On $q$-analogs of the Fano plane

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Network Coding and Designs
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What is this?

- Most frequent image in discrete math.
- Fano plane.
What is special about it?

- Smallest projective plane.
- Smallest non-trivial Steiner triple system.
Outline

Block designs and their $q$-analogs

Intersection numbers

Prescribed automorphisms

Subspace codes
Outline

Block designs and their $q$-analogs

Intersection numbers

Prescribed automorphisms

Subspace codes
Subset lattice

- Let $V$ be a $\nu$-element set.
- $(\binom{V}{k}) :=$ Set of all $k$-subsets of $V$.
- $\#(\binom{V}{k}) = \binom{\nu}{k}$.
- Subsets of $V$ form a distributive lattice (wrt. $\subseteq$).

Definition

$D \subseteq \binom{V}{k}$ is a $t$-$(\nu, k, \lambda)$ (block) design if each $T \in \binom{V}{t}$ is contained in exactly $\lambda$ blocks (elements of $D$).

- If $\lambda = 1$: $D$ Steiner system
- If $\lambda = 1$, $t = 2$ and $k = 3$: $D$ Steiner triple system STS$(\nu)$
Example

\[ V = \{1, 2, 3, 4, 5, 6, 7\} \]
\[ D = \{\{1, 2, 5\}, \{1, 4, 6\}, \{1, 3, 7\}, \{2, 3, 6\}, \]
\[ \text{\{2, 4, 7\}}, \{3, 4, 5\}, \{5, 6, 7\}\} \]

Fano plane \( D \) is a 2-(7, 3, 1) design, i.e an STS(7).
Lemma
Let $D$ be a $t$-$(v, k, \lambda)$ design and $i \in \{0, \ldots, t\}$. Then $D$ is also an $i$-$(v, k, \lambda_i)$ design with

$$\lambda_i = \frac{(\frac{v-i}{t-i})}{(\frac{k-i}{t-i})} \cdot \lambda.$$ 

In particular, $\#D = \lambda_0$.

Example
Fano plane STS(7) ($v = 7$, $k = 3$, $t = 2$, $\lambda = 1$):

$$\lambda_2 = 1, \quad \lambda_1 = 3, \quad \lambda_0 = 7$$

Corollary: Integrality conditions
If a $t$-$(v, k, \lambda)$ design exists, then

$$\lambda_0, \lambda_1, \ldots, \lambda_t \in \mathbb{Z} \quad \text{(Parameters are admissible)}$$
Lemma
STS($v$) admissible $\iff v \equiv 1, 3 \pmod{6}$.

STS($v$) for small $v$

- STS(3) = \{V\} exists trivially.
- Smallest non-trivial Steiner triple system: Fano plane STS(7).
- Next admissible case: STS(9) exists (affine plane of order 3).

Theorem (Kirkman 1847)
All admissible STS($v$) do exist.
Subspace lattice

- Let $V$ be a $v$-dimensional $\mathbb{F}_q$ vector space.
- Grassmannian $\binom{V}{k}_q := \text{Set of all } k\text{-dim. subspaces of } V$.
- Gaussian Binomial coefficient

\[
\#\binom{V}{k}_q = \binom{v}{k}_q = \frac{(q^v - 1)(q^{v-1} - 1) \cdot \ldots \cdot (q^{v-k+1} - 1)}{(q - 1)(q^2 - 1) \cdot \ldots \cdot (q^k - 1)}
\]

- Subspaces of $V$ form a modular lattice (wrt. $\subseteq$).
- Subspace lattice of $V = \text{projective geometry } PG(v-1, q)$
  - Elements of $\binom{V}{1}_q$ are points.
  - Elements of $\binom{V}{2}_q$ are lines.
  - Elements of $\binom{V}{3}_q$ are planes.
  - Elements of $\binom{V}{v-1}_q$ are hyperplanes.
- Fano plane is the projective geometry $PG(2, 2)$. 
**q-analogs in combinatorics**

Replace subset lattice by subspace lattice!

