$	$\begin{bmatrix} V \\ k \end{bmatrix}_q$
$\binom{v}{k}$	$\begin{bmatrix} v \\ k \end{bmatrix}_q$
cardinality	dimension
$\cap$	$\cap$
$\cup$	$+$

- The subset lattice corresponds to $q = 1$.
- Sometimes: Unary field $\mathbb{F}_1$.

Definition (block design, stated again)

Let V be a v-element set.
$D \subseteq \binom{V}{k}$ is a t-(v, k, λ) (block) design
if each $T \in \binom{V}{t}$ is contained in exactly λ elements of D.

q-analog of a design?

Definition (subspace design)

Let V be a v-dimensional $\mathbb{F}_q$ vector space.
$D \subseteq \begin{bmatrix} V \\ k \end{bmatrix}_q$ is a t-$(v, k, \lambda)_q$ (subspace) design
if each $T \in \begin{bmatrix} V \\ t \end{bmatrix}_q$ is contained in exactly λ elements of D.

- If $\lambda = 1$: D q-Steiner system
- If $\lambda = 1$, $t = 2$, $k = 3$: D q-Steiner triple system $\mathrm{STS}_q(v)$
- Geometrically:
  $\mathrm{STS}_q(v)$ is a set of planes in $\mathrm{PG}(v-1, q)$
  covering each line exactly once.

Lemma

Let D be a t-$(v, k, \lambda)_q$ design and $i, j \in \{0, \ldots, t\}$ with $i + j \leq t$. Then for all $I \in \begin{bmatrix} V \\ i \end{bmatrix}_q$ and $J \in \begin{bmatrix} V \\ v-j \end{bmatrix}_q$ with $I \subseteq J$

$$\lambda_{i,j} = \lambda \frac{\begin{bmatrix} v-i-j \\ k-i \end{bmatrix}_q}{\begin{bmatrix} v-t \\ k-t \end{bmatrix}_q}.$$

In particular, $\#D = \lambda_{0,0}$.

Corollary: Integrality conditions

If a t-$(v, k, \lambda)_q$ design exists, then all $\lambda_{i,j} \in \mathbb{Z}$.

Sufficient to check: $\lambda_i := \lambda_{i,0} \in \mathbb{Z}$ (Parameters admissible)

Lemma

$\mathrm{STS}_q(v)$ admissible $\iff v \equiv 1, 3 \pmod 6$.

$\mathrm{STS}_q(v)$ for small v

- $v = 3$: $\mathrm{STS}_q(3) = \{V\}$ exists trivially.
- $v = 7$: q-analog of the Fano plane $\mathrm{STS}_q(7)$.
  Existence undecided for every field order q.

  Most important open problem in q-analogs of designs.
- $v = 9$: $\mathrm{STS}_q(9)$: existence open for every q.
- $v = 13$: Only known non-trivial q-STS:
  $\mathrm{STS}_2(13)$ exists (Braun, Etzion, Östergård, Vardy, Wassermann 2013)

Status of the binary q-analog of the Fano plane.

$$\lambda_{0,0} = 381$$

$$\lambda_{1,0} = 21 \qquad \lambda_{0,1} = 45$$

$$\lambda_{2,0} = 1 \qquad \lambda_{1,1} = 5 \qquad \lambda_{0,2} = 5$$

- $\mathrm{STS}_2(7)$ consists of $\lambda_{0,0} = 381$ blocks (out of $\begin{bmatrix}7\\3\end{bmatrix}_2 = 11811$ planes).
- Huge search space ($\binom{11811}{381}$ has 730 digits).
- Heinlein, MK, Kurz, Wassermann 2019: Best known *packing* has size 333.
- Braun, MK, Nakić 2016; MK, Kurz, Wassermann 2018: $\mathrm{STS}_2(7)$ has at most 2 automorphisms.

For general q:

$$(q^2 - q + 1)\left[{7 \atop 1}\right]_q$$

$$q^4 + q^2 + 1 \qquad (q^3+1)(q^2+1)$$

$$1 \qquad q^2 + 1 \qquad q^2 + 1$$

Theme for remainder of the talk

- Let D be a $\mathsf{STS}_q(7)$.
- Fix a point P.
- What can be said
  about the "local" point of view of D from P?

