Construction of Optimal Linear Codes

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Thurnau April 2005

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Linear Code

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$$C = \{v\Gamma : v \in GF(q)^k\}$$

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The minimum distance of a linear code is the minimum number of nonzero entries (=weight) of all nonzero codewords.

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A linear code is called *optimal* if the minimum distance is at the upper bound, so no better linear code for (n, k, q) is possible, and the upper bound could be met.

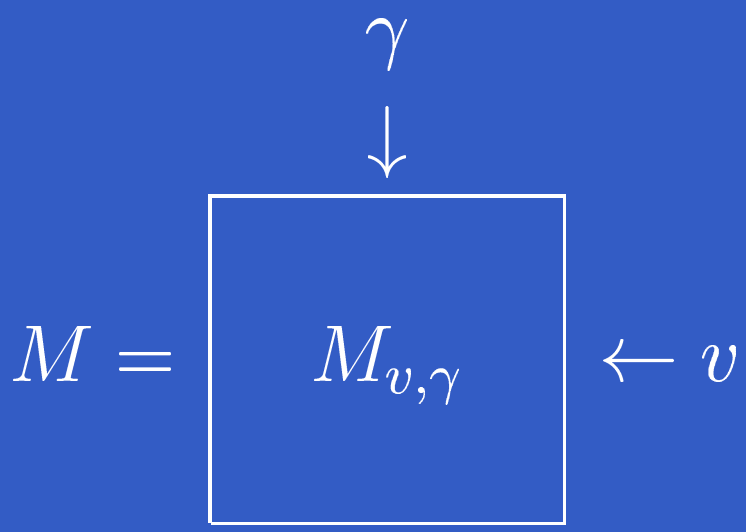
Minimum Distance Matrix

We build a matrix M whose columns are labeled by the possible columns γ of the generator matrix. Rows are labeled by the nonzero $v \in GF(q)^k$ which produce after the multiplication with the generator matrix the codewords of the code.

Minimum Distance Matrix

$$M = \square$$

Minimum Distance Matrix



Minimum Distance Matrix

$$M = \boxed{M_{v,\gamma}} \leftarrow v \qquad \gamma \downarrow$$

$$M_{v,\gamma} = \begin{cases} 1 & v\gamma = 0 \\ 0 & v\gamma \neq 0 \end{cases}$$

Diophantine System of Equations

We interested in an integral (or $0/1$) solution $x = (x_1, \ldots, x_{q^k-1})$ of the system

$$(1) \quad Mx \leq \begin{pmatrix} n-d \\ \vdots \\ n-d \end{pmatrix}$$

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A solution corresponds via selection of columns of the generator matrix to an (n, k, q) code with minimum distance $\geq d$.

Projective Geometry

As we are computing scalar products, the 0/nonzero property is invariant under scalar multiplication, so we can label rows and columns by $1-$dimensional subspaces of $GF(q)^k$.

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M is after this reduction the incidence matrix between the $1-$dimensional subspaces and the $(k-1)-$ dimensional subspaces of $GF(q)^k$.

Automorphisms

We now further reduce the size of the system of equations by prescribing a groups of automorphisms, this method corresponds to choosing complete orbits of subgroups of $GL(k,q)$ on the $1-$dimensional subspaces as possible columns of the generator matrix.

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This further reduces the number of columns, in our system of equations, as the dimension is now the number of orbits.

Reduction

The defining property of the incidence matrix

$$M_{U,V} = 1 \iff U \leq V$$

is invariant under the automorphisms.

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This also reduces the number of rows in the same way, the dimension is also the number of orbits.

Example

We computed a new $(103, 5, 8)$code with minimum distance 84.

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$$q^k - 1 \qquad \frac{q^k-1}{q-1}$$

$$\begin{pmatrix} 7 & & & & & \\ & & 7 & & & \\ & 7 & & & & \\ & & & & 7 & \\ & & & 7 & & \end{pmatrix} \begin{pmatrix} 4 & & & & & \\ & 5 & & & & \\ & & 6 & & & \\ & & & & 5 & \\ & & & & & 4 \end{pmatrix}$$

$$32767 \quad \rightarrow \quad 4681 \quad \rightarrow$$

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$$32767 \quad \rightarrow \quad 4681 \quad \rightarrow \quad 62$$

Searching for Groups

We use random subgroups of $GL(k, q)$.