<table>
<thead>
<tr>
<th>orig.</th>
<th>q-analog</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu )-element set ( V ) ( \nu )-dim. ( \mathbb{F}_q ) vector space ( V ) ( \binom{V}{k} )</td>
<td>( \nu )-dim. ( \mathbb{F}_q ) vector space ( V ) ( \binom{V}{k} ) ( [V]_q ) ( [k]_q )</td>
</tr>
<tr>
<td>cardinality</td>
<td>dimension</td>
</tr>
<tr>
<td>( \cap )</td>
<td>( \setminus )</td>
</tr>
<tr>
<td>( \cup )</td>
<td>( + )</td>
</tr>
</tbody>
</table>

- The subset lattice corresponds to \( q = 1 \).
- Sometimes: Unary field \( \mathbb{F}_1 \).
Definition (block design)
Let $V$ be a $v$-element set. $D \subseteq \binom{V}{k}$ is a $t-(v, k, \lambda)$ (block) design if each $T \in \binom{V}{t}$ is contained in exactly $\lambda$ elements of $D$.

$q$-analog of a design?

Definition (subspace design)
Let $V$ be a $v$-dimensional $\mathbb{F}_q$ vector space. $D \subseteq \left[ \binom{V}{k} \right]_q$ is a $t-(v, k, \lambda)_q$ (subspace) design if each $T \in \left[ \binom{V}{t} \right]_q$ is contained in exactly $\lambda$ elements of $D$.

- If $\lambda = 1$: $D$ $q$-Steiner system
- If $\lambda = 1$, $t = 2$, $k = 3$: $D$ $q$-Steiner triple system $\text{STS}_q(v)$
- Geometrically:
  $\text{STS}_q(v)$ is a set of planes in $\text{PG}(v - 1, q)$ covering each line exactly once.
Lemma
Let $D$ be a $t-(v, k, \lambda)_q$ design and $i \in \{0, \ldots, t\}$. Then $D$ is also an $i-(v, k, \lambda_i)_q$ design with

$$\lambda_i = \frac{\left\lceil \frac{v-i}{t-i} \right\rceil q}{\left\lceil \frac{k-i}{t-i} \right\rceil q} \cdot \lambda.$$ 

In particular, $\#D = \lambda_0$.

Corollary: Integrality conditions
If a $t-(v, k, \lambda)_q$ design exists, then

$$\lambda_0, \lambda_1, \ldots, \lambda_t \in \mathbb{Z} \quad \text{(Parameters are admissible)}$$
Lemma

$\text{STS}_q(\nu)$ admissible $\iff \nu \equiv 1, 3 \pmod{6}$.

$\text{STS}_q(\nu)$ for small $\nu$

- $\text{STS}_q(3) = \{V\}$ exists trivially.
- $q$-analog of the Fano plane $\text{STS}_q(7)$.
  Existence open for every field order $q$.

  Most important open problem in $q$-analogs of designs.

- $\text{STS}_q(9)$: existence open for every $q$.
- Only known non-trivial $q$-STS:
  $\text{STS}_2(13)$ exists (Braun, Etzion, Östergård, Vardy, Wassermann 2013)
Focus on binary $q$-analog of the Fano plane $\text{STS}_2(7)$.

$$\lambda_2 = 1, \quad \lambda_1 = 21, \quad \lambda_0 = 381$$

- $\text{STS}_2(7)$ consists of $\lambda_0 = 381$ blocks (out of $\binom{7}{3}_2 = 11811$ planes).
- Huge search space ($\binom{11811}{381}$ has 730 digits).

Need additional properties!

- Through each point $P$ there are $\lambda_1 = 21$ blocks. Image in $V/P \cong \text{PG}(5, 2)$ is a line spread. Mateva, Topalova 2009: $\exists 131044$ types of spreads. Problem still way too big.

- More refined information by intersection numbers (joint work with Mario Pavčević.)
Outline

Block designs and their $q$-analogs

Intersection numbers

Prescribed automorphisms

Subspace codes
Definition

In the following: $D$ a $t-(v, k, \lambda)_q$ design, $S$ a subspace of $V$, $s = \dim(S)$

- The $i$-th intersection number of $S$ in $D$ is
  \[ \alpha_i = \alpha_i(S) = \# \{ B \in D \mid \dim(B \cap S) = i \} \].

