Constructing Integral Pointsets in $\mathbb{Z}_n^m$

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Sofia October 2007

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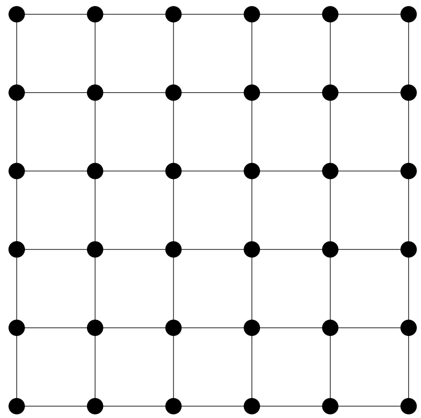
Overview

- The Problem
- Modelling
- Reducing the Size of the Problem
- other Problems

The Problem

- point sets in $\mathbb{Z}_n^m$
- $\mathbb{Z}_6^2$



- looking for sets with special properties

The Problem

- all points have pairwise integral distance
- given two points
  $x = (x_1, \ldots, x_m) \in \mathbb{Z}_n^m \quad , \quad y = (y_1, \ldots, y_m) \in \mathbb{Z}_n^m$
- if $\Sigma_{i=1}^{m}(x_i - y_i)^2$ is a square in $\mathbb{Z}_n$
- then x and y have integral distance
- integral point-sets, i.e. all pair of points have integral distance

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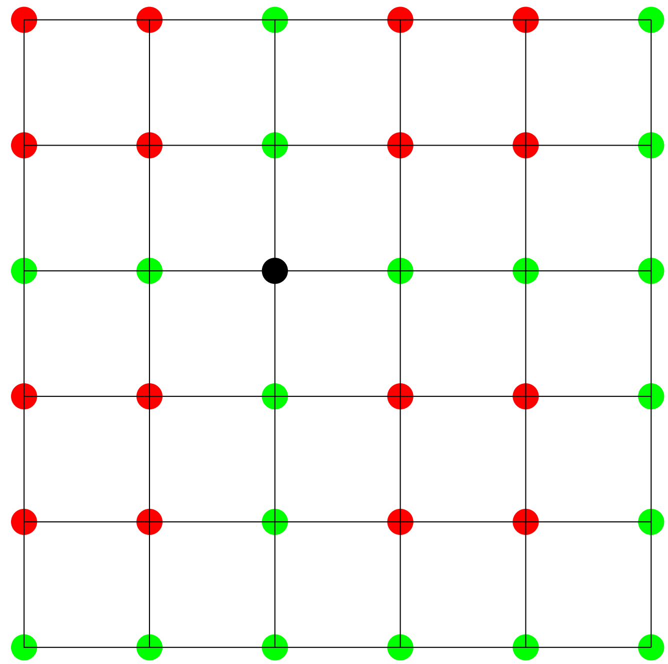
The Problem

- $\mathbb{Z}_6^2$



The Problem

- further properties for point-sets
- semi-general position
- no $(t+1)$ points on a $(t-1)$ hyperplane

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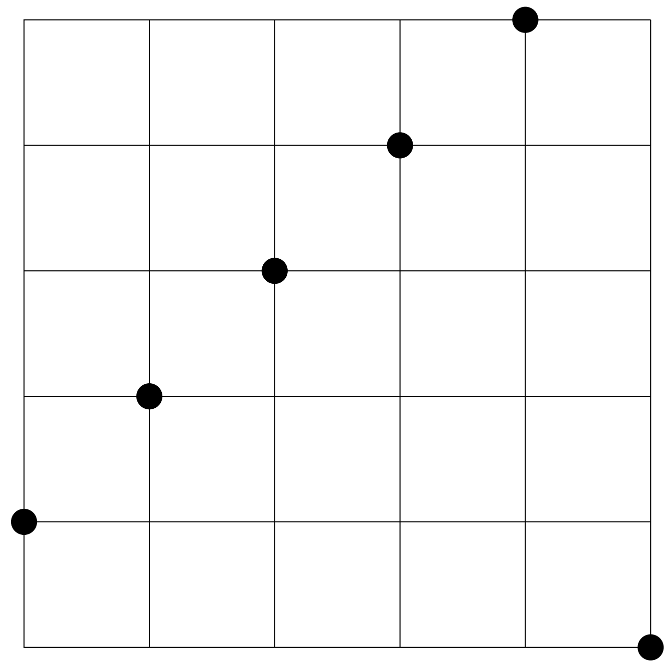
The Problem