Outline

Block designs and their q-analogs

Derived q-Fano planes and α-points

q-analogs of group divisible designs

- Let P be a point.
- For t-$(v, k, \lambda)_q$ design D:
  Derived design in P:

  $$\{B/P \mid B \in D \text{ with } P \subseteq B\} \subseteq V/P$$

  is $(t-1)$-$(v-1, k-1, \lambda)_q$ design.
- For $\mathrm{STS}_q(7)$:
  Derived design is 1-$(6, 2, 1)_q$ design.
- That is a set of $\lambda_{1,0} = q^4 + q^2 + 1$ lines in $\mathrm{PG}(5, q)$ covering all points exactly once.
- In other words:
  The derived design of $\mathrm{STS}_q(7)$ in a point P is a line spread of $\mathrm{PG}(5, q)$.

- Spread $\mathcal{S}$ called geometric if for all distinct $L_1, L_2 \in \mathcal{S}$: $\{L \in \mathcal{S} \mid L \subseteq L_1 + L_2\}$ is spread of the solid $L_1 + L_2$.
- P is called α-point of $\mathsf{STS}_q(7)$ if the derived design in P is a geometric spread.
- S. Thomas 1996: There exists a non-α-point.
- O. Heden, P. Sissokho 2016: For $q = 2$: Each hyperplane contains non-α-point.
- Goal: Investigate Heden-Sissokho result for general q!

- Assume that H is hyperplane containing only α-points.
- Fix a poor solid S in H (not containing any block).
- Let $\mathcal{F} = \{F \in \begin{bmatrix} H \\ 5 \end{bmatrix}_q \mid S \subseteq F\}$.
  We have $\#\mathcal{F} = q + 1$.
- For $F \in \mathcal{F}$, let

  $$\mathcal{L}_F := \{B \cap S \mid B \in D \text{ and } B + S = F\}.$$

  Dimension formula:
  $\dim(B \cap S) = \dim(B) + \dim(S) - \dim(F) = 3 + 4 - 5 = 2.$
  So $\mathcal{L}_F$ is a set of lines in S.
- Lemma $\mathcal{L}_F$ is a line spread of S.

Conclusion

$\mathcal{L} := \biguplus_{F \in \mathcal{F}} \mathcal{L}_F$ is a set of $(q+1)(q^2+1)$ lines in $\mathrm{PG}(3, q)$ admitting a partition into $q + 1$ line spreads.

Lemma

For each point P in S, the $q+1$ lines in $\mathcal{L}$ passing through P span only a plane E_P.
(In other words, the lines form a pencil in E_P through P.)

Corollary

$(\begin{bmatrix} S \\ 1 \end{bmatrix}_q, \mathcal{L})$ is a generalized quadrangle.

Classification

Classification of projective generalized quadrangles:
(F. Buekenhout, C. Lefèvre 1974)
$\implies$ $(\begin{bmatrix} S \\ 1 \end{bmatrix}_q, \mathcal{L})$ is symplectic generalized quadrangle $W(q)$.

- By property of $\mathcal{L}$:
  The lines of $W(q)$ admit a partition into $q+1$ line spreads.
- Equivalently: The points of the parabolic quadric $Q(4, q)$ admit a partition into ovoids.
- Not possible for even q.
  - Payne, Thas: Finite generalized quadrangles, 3.4.1(i)
- Not possible for prime q.
  - Ball, Govaerts, Storme 2006:
    Each ovoid in $Q(4, q)$ is an elliptic quadric.
  - Any two of them have non-trivial intersection.

Theorem

Let q be prime or even and D a $\mathsf{STS}_q(7)$.
Then each hyperplane contains a non-α-point of D.

Research problem

Investigate the remaining q (i.e. q a proper odd prime power).

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joint work with S. Kurz, A. Wassermann.

Definition (Classical group divisible design)

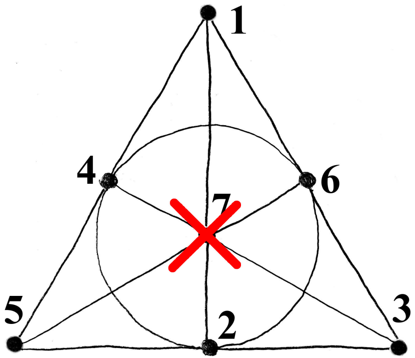
Let V be a finite set of size v.
$(\mathcal{G}, \mathcal{B})$ is a (v, k, λ, g) group divisible design (gdd), if

- $\mathcal{G} \subseteq \binom{V}{g}$ is a partition of V.
- $\mathcal{B} \subseteq \binom{V}{k}$
- such that each $T \in \binom{V}{2}$ is either contained in a group, or in exactly λ blocks.