- The intersection vector of $S$ in $D$ is
  \[ (\alpha_0(S), \alpha_1(S), \ldots, \alpha_k(S)) \]
Theorem ($q$-analog of Mendelsohn equations 1971)

For $i \in \{0, \ldots, t\}$

\[
\sum_{j=i}^{s} \binom{j}{i}_q \alpha_j = \binom{s}{i}_q \lambda_i
\]

(Linear system of equations for the intersection vector)

Proof.

Double count

\[
X = \left\{ (I, B) \in \binom{V}{i}_q \times D \mid I \leq B \cap S \right\}
\]

- $\binom{s}{i}_q$ possibilities for $I$.
  For each $I$, $\lambda_i$ blocks $B$ with $I \leq B$.
  \[
  \implies \#X = \binom{s}{i}_q \lambda_i.
  \]

- For fixed block $B$, there are $\binom{\dim(B \cap S)}{i}_q$ suitable $I$.
  \[
  \implies \#X = \sum_{j=i}^{s} \binom{j}{i}_q \alpha_j.
  \]
Theorem ($q$-analog of Köhler equations 1988)

For $i \in \{0, \ldots, t\}$

$$\alpha_i = \begin{bmatrix} s \\ i \end{bmatrix}_q \sum_{j=i}^t (-1)^{j-i} q^{(j-i)/2} \begin{bmatrix} s-i \\ j-i \end{bmatrix}_q \lambda_j$$

$$+ (-1)^{t+1-i} q^{(t+1-i)/2} \sum_{j=t+1}^s \begin{bmatrix} j \\ i \end{bmatrix}_q \begin{bmatrix} j-i-1 \\ t-i \end{bmatrix}_q \alpha_j.$$

(Parameterization of $\alpha_0, \alpha_1 \ldots, \alpha_t$ by $\alpha_{t+1}, \alpha_{t+2}, \ldots, \alpha_k$)

History

- For classical designs by Köhler in 1988, long and complicated induction proof.
- Simpler proof by de Vroedt in 1991.
- Can be simplified further!
  Idea: Apply Gauss reduction to the Mendelsohn equations. Works in the $q$-analog situation, too.
Corollary
Intersection vector is uniquely determined for $\dim(S) \leq t$ and $\dim(S) \geq v - t$.

Example
Köhler equations for STS$_2(7)$ with $s = 4$.

$$\begin{align*}
\alpha_0 &= 136 - 8\alpha_3 \\
\alpha_1 &= 210 + 14\alpha_3 \\
\alpha_2 &= 35 - 7\alpha_3
\end{align*}$$

$\alpha_3 \in \{0, 1\}$, since otherwise

- $S$ contains two blocks $B_1, B_2$.
- By the dimension formula, $\dim(B_1 \cap B_2) \geq 2$. Contradiction.

$\implies$ Two possible intersection vectors:

$$(136, 210, 35, 0) \quad \text{and} \quad (128, 224, 28, 1)$$
Example (cont.)

- Distribution of the 4-dim subspaces $S$ to the two intersection numbers?
  (total: $\binom{7}{4}^2 = 11811$ subspaces $S$)

- Double counting:
  $(136, 210, 35, 0)$ occurs 6096 times,
  $(128, 224, 28, 1)$ occurs 5715 times.
Similarly, compute the intersection vectors for all possible values of $s$.

<table>
<thead>
<tr>
<th>$s$</th>
<th>Intersection Vector</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>$(0, 0, 0, 381)$</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>$(0, 0, 336, 45)$</td>
<td>127</td>
</tr>
<tr>
<td>5</td>
<td>$(0, 256, 120, 5)$</td>
<td>2667</td>
</tr>
<tr>
<td>4</td>
<td>$(128, 224, 28, 1)$</td>
<td>5715</td>
</tr>
<tr>
<td>4</td>
<td>$(136, 210, 35, 0)$</td>
<td>6096</td>
</tr>
<tr>
<td>3</td>
<td>$(240, 140, 0, 1)$</td>
<td>381</td>
</tr>
<tr>
<td>3</td>
<td>$(248, 126, 7, 0)$</td>
<td>11430</td>
</tr>
<tr>
<td>2</td>
<td>$(320, 60, 1, 0)$</td>
<td>2667</td>
</tr>
<tr>
<td>1</td>
<td>$(360, 21, 0, 0)$</td>
<td>127</td>
</tr>
<tr>
<td>0</td>
<td>$(381, 0, 0, 0)$</td>
<td>1</td>
</tr>
</tbody>
</table>

