THE ALGEBRA OF FLOWS IN GRAPHS

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ABSTRACT. We define a contravariant functor $K^{\cdot}$ from the category of finite graphs and graph morphisms to the category of finitely generated graded abelian groups and homomorphisms. For a graph X, an abelian group B, and a nonnegative integer j, an element of $\mathrm{Hom}(K^j(X), B)$ is a coherent family of B-valued flows on the set of all graphs obtained by contracting some $(j-1)$-set of edges of X; in particular, $\mathrm{Hom}(K^1(X), \mathbb{R})$ is the familiar (real) "cycle-space" of X. We show that $K^{\cdot}(X)$ is torsion-free and that its Poincaré polynomial is the specialization $t^{\#V(X)-1}T_X(1/t, 1+t)$ of the Tutte polynomial of X. Functoriality of $K^{\cdot}$ induces a functorial coalgebra structure on $K^{\cdot}(X)$; dualizing, for any ring B we obtain a functorial B-algebra structure on $\mathrm{Hom}(K^{\cdot}(X), B)$. When B is commutative we present this algebra as a quotient of a divided power algebra, leading to some interesting inequalities on the coefficients of the above Poincaré polynomial. We also provide a formula for the theta function of the lattice of integer-valued flows in X, and conclude with ten open problems.

0. INTRODUCTION.

Two ideas for defining algebraic invariants of graphs have been particularly successful up to now: these are the spectral and the K-theoretic approaches. The spectral theory begins by associating a (usually Hermitian) matrix M with a graph X, and proceeds by relating combinatorial structure in X to the spectral decomposition of M [3, 7, 11, 14]. The K-theoretic approach is the theory of the Tutte polynomial and its many interesting specializations [3, 8, 19, 20, 23]. A third method which seems promising but has been relatively neglected is the categorical approach. This idea is to define a functor from the category of graphs and graph morphisms into some (algebraic, geometric, topological,...) category, and to use these other structures to analyze the category of graphs. The most notable example of this approach so far is Walker's functorial setting [24] for Lovász's proof [15] of the Kneser conjecture; see also Björner's survey [5].

Here we define such a functor $K^{\cdot}$ into the category of finitely generated graded abelian groups and homomorphisms. The definition of $K^{\cdot}(X)$ is a formalization of Kirchhoff's Laws on X and all of its images under repeated contractions of edges; for this reason we refer to $K^{\cdot}(X)$ as the "Kirchhoff group" of X. Imagine that the edges of X represent pipes, all of the same cross-sectional area, which are full of water (an incompressible fluid) and connected at the vertices. A flow of water in this system is represented by assigning a (real) velocity parallel to each edge; Kirchhoff's Laws are the linear equations on these velocities which require that

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mass be conserved at each vertex. The Kirchhoff group is constructed so that the vector space of all real-valued flows in X turns out to be $\mathrm{Hom}(K^1(X), \mathbb{R})$. For any abelian group B and any nonnegative integer j, the elements of $\mathrm{Hom}(K^j(X), B)$ can be interpreted as coherent families of B-valued flows on the set of images of X after contracting each $(j-1)$-set of edges, as explained in Section 1. The number $d_j(X) := \mathrm{rank}\, K^j(X)$ is thus the dimension of the vector space of such coherent families of real-valued flows.

The Kirchhoff group turns out to be a remarkable invariant of graphs, and its algebraic properties impose several inequalities on the numbers $d_j(X)$. This is quite interesting since, as we see in Theorem 3.1, if X is a graph with n vertices and m edges then

$$d_0 + d_1 t + \cdots + d_m t^m = t^{n-1} T_X\left(\frac{1}{t}, 1+t\right), \tag{0.1}$$

in which $T_X(x,y)$ is the Tutte polynomial of X. To be specific about these inequalities, for positive integers a and j there is a unique expression

$$a = \binom{a_j}{j} + \binom{a_{j-1}}{j-1} + \cdots + \binom{a_i}{i}$$

such that $a_j > a_{j-1} > \cdots > a_i \geq i > 0$. The *$j$-th pseudopower of a* is

$$\psi_j(a) := \binom{a_j+1}{j+1} + \binom{a_{j-1}+1}{j} + \cdots + \binom{a_i+1}{i+1},$$

and we also put $\psi_j(0) := 0$. In Corollary 4.6 we show that if X has k connected components and ℓ cut-edges then

$$\begin{array}{l} d_0 = 1,\ d_1 = m-n+k, \text{ and } d_{m-\ell} = 1, \\ d_j \neq 0 \text{ if and only if } 0 \leq j \leq m-\ell, \\ \text{and if } 1 \leq j \leq m-\ell-1 \text{ then } 0 < d_{j+1} \leq \psi_j(d_j). \end{array} \tag{0.2}$$