Example

A $(6, 3, 1, 2)$-gdd. (So: $v = 6$, $k = 3$, $\lambda = 1$, $g = 2$)

$$V = \{1, 2, 3, 4, 5, 6\}$$
$$\mathcal{G} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$$
$$\mathcal{B} = \{\{1, 3, 6\}, \{1, 4, 5\}, \{2, 3, 5\}, \{2, 4, 6\}\}$$



Definition (q-analog of group divisible design)

Let V be a v-dimensional $\mathbb{F}_q$ vector space.
$(\mathcal{G}, \mathcal{B})$ is a $(v, k, \lambda, g)_q$ group divisible design (gdd), if

- $\mathcal{G} \subseteq \begin{bmatrix} V \\ g \end{bmatrix}_q$ is a spread of V.
- $\mathcal{B} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}_q$
- such that each $T \in \begin{bmatrix} V \\ 2 \end{bmatrix}_q$ is either contained in a group, or in exactly λ blocks.

Lemma

Let D be a 2-$(v, k, 1)_q$ Steiner system on V and $P \in \begin{bmatrix} V \\ 1 \end{bmatrix}_q$.
Projection mod P is

$$\pi : \mathrm{PG}(V) \to \mathrm{PG}(V/P), \quad U \mapsto (U + P)/P$$

Set

$$\mathcal{G} := \{\pi(B) \mid B \in D \text{ and } P \subseteq B\}$$
$$\mathcal{B} := \{\pi(B) \mid B \in D \text{ and } P \not\subseteq B\}$$

Then $(\mathcal{G}, \mathcal{B})$ is a $(v - 1, k, q^2, k - 1)_q$-gdd.

Application to q-Fano plane

Existence of $\mathrm{STS}_q(7) \implies$ Existence of $(6, 3, q^2, 2)_q$-gdd

Admissibility of the parameters

1. Spread $\mathcal{G}$ exists $\iff g \mid v$
2. For all blocks $B \in \mathcal{B}$ and $G \in \mathcal{G}$:
   $\dim(B \cap G) \leq 1 \quad$ (B scattered wrt $\mathcal{G}$) $\implies k \leq v - g$
3. Double count incidences (B, T) with $B \in \mathcal{B}$ and $T \in \begin{bmatrix} B \\ 2 \end{bmatrix}_q$

$$\implies \#\mathcal{B} = \lambda \cdot \frac{\begin{bmatrix} v \\ 2 \end{bmatrix}_q - \begin{bmatrix} v \\ 1 \end{bmatrix}_q / \begin{bmatrix} g \\ 1 \end{bmatrix}_q \cdot \begin{bmatrix} g \\ 2 \end{bmatrix}_q}{\begin{bmatrix} k \\ 2 \end{bmatrix}_q} \in \mathbb{Z}$$

4. Fix $P \in \begin{bmatrix} V \\ 1 \end{bmatrix}_q$, let $r = \#\{B \in \mathcal{B} \mid P \subseteq B\}$ replication number.
   Double count incid. (B, T) with $B \in \mathcal{B}$, $T \in \begin{bmatrix} B \\ 2 \end{bmatrix}_q$, $P \subseteq T$

$$\implies r = \lambda \cdot \frac{\begin{bmatrix} v-1 \\ 1 \end{bmatrix}_q - \begin{bmatrix} g-1 \\ 1 \end{bmatrix}_q}{\begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q} \in \mathbb{Z}$$

1. – 4. are counterparts of conditions for classical gdds.

New admissibility condition (no classical counterpart):

Lemma

$q^{k-g} \mid \lambda$

Proof.

- Let P be a point.
- There is a unique $G \in \mathcal{G}$ passing through P.
- Let G' be image of G mod P.
- Points outside of G' are covered λ times by the images of the blocks ($k-1$-subspaces).
- $\implies$ λ-fold repetition of the complement of G' is q^{k-2}-divisible.
- $\implies$ λ-fold repetition of G' is q^{k-2}-divisible.
- G' is exactly q^{g-2}-divisible, so $q^{k-g} \mid \lambda$.