- $\mathbb{Z}_6^2$

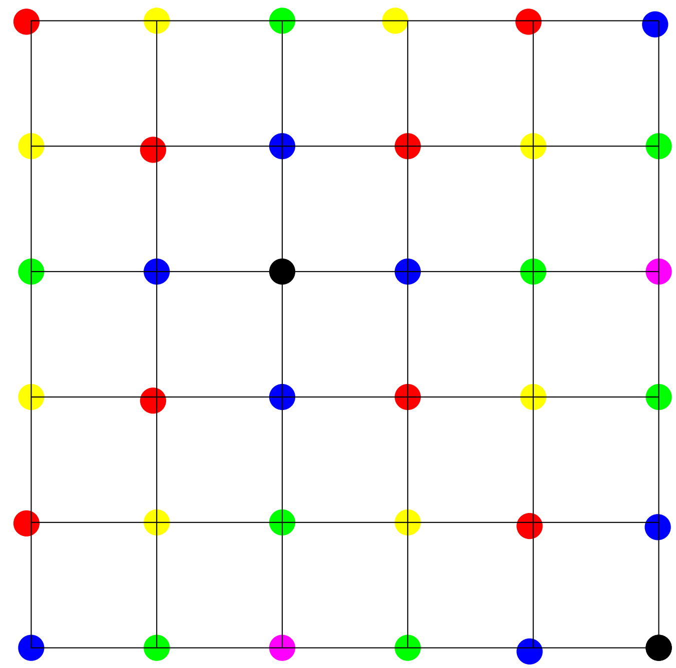


The Problem

- point-sets with further properties
- general position
- := semi-general
  and no $(t+2)$ points on a $(t-1)$ hypersphere

The Problem

- $\mathbb{Z}_6^2$



4 circles with 8 points, 2 circles with 2 points

Modelling

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Modelling

- search for integral point-sets
- becomes a $0-1$ solution of a Diophantine system of equations
- for each point there is a variable x_i
- a solution with $x_i = 0$ says this point is not in our point-set
- a solution with $x_i = 1$ says this point is in our point-set

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- prescribe the number s of points
- $\sum_{i=1}^{n^m} x_i = s$
- any solution is a point-set with the only property, there are s points in it.

- integral distance
- for each point (=variable) x_i there are a_i points $\{y_1,\dots,y_{a_i}\}\subset\{x_1,\dots,x_{n^m}\}$ which have not integral distance
- for each point add the inequality

$$a_ix_i+y_1+\ldots+y_{a_i}\le a_i.$$

- a $0/1$ solution with $x_i=1$ has no further points in the solution which have no integral distance to x_i.

- semi-general position
- for each $(t-1)-$hyperplane h_i build by the points $\{y_1, \ldots, y_{r_i}\} \subset \{x_1, \ldots, x_{n^m}\}$
- add the inequality:

$$y_1 + \ldots + y_{r_i} \leq t + 1$$

- in a $0/1$ solution there are at most $t+1$ points on each hyperplane h_i

- general position, further inequalities
- for each $(t-1)-$hypersphere s_i build by the points $\{y_1, \ldots, y_{t_i}\} \subset \{x_1, \ldots, x_{n^m}\}$
- add the inequality:

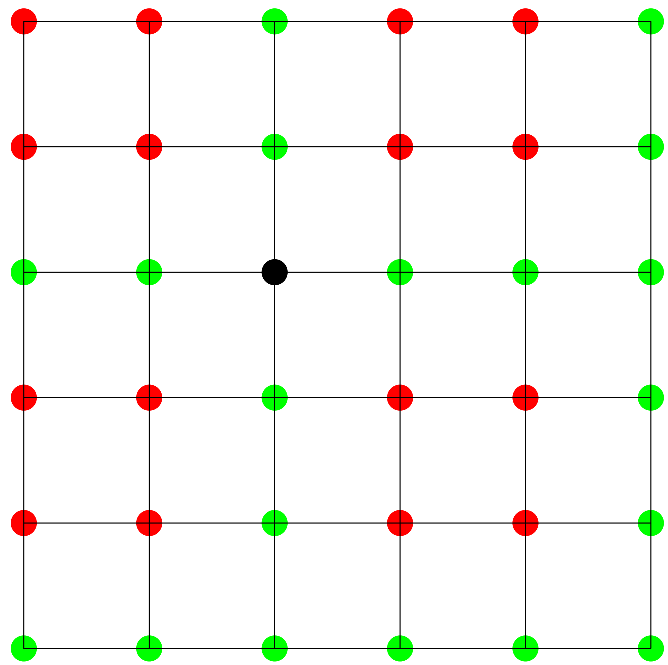
$$y_1 + \ldots + y_{t_i} \leq t + 2$$

- in a $0/1$ solution there are at most $t+2$ points on each hypersphere h_i

- $\mathbb{Z}_6^2$
- variables $x_1, \ldots, x_{36}$
- looking for a point-set with 4 points:

$$x_1 + \ldots + x_{36} = 4.$$

- integral distance



$$16x_{15} + x_1 + x_2 + x_4 + x_5 + x_7 + \ldots + x_{29} \leq 16$$

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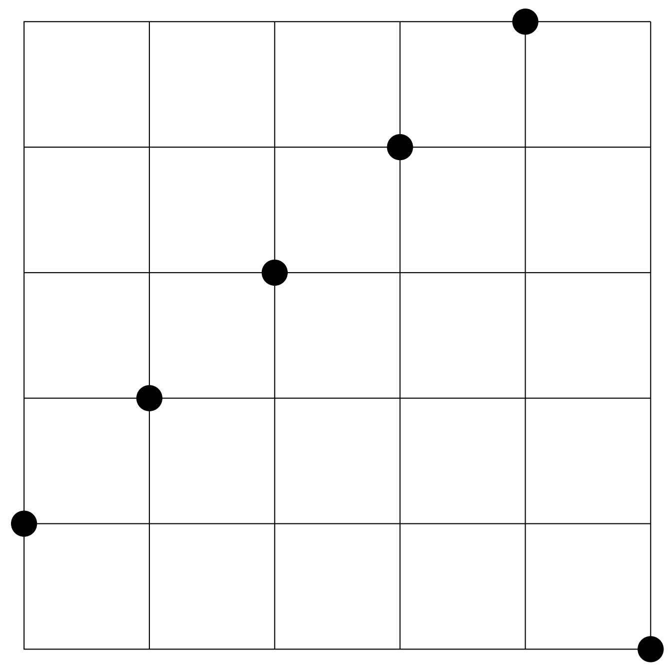
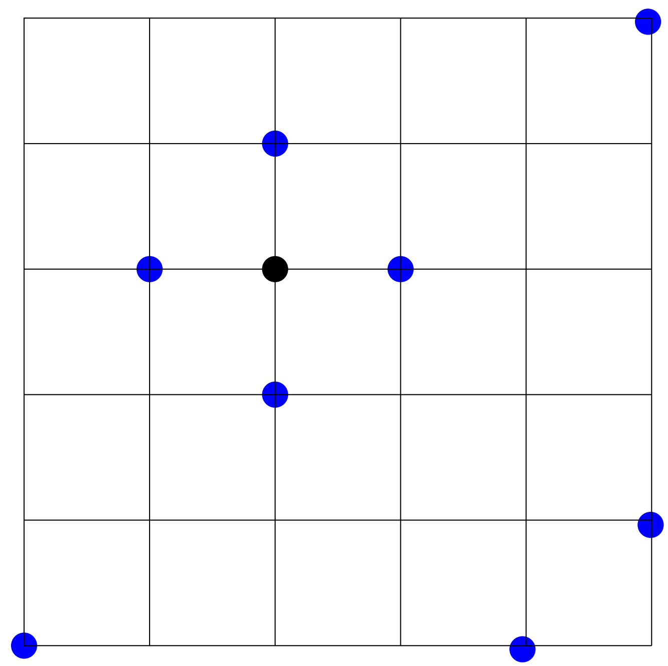
Modelling