In Corollary 4.7 we show that if Y is a maximal forest in X and the fundamental cycles of X with respect to Y have lengths $r_1, ..., r_d$, then for each $0 \leq j \leq m-\ell$,

$$d_j \leq [t^j] \prod_{i=1}^{d} \left(1 + t + t^2 + \cdots + t^{r_i}\right), \tag{0.3}$$

with the notation on the right side indicating the coefficient of t^j in the polynomial shown. In Corollary 4.9 we show that

$$d_j = \binom{d_1+j-1}{j} \text{ for all } 0 \leq j \leq g(X), \tag{0.4}$$

in which $g(X)$ denotes the girth of X. In Corollary 4.11 we show that

$$\begin{array}{l} d_0 \leq d_1 \leq \cdots \leq d_{\lfloor (m-\ell)/2 \rfloor}, \\ \text{and if } 0 \leq j \leq (m-\ell)/2 \text{ then } d_j \leq d_{m-\ell-j}. \end{array} \tag{0.5}$$

All of these statements are just coarse numerical consequences of the algebraic structure of Kirchhoff groups, the description of which is the main purpose of this paper.

In Section 1 we formalize the definitions we require from graph theory and define the Kirchhoff group $K^{\cdot}(X)$. This is somewhat involved, in that we must orient each edge of X arbitrarily in order to write down Kirchhoff's Laws and then, to obtain a group which is independent of the orientation, symmetrize the construction over all orientations. As a warm-up exercise we show in Theorem 1.2 that if X and Y have at most one vertex in common then $K^\cdot(X \cup Y) = K^\cdot(X) \otimes K^\cdot(Y)$. Theorem 1.3 is a technical but essential fact.

In Theorem 2.1 and the following remarks we show that the Kirchhoff group is a contravariant functor. By Proposition 2.2, every graph morphism $f : Y \to X$ can be factored as a composition of morphisms each of which has one of four very simple forms. Propositions 2.3 to 2.6 describe the structure of the group homomorphism $f^* : K^\cdot(X) \to K^\cdot(Y)$ in each of these cases.

Theorem 3.1 uses the foregoing to show that Kirchhoff groups are torsion-free, and relates them to the Tutte polynomial as in (0.1). Since $K^\cdot(X)$ is torsion-free, the ranks $d_j(X)$ determine $K^\cdot(X)$ up to isomorphism. However, functoriality of $K^\cdot$ induces a functorial coalgebra structure on $K^\cdot(X)$, and the situation becomes much more interesting.

For any ring B, the coalgebra structure on $K^\cdot(X)$ induces a B-algebra structure on $\mathrm{Hom}(K^\cdot(X), B)$, and this is functorial. We use the notation $\Phi.(X, B)$ for $\mathrm{Hom}(K^\cdot(X), B)$, and refer to it as the "B-circulation algebra of X". For commutative B, Theorem 4.4 gives a spanning set for $\Phi.(X, B)$ as a B-module; if B is also a $\mathbb{Q}$-algebra then Corollary 4.5 identifies a subset of $\Phi_1(X, B)$ which generates $\Phi.(X, B)$ as a B-algebra. This implies most of (0.2) above (some parts require separate arguments). The inequalities (0.3) also follow, as does a sharpening of (0.2) by applying the Clements-Lindström Theorem. For a commutative ring B, Theorem 4.8 presents $\Phi.(X, B)$ as a quotient of a divided power algebra, and the equalities (0.4) follow from the form of this presentation. Theorem 4.10 describes a situation in which there are monomorphisms $\Phi_j(X, B) \to \Phi_{m-\ell-j}(X, B)$ for $0 \le j \le (m - \ell)/2$ which, if B is a commutative $\mathbb{Q}$-algebra, factor through all the intermediate graded components of $\Phi.(X, B)$. This implies the inequalities (0.5) above.