How do the different $S$ relate to each other?
Theorem (K., Pavčević 2015)

The "intersection structure" of a 2-analog of the Fano plane is

\[ s = 5 \quad (0, 256, 120, 5)^{2667} \]

\[ s = 4 \quad (128, 224, 28, 1)^{5715} \quad (136, 210, 35, 0)^{6096} \]

\[ s = 3 \quad (240, 140, 0, 1)^{381} \quad (248, 126, 7, 0)^{11430} \]

\[ s = 2 \quad (320, 60, 1, 0)^{2667} \]
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Fundamental theorem of projective geometry

For \( \nu \geq 3 \), the automorphism group of the subspace lattice of \( V \) is \( \mathrm{PGL}(\nu, q) \).

For \( q = 2 \), simply \( \mathrm{PGL}(\nu, 2) = \mathrm{GL}(\nu, 2) \).

Idea

- Pick some subgroup \( G < \mathrm{GL}(7, 2) \).
- Focus on \( G \)-invariant \( q \)-analogues of the Fano plane.
  \( \leadsto \) problem gets much smaller.
- If lucky: Find one!
- Otherwise: Systematically narrow down the possible automorphism groups.

Observation

- Conjugate subgroups \( G \) yield isomorphic designs.
- \( \iff \) Only the conjugation type of \( G \) matters.
Theorem (Braun, K., Nakić 2016)

The order of the automorphism group of a binary q-analog of the Fano is

- 1 or
- 2 (1 remaining type) or
- 3 (2 remaining types) or
- 4 (1 remaining type)

Progress

(joint work with Sascha Kurz and Alfred Wassermann)

- Parallel computing: Order 4 not possible.
- Theory and parallel computing: Order 3 not possible.
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- Parallel computing: Order 4 not possible.
- Theory and parallel computing: Order 3 not possible.
Lemma

Elements of order 3 in $\text{GL}(v, 2)$ are represented by

$$A_{v,f} := \begin{pmatrix} (1&1) \\ (1&0) \\ \ddots \\ (1&1) \end{pmatrix}$$

with $f \in \{0, \ldots, v - 1\}$, $v - f$ even.

Proof.

Let $A \in \text{GL}(v, 2)$ of order 3.

$\implies A^3 = I_v.$

$\implies m_A \mid X^3 - 1 = (X^2 + X + 1)(X - 1).$

Enumerate the possible rational normal forms.
Example

Types of elements of order 3 in $\text{GL}(7, 2)$:

$A_{7,1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$A_{7,3} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$A_{7,5} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$
Example (GL(3, 2))

Single element type of order 3: \[ A_{3,1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 \end{pmatrix} \]

▶ 1 fixed point
▶ 2 orbits of size 3 falling into:
  ▶ 1 orbit line
  ▶ 1 orbit triangle
Example (GL(3, 2))

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  - 1 orbit triangle
Action of $A_{v,f}$ on the point set $[\left[ V \right]_q]

- $2^f - 1$ fixed points
  (points of the form $\langle (0, \ldots, 0, *, \ldots, *) \rangle$)
- $2^{v-f} - 1$
  orbit lines
  (points of the form $\langle (*, \ldots, *, 0, \ldots, 0) \rangle$)
- $\frac{(2^{v-f} - 1)(2^f - 1)}{3}$ orbit triangles

Example

<table>
<thead>
<tr>
<th>$v$</th>
<th>$f$</th>
<th>fixed points</th>
<th>orbit triangles</th>
<th>orbit lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>1</td>
<td>21</td>
<td>21</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>7</td>
<td>35</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td>31</td>
<td>31</td>
<td>1</td>
</tr>
</tbody>
</table>
Fixed planes

- Let $G = \langle A_v, t \rangle$
- Let $E \in \mathbb{V}_3^q$ be a fixed plane (i.e. $E^G = E$)
- Then $G|_E$ is well-defined
- $\#G|_E \in \{1, 3\}$
- $\#G|_E = 1 \implies E$ has 7 fixed points (type 7)
- $\#G|_E = 3 \implies E$ has 1 fixed point, 1 orbit line and 1 orbit triangle (type 1)
Counting fixed planes

How many fixed planes of type 1 and 7?

