Constructing Integral Pointsets in $\mathbb{Z}^m_n$

Axel Kohnert
Sofia October 2007

Bayreuth University Germany
axel.kohnert@uni-bayreuth.de
Overview

- The Problem
- Modelling
- Reducing the Size of the Problem
- other Problems
The Problem
The Problem

- point sets in $\mathbb{Z}_n^m$
- $\mathbb{Z}_6^2$
- looking for sets with special properties
The Problem

- all points have pairwise integral distance
- given two points
  \[ x = (x_1, \ldots, x_m) \in \mathbb{Z}_n^m, \quad y = (y_1, \ldots, y_m) \in \mathbb{Z}_n^m \]
- if \( \sum_{i=1}^{m} (x_i - y_i)^2 \) is a square in \( \mathbb{Z}_n \)
- then \( x \) and \( y \) have integral distance
- integral point-sets, i.e. all pair of points have integral distance
The Problem

\[ \mathbb{Z}_6^2 \]
The Problem

- further properties for point-sets
- semi-general position
- no \((t + 1)\) points on a \((t - 1)\) hyperplane
The Problem

• $\mathbb{Z}_6^2$
The Problem

- point-sets with further properties
- general position
- \( := \) semi-general and no \((t + 2)\) points on a \((t - 1)\) hypersphere
The Problem

- $Z_6^2$

4 circles with 8 points, 2 circles with 2 points
Modelling

- search for integral point-sets
- becomes a $0 - 1$ solution of a Diophantine system of equations
- for each point there is a variable $x_i$
- a solution with $x_i = 0$ says this point is not in our point-set
- a solution with $x_i = 1$ says this point is in our point-set
• prescribe the number \( s \) of points

\[
\sum_{i=1}^{m} x_i = s
\]

• any solution is a point-set with the only property, there are \( s \) points in it.
• integral distance

• for each point (=variable) \( x_i \) there are \( a_i \) points \( \{ y_1, \ldots, y_{a_i} \} \subset \{ x_1, \ldots, x_{n^m} \} \) which have not integral distance

• for each point add the inequality

\[
a_i x_i + y_1 + \ldots + y_{a_i} \leq a_i.
\]

• a 0/1 solution with \( x_i = 1 \) has no further points in the solution which have no integral distance to \( x_i \).
• semi-general position

• for each \((t - 1)\)–hyperplane \(h_i\) build by the points \(\{y_1, \ldots, y_{r_i}\} \subseteq \{x_1, \ldots, x_{nm}\}\)

• add the inequality:

\[
y_1 + \ldots + y_{r_i} \leq t + 1
\]

• in a 0/1 solution there are at most \(t + 1\) points on each hyperplane \(h_i\)
Modelling

- general position, further inequalities
- for each \((t - 1)\)–hypersphere \(s_i\) build by the points \(\{y_1, \ldots, y_{t_i}\} \subset \{x_1, \ldots, x_{n^m}\}\)
- add the inequality:

\[
y_1 + \ldots + y_{t_i} \leq t + 2
\]
- in a 0/1 solution there are at most \(t + 2\) points on each hypersphere \(h_i\)
- $\mathbb{Z}_6$
- variables $x_1, \ldots, x_{36}$
- looking for a point-set with 4 points:

$$x_1 + \ldots + x_{36} = 4.$$
- integral distance

\[ 16x_{15} + x_1 + x_2 + x_4 + x_5 + x_7 + \ldots + x_{29} \leq 16 \]
• semigeneral = no 3 on a line

\[ x_5 + x_{10} + x_{15} + x_{20} + x_{25} + x_{36} \leq 2 \]
• general = no 4 on a circle

\[ x_6 + x_9 + x_{14} + x_{16} + x_{21} + x_{30} + x_{31} + x_{35} \leq 3 \]
• size of the problem
• one equation for the number of points
• for each point one inequality for the integral distance
• for each hyperplane one inequality for semi-general
• for each hypersphere one inequality for general
Reduce the Problem
Reduce the Problem

\[ \mathbb{Z}_6^2 \]

- 1 equation for the number of points
- 36 point inequalities for integral distance
- 42 line inequalities
- \(18 \times 4 = 72\) circle inequalities
Reduce the Problem

• no longer searching for an arbitrary solution
• decompose the set of all points in disjoint subsets
• search for solutions built from subsets
• = adding up columns in the system of equations
• still too many rows
Reduce the Problem

- prescribing automorphisms of $\mathbb{Z}_n^m$
  - $\phi_1(x, y) = (x + 2, y)$ and $\phi_2(x, y) = (x, y + 2)$
Reduce the Problem

• prescribing automorphisms of $\mathbb{Z}_n^m$
• columns are the orbits of the automorphisms
• but an automorphism $\phi$ is incidence - preserving
• point $p$ in hyperplane $h \iff \phi(p) \in \phi(h)$
• point $p$ in hypersphere $s \iff \phi(p) \in \phi(s)$
• $x$ and $y$ in integral distance $\iff \phi(x)$ and $\phi(y)$ are in integral distance
Reduce the Problem

- rows corresponding to points/hyperplanes/hyperspheres in the same orbit are identical
- automorphisms also reduce the number of rows
- size of the system of equations is now the number of orbits
Last Words
Other Problems

- other rings
- other properties
- other methods for reduction
Thank you very much for your attention.