There is a natural Euclidean bilinear form $\langle \cdot, \cdot \rangle$ on $\Phi.(X, \mathbb{R})$, and it is integer-valued when restricted to $\Phi.(X, \mathbb{Z})$. The theta function of the lattice $\Phi_j(X, \mathbb{Z})$ is the generating function for coherent families φ of integer-valued flows on the set of $(j-1)$-fold contractions of X with respect to the squared norm $\langle \varphi, \varphi \rangle$. Theorem 5.4 gives a formula for this theta function in the simplest nontrivial case, for $\Phi_1(X, \mathbb{Z})$.

In Section 6 we conclude with ten open problems of varying feasibility and interest.

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1. The Kirchhoff group of a graph.

The Kirchhoff group of a graph is a universal test object. Informally, it is a collection of ammeters, one for each edge of each contraction of X. The edges of the contractions of X are identified with one another in a natural way, so the readings on these ammeters must agree. Kirchhoff's Laws specify further linear equations which must be satisfied by the readings on these ammeters if they are measuring a collection of flows in the contractions of X; these are the relations which are built into the definition.

Let $X = (V, A, o, t, \bar{\cdot})$ be a general finite undirected graph, by which we mean a finite set V of *vertices*, a finite set A of *arcs* (directed edges), an *origin function* $o : A \to V$, a *terminus function* $t : A \to V$, and an *arc reversal function* $\bar{\cdot} : A \to A$, subject to the conditions that for all $a \in A$, $\overline{a} \neq a$ and $\overline{\overline{a}} = a$ and $o(\overline{a}) = t(a)$. It follows that for all $a \in A$, $t(\overline{a}) = o(a)$. The set E of *edges* of X is the set of orbits of $\bar{\cdot}$ acting on A. For $a \in A$ let $\underline{a} := \{a, \overline{a}\}$ be the corresponding edge. For $v \in V$ let $X(v) := \{a \in A : \ t(a) = v\}$. An *orientation* of X is a function $\omega : E \to A$ such that $\omega(e) \in e$ for each $e \in E$; the set of all orientations of X is denoted by Ω. For $\omega \in \Omega$ and $a \in A$, the *sign of a relative to ω* is

$$s_\omega(a) := \begin{cases} \ \ 1 & \text{if } \omega(\underline{a}) = a, \\ -1 & \text{if } \omega(\underline{a}) = \overline{a}. \end{cases}$$

For $e \in E$, the *contraction of e in X* is the graph $X_e = (V', A', o', t', \tilde{\cdot})$ defined as follows. Choose $a \in e$ (the choice does not matter). Let $V' := (V \smallsetminus \{o(a), t(a)\}) \cup \{e\}$, and define $\pi : V \to V'$ by $\pi(v) := v$ if $v \notin \{o(a), t(a)\}$, and $\pi(v) := e$ otherwise. Let $A' := A \smallsetminus \{a, \overline{a}\}$, let $o' := \pi \circ o|_{A'}$ and $t' := \pi \circ t|_{A'}$, and let $\tilde{\cdot} := \bar{\cdot}|_{A'}$. Any $\omega \in \Omega$ induces an orientation on X_e (by restriction); somewhat improperly we shall also denote this orientation of X_e by ω. For distinct edges $c, e \in E$ one sees that $(X_e)_c$ is naturally isomorphic with $(X_c)_e$, and it follows that for every subset $\sigma \subseteq E$, the contracted graph X_σ is well-defined. (It may be helpful to think of a graph X as a finite CW-complex of dimension at most one, in which case X_σ is the quotient complex of X modulo the subcomplex Z with vertices V and edges σ; in particular, $V(X_\sigma)$ is the set of connected components of this subgraph Z of X.)

