A remark on two partition identities

by Thomas Scharf, University of Bayreuth

Recently J.Thévenaz rose two questions on certain partition identities [Th]. In this note we will give a positive answer two both his questions.

Though the results themselves can be treated in a more elementary way, we would like to take a more general viewpoint, i.e. that of symmetric functions, or more disguised that of Lambda-Rings or polynomial functors. For notation we refer to the book of I.G.Macdonald [Mcd].

Let $\Lambda(X) := \bigoplus_n \Lambda_n(X)$ be the algebra of symmetric functions over a countably infinite alphabet $X := \{x_1, x_2, \ldots\}$ graded by total degree. We will only need that the underlying ring R contains the field of rational numbers. For each $d \in \mathbf{N}^*$ let

$$\ell_d := \frac{1}{d}\sum_{t|d}\mu(t)p_t^{d/t},\ \tilde{\ell}_d := \frac{1}{d}\sum_{t|d}\mu(t)(p_t^{d/t}-1),$$

where μ is the number theoretic Moebius function and the p_i are the symmetric power sums. Hence we have $\tilde{\ell}_d = \ell_d,\ d > 1,\ \tilde{\ell}_1 = \ell_1 - 1 = p_1 - 1$.

If we consider the algebra homomorphism $\Lambda(X) \to \mathbf{Q}[q], F(X) \mapsto F(q)$, which replaces each p_i by a fixed indeterminate/constant q, we see that $\tilde{\ell}_d(q)$ is just the polynomial $\tilde{P}_d$ of [Th].

This mapping behaves nicely with respect to the plethysm of symmetric functions [Mcd], that is $(F \circ G)(q) = F(G(q))$. From this we get

$$(h_m \circ \ell_d)(q) = \frac{1}{m!}\sum_{\pi \in S_m} \ell_d(q)^{l(\pi)},$$

where $l(\pi)$ denotes the number of cycles of π, which is well-known to be a binomial coefficient [Mcd]. Thus we get

$$(h_m \circ \ell_d)(q) = \binom{\tilde{P}_d + m - 1}{m} \text{ and } (e_m \circ \ell_d)(q) = \binom{\tilde{P}_d}{m}.$$

Hence the left sides of theorems A and B of [Th] may be written as

$(*)\ \sum_{a \mapsto n}\prod_i \binom{\tilde{P}_i + a_i - 1}{a_i} = \sum_{a \mapsto n}\prod_i (h_{a_i} \circ \tilde{\ell}_i)(q)$,

and

$$\sum_{a \mapsto n}(-1)^{\sum_j a_j}\prod_i \binom{\tilde{P}_i}{a_i} = \sum_{a \mapsto n}\prod_i (((-1)^{a_i}e_{a_i}) \circ \tilde{\ell}_i)(q),$$

respectively. Here $a \mapsto n$ means summing over all cycle types of n, i.e. over all sequences $a := (a_1, \ldots, a_n)$ of non-negative integers with $\sum_i ia_i = n$.

The similarity of the formulas can be made more precise. One can – for each $a \mapsto n$ – define operators $\hat{\phi}_{[a]}$ on $\Lambda(X)$ such that

$$\hat{\phi}_{[a]}h_n = \prod_i (h_{a_i} \circ \tilde{\ell}_i) \text{ and } \hat{\phi}_{[a]}(-1)^n e_n = \prod_i ((-1)^{a_i}e_{a_i} \circ \tilde{\ell}_i)$$

(see [S-T], for details).
There is a standard involution on $\Lambda(X)$, namely $F(X) \mapsto F(-X)$, in the sense of Lambda-rings, which obeys $h_n(-X) = (-1)^n e_n(X)$. Hence it is just this involution which interchanges the expressions .
And indeed they are inverse identities in the following sense (cf. [L-S],e.g., for the general situation). Let t be an indeterminate. Then it is well known that

$$\sigma_t := \prod_i (1 - x_i t)^{-1} = \sum_n h_n t^n, \quad \lambda_{-t} := \prod_i (1 + x_i t) = \sum_n (-1)^n e_n t^n,$$

whence

$$\sigma_t \cdot \lambda_{-t} = 1 = \lambda_{-t} \cdot \sigma_t.$$

It can also be shown that $\hat{\phi} := \sum_n \sum_{a \dashv\vdash n} \hat{\phi}_{[a]}$ is a well-defined algebra homomorphism of $\Lambda(X)$ and it is easy to see that the product of the "generating functions" of the left sides is $\hat{\phi}(\sigma_t) \cdot \hat{\phi}(\lambda_{-1}) = \hat{\phi}(1)t^0 = 1$.
This can, of course, also be seen directly. For the theorems are equivalent to the following identities of formal power series:

$$\prod_{i>0} (\frac{1}{1-t^i})^{\tilde{P}_i} = \frac{1-t}{1-qt}, \text{ and } \prod_{i>0} (1-t^i)^{\tilde{P}_i} = \frac{1-qt}{1-t}.$$

And, indeed, the first equation is equivalent to *Gauss' cyclotomic identity* [Ga]

$$\prod_{i>0} (\frac{1}{1-t^i})^{\ell_i(q)} = \frac{1}{1-qt}.$$

But also a direct proof of the first (and hence second equation) along the lines in [Th] is possible, as was shown to me by V.Welker [We].
We only need an easier proof of the fact that the number of semisimple conjugacy classes, i.e. of those whose elements are diagonalizable in a certain field extension of $GF(q)$, is equal to $q^n - q^{n-1}$. For this we only have to note that these classes are in bijection to their characteristic polynomials. These polynomials are normed of degree n with non-zero constant term. But each of these polynomials is characteristic polynomial by the Jacobi normal form. Hence the number of semisimple classes is just the number of these polynomials which is obviously $q^n - q^{n-1}$.

References

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