$\square$

Lemma

Let $\mathcal{G} \subseteq \begin{bmatrix} V \\ g \end{bmatrix}_q$ be spread, G subgroup of $\mathrm{P\Gamma L}(v, q)_{\mathcal{G}}$.
If action of G on $\begin{bmatrix} V \\ 2 \end{bmatrix}_q \setminus \bigcup_{U \in \mathcal{G}} \begin{bmatrix} U \\ 2 \end{bmatrix}_q$ is transitive
$\implies$ For any union $\mathcal{B}$ of G-orbits on the scattered k-subspaces $(\mathcal{G}, \mathcal{B})$ is a $(v, k, \lambda, g)_q$-gdd (with suitable λ).

Proof.

Use transitivity. $\square$

Remark on the principle

- Powerful construction method for classical designs.
- Does not work for subspace designs
  (lack of suitable groups).

Now:

- $v = g \cdot s$
- $V = (\mathbb{F}_{q^g})^s$
- $\mathcal{G} = \begin{bmatrix} V \\ 1 \end{bmatrix}_{q^g}$ Desarguesian $(g-1)$-spread.
- $\forall U \leq_{\mathbb{F}_q} V : \dim_{\mathbb{F}_{q^g}}(\langle U \rangle_{\mathbb{F}_{q^g}}) \leq \dim_{\mathbb{F}_q}(U)$.
  In case of equality: U fat
- Let $\mathcal{F}_k$ be set of fat k-subspaces.
- Lines covered by elements of $\mathcal{G}$ = non-fat 2-subspaces.

Lemma

Action of $\mathrm{SL}(s, q^g)/(\mathbb{F}_q^\times \cap \mathrm{SL}(s, q))$ on $\mathcal{F}_k$

- for $k < s$: is transitive
- for $k = s$: $\frac{q^g-1}{q-1}$ orbits of equal length

Theorem

Let $g \geq 2$ and $s \geq 3$.

- Case $k \in \{3, \ldots, s-1\}$:
  $(\mathcal{G}, \mathcal{F}_k)$ is $(gs, k, \lambda, g)_q$-gdd with

$$\lambda = q^{(g-1)(\binom{k}{2}-1)} \prod_{i=2}^{k-1} \frac{q^{g(s-i)} - 1}{q^{k-i} - 1}.$$

- Case $k = s$:
  For all $\alpha \in \{1, \ldots, \frac{q^g-1}{q-1}\}$ and any union $\mathcal{B}$ of α orbits of the action of $\mathrm{SL}(s, q^g)/(\mathbb{F}_q^\times \cap \mathrm{SL}(s, q))$ on $\mathcal{F}_s$:
  $(\mathcal{G}, \mathcal{F}_k)$ is a $(gs, s, \lambda, g)_q$-gdd with

$$\lambda = \alpha q^{(g-1)(\binom{k}{2}-1)} \prod_{i=2}^{s-2} \frac{q^{gi} - 1}{q^i - 1}.$$

Remark

- Theorem with $g = 2$, $k = s = 3$, $\alpha = 1$:
  $\rightsquigarrow \exists (6, 3, q^2, 2)_q$ gdds
- We have seen: gdds with these parameters would arise from q-analog of the Fano plane $\mathrm{STS}_q(7)$.
- First $(6, 3, q^2, 2)_q$-gdds
  constructed by Etzion, Hooker 2018.
- If $\mathrm{STS}_q(7)$ exists $\implies$ $(6, 3, q^2, 2)_q$-gdds exist for non-Desarguesian spreads, too. (α-points!)
  Found computationally for $q = 2$.

Conclusion for binary q-analog of the Fano plane

- $\mathrm{STS}_2(7)$ cannot look too nice.
  (at most 2 automorphisms; result on α-points)
  $\rightsquigarrow$ Might be seen as sign for non-existence.
- So far, all "local" investigations lead to consistent answers.
  $\rightsquigarrow$ Might be seen as sign for existence.

Open problems

- Further investigate α-points.
- Computational evidence:
  - For the Desarguesian spread: $(6, 3, \lambda, 2)_2$ exists $\iff \lambda \in \{2, 4, 6, 8, 10, 12\}$
  - For the 131.043 non-Desarguesian spreads: $(6, 3, \lambda, 2)_2$ exists only for $\lambda \in \{4, 8, 12\}$.

  Explain this!
- For any of the 8 solid spreads in $\mathrm{PG}(7, 2)$: No $(8, 4, 7, 4)_2$ does exist. Explanation?

Invitation!

Conference ALCOMA 20
(Algebraic Combinatorics and Applications)

- 2020-3-29 – 2020-4-4
- Kloster Banz, Lichtenfels, Germany
- `https://alcoma20.uni-bayreuth.de/`

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