The set of all j-element subsets of E is denoted by $\mathcal{P}_j(X)$, and $\mathcal{P}(X) := \bigcup_j \mathcal{P}_j(X)$. Ordered by inclusion, $\mathcal{P}(X)$ is of course a Boolean lattice. For each integer j, let $F^j(X)$ be the free abelian group with $\{X_\sigma : \ \sigma \in \mathcal{P}_j(X)\}$ as a basis:

$$F^j(X) := \mathbb{Z}\{X_\sigma : \ \sigma \in \mathcal{P}_j(X)\}.$$

Then $F^{\cdot}(X) := \bigoplus_j F^j(X)$ is a graded finitely generated free abelian group. Homomorphisms between graded groups are required to preserve the grading. For a graded abelian group $M^{\cdot}$ and integer r, $M^{\cdot}(r)$ denotes $M^{\cdot}$ with the grading shifted by r: the j-th graded component of $M^{\cdot}(r)$ is the $(j+r)$-th graded component of $M^{\cdot}$. Frequently, we define a homomorphism on a free module over a ring B by defining its values on a basis only; it is to be understood that the definition is implicitly extended B-linearly to the whole module. (The case $B = \mathbb{Z}$ is that of free abelian groups.)

For each $\sigma \in \mathcal{P}_j(X)$ there is a natural inclusion

$$\hat{\eta}_{X,\sigma} : \ F^{\cdot}(X_\sigma)(-j) \hookrightarrow F^{\cdot}(X) \tag{1.1}$$

induced by the identifications $(X_\sigma)_\tau = X_{\sigma\cup\tau}$ for all $\tau \in \mathcal{P}(X_\sigma)$. If $\sigma, \rho \in \mathcal{P}(X)$ are disjoint then clearly

$$\hat{\eta}_{X,\sigma} \circ \hat{\eta}_{X_\sigma,\rho} = \hat{\eta}_{X,\sigma\cup\rho}. \tag{1.2}$$

Fix an orientation $\omega \in \Omega$. For each vertex $v \in V$ define a relation

$$R_\omega(X, v) := \sum_{a\in X(v)} s_\omega(a) X_{\underline{a}}$$

in $F^\cdot(X)$, and let $\mathcal{R}_\omega(X) := \{R_\omega(X,v) : v \in V(X)\}$. These are Kirchhoff's Laws on X. For each integer j define a subgroup of $F^j(X)$ by

$$N^j_\omega(X) := \mathrm{span}_{\mathbb{Z}} \bigcup_{\sigma \in \mathcal{P}_{j-1}(X)} \hat{\eta}_{X,\sigma}(\mathcal{R}_\omega(X_\sigma)).$$

Then $N^\cdot_\omega(X) := \bigoplus_j N^j_\omega(X)$ is a graded subgroup of $F^\cdot(X)$. The *Kirchhoff group of X relative to ω* is the quotient

$$K^\cdot_\omega(X) \quad := \quad F^\cdot(X)/N^\cdot_\omega(X). \tag{1.3}$$

To define a group $K^\cdot(X)$ which depends naturally on X, and not on any particular orientation of X, we symmetrize the above construction using automorphisms $\psi^\omega_\varepsilon :\ F^\cdot(X) \to F^\cdot(X)$ for all $\omega, \varepsilon \in \Omega$, defined as follows. For each $\sigma \in \mathcal{P}(X)$, let $m_\sigma(\omega,\varepsilon) := \#\{e \in \sigma : \omega(e) \neq \varepsilon(e)\}$, and define $\psi^\omega_\varepsilon(X_\sigma) := (-1)^{m_\sigma(\omega,\varepsilon)} X_\sigma$. Let $\Psi := \{\psi^\omega_\varepsilon : \omega,\varepsilon \in \Omega\}$. (In fact, Ψ is an elementary abelian group of order $2^{\#E}$, but we do not use this fact.) One easily sees that for all $\omega \in \Omega$, $\psi^\omega_\omega = 1$, and that for all $\omega,\varepsilon,\nu \in \Omega$, $\psi^\varepsilon_\nu \circ \psi^\omega_\varepsilon = \psi^\omega_\nu$. Let $F^\cdot_\Omega(X)$ be the subgroup of $\bigoplus_{\omega\in\Omega} F^\cdot(X)$ consisting of all elements $(T_\omega : \omega \in \Omega)$ such that $T_\varepsilon = \psi^\omega_\varepsilon(T_\omega)$ for all $\omega,\varepsilon \in \Omega$. For each $\omega \in \Omega$ there is an isomorphism