- semigeneral = no 3 on a line



$$x_5 + x_{10} + x_{15} + x_{20} + x_{25} + x_{36} \leq 2$$

- general = no 4 on a circle



$$x_6 + x_9 + x_{14} + x_{16} + x_{21} + x_{30} + x_{31} + x_{35} \leq 3$$

Modelling

- size of the problem
- one equation for the number of points
- for each point one inequality for the integral distance
- for each hyperplane one inequality for semi-general
- for each hypersphere one inequality for general

Reduce the Problem

Reduce the Problem

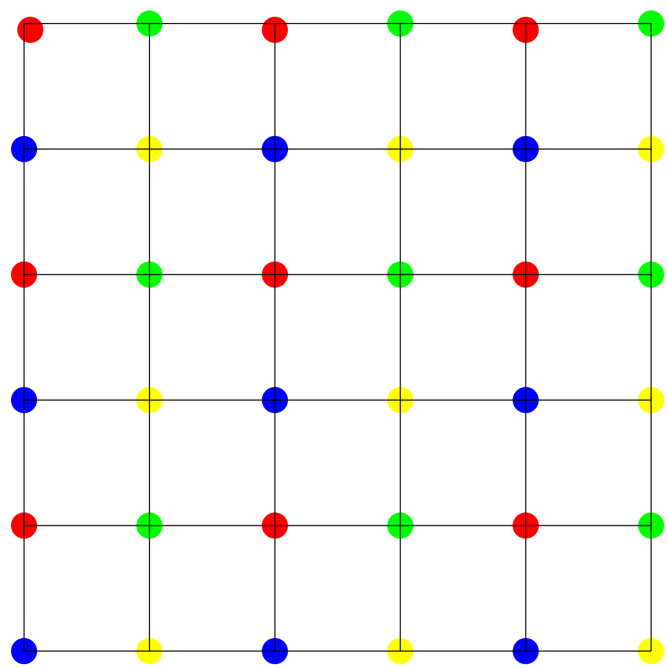
- $\mathbb{Z}_6^2$
- 1 equation for the number of points
- 36 point inequalities for integral distance
- 42 line inequalities
- $18 * 4 = 72$ circle inequalities

Reduce the Problem

- nolonger searching for an arbitrary solution
- decompose the set of all points in disjoint subsets
- search for solutions built from subsets
- = adding up columns in the system of equations
- still too many rows

Reduce the Problem

- prescribing automorphisms of $\mathbb{Z}_n^m$
- $\phi_1(x, y) = (x + 2, y)$ and $\phi_2(x, y) = (x, y + 2)$



Reduce the Problem

- prescribing automorphisms of $\mathbb{Z}_n^m$
- columns are the orbits of the automorphisms
- but an automorphism ϕ is incidence - preserving
- point p in hyperplane $h \iff \phi(p) \in \phi(h)$
- point p in hypersphere $s \iff \phi(p) \in \phi(s)$
- x and y in integral distance $\iff \phi(x)$ and $\phi(y)$ are in integral distance

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Reduce the Problem

- rows corresponding to points/hyperplanes/hyperspheres in the same orbit are identical
- automorphisms also reduce the number of rows
- size of the system of equations is now the number of orbits

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Last Words

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Other Problems

- other rings
- other properties
- other methods for reduction

Thank you very much for your attention.