▷ Type 7:

3-subspaces of the $f$-dim space of fixed points.

$\sim \begin{bmatrix} f \\ 3 \end{bmatrix}_2$

▷ Type 1:

Uniquely spanned by an orbit triangle.

$\sim \#\text{orbit triangles} = \frac{(2^f - 1)(2^v - f - 1)}{3}$

Example

<table>
<thead>
<tr>
<th>$v$</th>
<th>$f$</th>
<th>#f.p.</th>
<th>#o.t. = #T1</th>
<th>#o.l.</th>
<th>#T7</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>1</td>
<td>21</td>
<td>21</td>
<td>0</td>
</tr>
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<td>1</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td>31</td>
<td>31</td>
<td>1</td>
<td>155</td>
</tr>
</tbody>
</table>
Fixed blocks

- Let $D$ be a $G$-invariant STS$_2(\nu)$.
- $\mathcal{F}_1 := \text{set of fixed blocks of } D \text{ of type 1}$
- $\mathcal{F}_7 := \text{set of fixed blocks of } D \text{ of type 7}$

Double count $X = \{(L, B) \mid L \text{ orbit line}, B \in \mathcal{F}_1, L < B\}$.

1. $\#X = \#\mathcal{F}_1 \cdot 1$
2. Let $L$ be an orbit line.
   - $D$ Steiner system $\implies \exists$ unique $B \in D$ with $L < B$.
   - For all $g \in G$: $B^g > L^g = L$.
   - Uniqueness of $B$ $\implies$ $B$ is fixed block.
   - $B$ contains orbit line $L$ $\implies$ $B$ of type 1.

So: $\#X = \#(\text{orbit lines}) \cdot 1$.

$$\implies \#\mathcal{F}_1 = \#\text{orbit lines} = \frac{2^{\nu-f} - 1}{3}$$

Similarly: $\#\mathcal{F}_7 = \frac{(2^f - 1)(2^{f-1} - 1)}{21}$
Example

\[\begin{array}{c|ccccc}
  v & f & \#f.p. & \#o.l. = \#F_1 & \#o.t. = \#T1 & \#T7 & \#F_7 \\
  \hline
  7 & 1 & 1 & 21 & 21 & 0 & 0 \\
  7 & 3 & 7 & 5 & 35 & 1 & 1 \\
  7 & 5 & 31 & 1 & 31 & 155 & 155/7 \\
\end{array}\]

Conclusion

- \#F_7 must be integral
  \[\implies\] The group \(\langle A_{7,5} \rangle\) is not possible!
- For \(f = 3\), the T7-plane is contained in \(D\).
- For \(f = 1\), all 21 T1-planes are contained in \(D\).
Example

<table>
<thead>
<tr>
<th>$v$</th>
<th>$f$</th>
<th>#f.p.</th>
<th>#o.l. = $#F_1$</th>
<th>#o.t. = $#T_1$</th>
<th>#T7</th>
<th>$#F_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>1</td>
<td>1</td>
<td>21</td>
<td>21</td>
<td>0</td>
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Conclusion

- $\#F_7$ must be integral
  \[ \Rightarrow \text{The group } \langle A_{7,5} \rangle \text{ is not possible!} \]
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- For $f = 1$, all 21 T1-planes are contained in $D$. 
### Example

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<th>(#\text{o.l.} = #F_1)</th>
<th>(#\text{o.t.} = #T_1)</th>
<th>(#T_7)</th>
<th>(#F_7)</th>
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Example

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<th>f</th>
<th>#f.p.</th>
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<th>#o.t. = #T_1</th>
<th>#T_7</th>
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<td>1</td>
<td>31</td>
<td>155</td>
<td>155/7</td>
</tr>
</tbody>
</table>