$$\hat{\kappa}_\omega :\ F^\cdot(X) \to F^\cdot_\Omega(X) \tag{1.4}$$

defined by $\hat{\kappa}_\omega(T) := (\psi^\omega_\varepsilon(T) : \varepsilon \in \Omega)$ for all $T \in F^\cdot(X)$. These are compatible with Ψ in the sense that for any $\omega,\varepsilon \in \Omega$, $\hat{\kappa}_\omega = \hat{\kappa}_\varepsilon \circ \psi^\omega_\varepsilon$. For any $\omega,\varepsilon \in \Omega$, any $\sigma \in \mathcal{P}(X)$, and any $v \in V(X_\sigma)$, one sees that $\psi^\omega_\varepsilon(R_\omega(X_\sigma,v)) = (-1)^{m_\sigma(\omega,\varepsilon)} R_\varepsilon(X_\sigma,v)$. From this it follows that for any $\omega,\varepsilon \in \Omega$, $\psi^\omega_\varepsilon(N^\cdot_\omega(X)) = N^\cdot_\varepsilon(X)$. Therefore, the subgroup $N^\cdot_\Omega(X) := \hat{\kappa}_\omega(N^\cdot_\omega(X))$ of $F^\cdot_\Omega(X)$ is independent of the choice of $\omega \in \Omega$. Finally, we define *the Kirchhoff group of X* to be the quotient

$$K^\cdot(X) := F^\cdot_\Omega(X)/N^\cdot_\Omega(X).$$

From (1.4) we obtain a family of isomorphisms

$$\kappa_\omega :\ K^\cdot_\omega(X) \to K^\cdot(X) \tag{1.5}$$

for $\omega \in \Omega$ which are compatible with Ψ. If $\mu_\omega : K^\cdot_\omega(X) \to M^\cdot$ is a family of homomorphisms indexed by Ω and compatible with Ψ, then the homomorphism $\mu := \mu_\omega \circ \kappa_\omega^{-1}$ is independent of the choice of $\omega \in \Omega$. By virtue of this, in any particular argument one is free to choose $\omega \in \Omega$ arbitrarily, and to work with $K^\cdot_\omega(X)$ instead of $K^\cdot(X)$, as long as one checks that the construction is compatible with Ψ.

A word of explanation is in order as to what the Kirchhoff group really means. For each $\sigma \in \mathcal{P}(X)$ and $e \notin \sigma$, think of $X_{\sigma\cup\{e\}}$ as representing the edge e in X_σ. Each X_τ thus represents one edge in each of $\#\tau$ different contractions of X. Fixing a reference orientation $\omega \in \Omega$, the set $\mathcal{R}_\omega(X_\sigma)$ is a formulation of Kirchhoff's Laws on X_σ. Thus, for any integer j and abelian group B, an element of $\mathrm{Hom}(K^j(X),B)$ is a coherent family of B-valued flows on $\{X_\sigma : \sigma \in \mathcal{P}_{j-1}(X)\}$. This $\mathrm{Hom}(K^j(X),B)$ is naturally an abelian group, and moreover if B is a ring then it is naturally a B-module. In particular, $\mathrm{Hom}(K^1(X),\mathbb{R})$ is the vector space of real-valued flows on X; that is, the (real) *cycle space* of X.

The following very simple lemma is a key ingredient of many proofs in this paper. For a proposition P we define the *truth value of P* to be $\mathtt{tv}[P] := 1$ if P is true and $\mathtt{tv}[P] := 0$ if P is false.

Lemma 1.1. *Let X be a graph. For any $\omega \in \Omega$ and $U \subseteq V$,*

$$\sum_{v \in U} R_\omega(X, v) = \sum_{a \in X(U)} s_\omega(a) X_{\underline{a}},$$

in which $X(U) := \{a \in A : \ t(a) \in U \text{ and } o(a) \notin U\}$.