- Permutation groups
- Blockdiagonal
- Monomial
- random cyclic generator

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Limits on orbit sizes, number of orbits, ...

Results

Using this method we computed over 400 new codes for $q \in \{2, 3, 4, 5, 7, 8, 9\}$, i.e. codes better than the previous lower bound.

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Among these there are more than 50 optimal codes.

Results

Ternary Codes (q=3)

k	n	d	k	n	d	k	n	d
6	191	**126**	7	46	26	8	64	37
	201	131		59	34		65	37
	202	132		60	**36**		200	126
	217	142		61	36		205	128
	219	**144**		222	144		224	141
				243	156		225	141
							226	142
							227	143
							228	144

Last update on March, 22 2005

Results

Code details

best found code with parameters
q=3 k=7 n=60
minimum distance = 36

this is new optimal code

the previous bounds were 34/36
this is a projective code

We used the prescribed group of **automorphisms** with the following generators

0 0 2 0 0 0 0	1 0 0 0 0 0 0
2 0 0 0 0 0 0	0 2 0 0 0 0 0
0 2 0 0 0 0 0	0 0 2 0 0 0 0
0 0 0 0 0 2 0	0 0 0 1 0 0 0
0 0 0 2 0 0 0	0 0 0 0 1 0 0
0 0 0 0 2 0 0	0 0 0 0 0 2 0
0 0 0 0 0 0 2	0 0 0 0 0 0 2

Results

This group make 67 orbits of sizes:

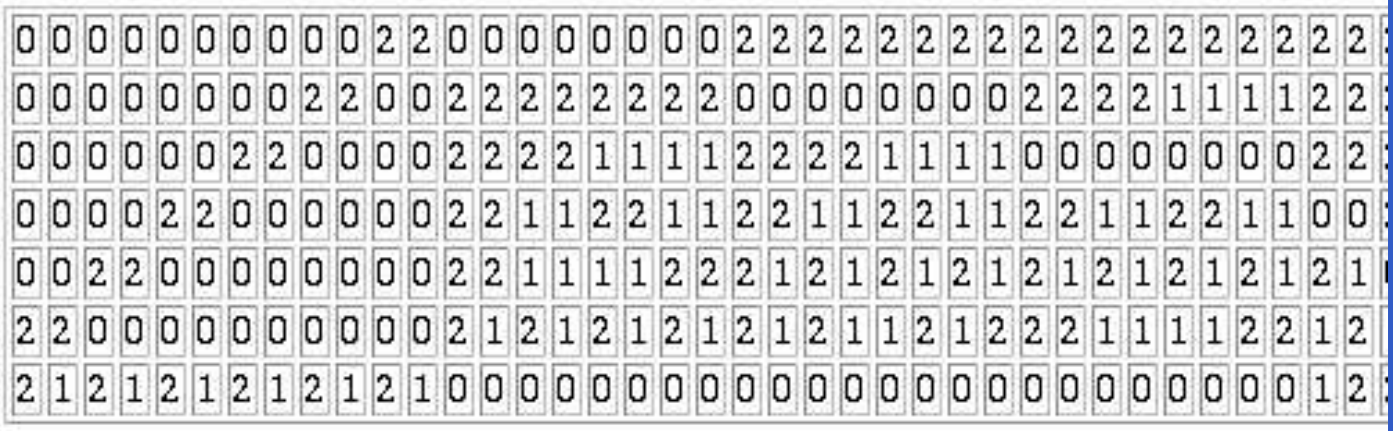
1	3	6	6	12	4	4	4	3	6	6	12	6	12	12	24	6	12	12	24	12	24	24	24	24	6	12	12	24	12	12	12	24	24	24

The solution of the corresponding linear system of equations was found after less than 22 seconds:

0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0

Results

This produces the following **generator matrix**



Which is a code with the following weight distribution

$1y^{60}+742x^{36}y^{24}+444x^{39}y^{21}+564x^{42}y^{18}+400x^{45}y^{15}+24x^{48}y^{12}+12x^{51}y^{9}$

Last Page

Thank you very much for your attention.

- M. Braun, A. Kohnert, A. Wassermann: Optimal Linear Codes From Matrix Groups, submitted, 2004
- list of new codes including generator matrix and weight distribution:
  http://linearcodes.uni-bayreuth.de
- A. E. Brouwer has current bounds:
  http://www.win.tue.nl/~aeb/voorlincod.html