Conclusion

- #F_7 must be integral
  → The group ⟨A_{7,5}⟩ is not possible!
- For f = 3, the T7-plane is contained in D.
- For f = 1, all 21 T1-planes are contained in D.
The case $v = 7, f = 3$

- $\#\mathcal{F}_7 = 1$
- $\lambda_1 = 21$
- Orbit lengths 1 or 3 $\implies$ at least 2 fixed blocks!
- In total: At least 14 fixed blocks.
- But $\#\mathcal{F}_1 = 5$. Contradiction!
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The case $v = 7$, $f = 1$

- We didn’t find a theoretic argument to exclude $G = \langle A_{7,1} \rangle$.
- We know: $D$ contains the set $S$ of 21 T1-blocks. They all pass through $P = \langle (0, 0, 0, 0, 0, 0, 1) \rangle$. In $V/P \cong \text{PG}(5, 2)$, they form a Desarguesian line spread.
- Problem: Out of 3720 orbits of length 3, select 120 such that together with $S$, they form an STS$_2(7)$. Huge search space!
- Normalizer $N(G)$ of order 362880 acts on the search space.
- Orderly generation (wrt $N(G)$) to reduce the number of cases.
- Parallel computation on the Bayreuth Linux cluster.
- Finally: There is no $G$-invariant STS$_2(7)$. 
Theorem (K., Kurz, Wassermann)

The automorphism group of a binary $q$-analog of the Fano plane is

- trivial or
- of order 2 and conjugate to

\[
\left\langle \begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{pmatrix} \right\rangle.
\]
Implications on the existence of a $\text{STS}_2(7)$

- Won’t be very symmetric.
- Many “natural” approaches for the construction won’t work.
- Still: Vast part of the search space remains untouched.
- Further theoretical insight is needed to reduce the complexity to a computationally feasible level.
- Problem is still wide open!
Outline

Block designs and their $q$-analogs

Intersection numbers

Prescribed automorphisms

Subspace codes
Definition (Steiner system)
\[ D \subseteq \binom{V}{k}_q \text{ is a } t-(v, k, 1)_q \text{ Steiner system} \]
if each \( T \in \binom{V}{t}_q \) is contained in exactly one element of \( D \).

Definition ((constant dimension) subspace code)
\[ C \subseteq \binom{V}{k}_q \text{ is a } (v, 2(k - t + 1); k)_q \text{ subspace code} \]
if each \( T \in \binom{V}{t}_q \) is contained in at most one element of \( C \).

- \( q \)-Fano setting: \((7, 4; 3)_q \) subspace code \( C \).
- For \( q = 2 \):
  - \# \( C \leq 381 \)
  - \# \( C = 381 \) \iff \( C \) is a \( \text{STS}_2(7) \)
- Find maximum size \( A_q(7, 4; 3) \) of \((7, 4; 3)_q \) subspace code!
History

- Silberstein 2008: $A_2(7, 4; 3) \geq 289$
  Based on lifted rank metric codes.
- Vardy 2008: $A_2(7, 4; 3) \geq 294$
- Kohnert, Kurz 2008: $A_2(7, 4; 3) \geq 304$
  Prescribe group of order 21
- Braun, Reichelt 2012: $A_2(7, 4; 3) \geq 329$
  Prescribe group of order 15, modify large solutions.
- Liu, Honold 2014; Honold, K. 2015:
  explicit construction of $\# C = 329$
  expurgation and augmentation of the lifted Gabidulin code

Also: $A_3(7, 4; 3) \geq 6977$ for $q = 3$
(STS$_3(7)$ would have size 7651.)
Recent approach
joint work with Daniel Heinlein, Sascha Kurz and Alfred Wassermann.

- Systematically check $G < \text{GL}(7, 2)$ for admitting large $G$-invariant codes.
- Found $\#G = 64$ admitting $\#C = 319$.
- ... having a subgroup of order 32 admitting $\#C = 327$.
- ... having a subgroup of order 16 admitting $\#C = 329$.
- ... having a subgroup of order 4 admitting $\#C = 333$.

- Code provided at subspacecodes.uni-bayreuth.de

Thank you!