Proof. For each $e \in E$, X_e appears on the left side with coefficient $\mathtt{tv}[t(\omega(e)) \in U] - \mathtt{tv}[o(\omega(e)) \in U]$, from which the result follows. □

For any graph X, the assignment $X_\varnothing \mapsto 1$ defines a natural isomorphism $F^0(X) = \mathbb{Z}$. Since $X_\varnothing$ is fixed by every element of Ψ, its image $u_X \in K^0(X)$ is well-defined. This gives a natural isomorphism $K^0(X) = \mathbb{Z}$ by sending u_X to 1. If X and Y are graphs with disjoint edge-sets, then the bilinear map $F^\cdot(X) \times F^\cdot(Y) \to F^\cdot(X \cup Y)$ defined by $(X_\sigma, Y_\tau) \mapsto (X \cup Y)_{\sigma \cup \tau}$ for all $\sigma \in \mathcal{P}(X)$ and $\tau \in \mathcal{P}(Y)$ induces a natural isomorphism $F^\cdot(X) \otimes F^\cdot(Y) = F^\cdot(X \cup Y)$. (Tensor products are over $\mathbb{Z}$ unless otherwise noted.)

Theorem 1.2. *Let X and Y be graphs with disjoint edge-sets, and with at most one vertex in common. Then $K^\cdot(X \cup Y) = K^\cdot(X) \otimes K^\cdot(Y)$.*

Proof. Fix reference orientations $\omega \in \Omega(X)$ and $\gamma \in \Omega(Y)$; these determine an orientation $\nu \in \Omega(X \cup Y)$. Consider a relation $R_\omega(X_\sigma, v)$ (so $\sigma \in \mathcal{P}(X)$ and $v \in V(X_\sigma)$) for which v is not the image in $V(X_\sigma)$ of any vertex in $V(Y)$, and let $\tau \in \mathcal{P}(Y)$. Then

$$\begin{aligned} R_\omega(X_\sigma, v) \otimes Y_\tau &= \sum_{a \in X_\sigma(v)} s_\omega(a)(X \cup Y)_{\sigma \cup \{\underline{a}\} \cup \tau} \\ &= \sum_{a \in (X \cup Y)_{\sigma \cup \tau}(v)} s_\omega(a)(X \cup Y)_{\sigma \cup \tau \cup \{\underline{a}\}} \\ &= R_\nu((X \cup Y)_{\sigma \cup \tau}, v), \end{aligned}$$

which is in $N^\cdot_\nu(X \cup Y)$. But there is at most one vertex $w \in V(X_\sigma)$ which is the image of some vertex of $V(Y)$. By Lemma 1.1, $\sum_{v \in V(X_\sigma)} R_\omega(X_\sigma, v) = 0$, and so if such a w exists then $R_\omega(X_\sigma, w) \otimes Y_\tau$ is also in $N^\cdot_\nu(X \cup Y)$. By the symmetry exchanging X and Y we conclude that $N^\cdot_\omega(X) \otimes F^\cdot(Y) + F^\cdot(X) \otimes N^\cdot_\gamma(Y) \subseteq N^\cdot_\nu(X \cup Y)$, and so we have an epimorphism

$$\phi : K^\cdot_\omega(X) \otimes K^\cdot_\gamma(Y) \to K^\cdot_\nu(X \cup Y).$$

Now consider any $\rho \in \mathcal{P}(X \cup Y)$, and let $\sigma := \rho \cap E(X)$ and $\tau := \rho \cap E(Y)$. For v a vertex of $(X \cup Y)_\rho$ which is not in $V(X_\sigma) \cap V(Y_\tau)$ we may assume that $v \in V(X_\sigma)$, by symmetry. Then $R_\omega(X_\sigma, v) \otimes Y_\tau = R_\nu((X \cup Y)_\rho, v)$. But there is at most one vertex w in $V(X_\sigma) \cap V(Y_\tau)$, and by Lemma 1.1, $\sum_{w \in V((X \cup Y)_\rho)} R_\nu((X \cup Y)_\rho, v) = 0$. This shows that $N^\cdot_\nu(X \cup Y) \subseteq N^\cdot_\omega(X) \otimes F^\cdot(Y) + F^\cdot(X) \otimes N^\cdot_\gamma(Y)$, and so ϕ is an isomorphism. To complete the proof one must check that the construction of ϕ is compatible with $\Psi(X)$ and $\Psi(Y)$ and the isomorphisms $\kappa_\nu : K^\cdot_\nu(X \cup Y) \to K^\cdot(X \cup Y)$, but this is straightforward. □

Now fix an $\omega \in \Omega$, and consider one of the relations $R_\omega((X_\sigma)_\tau, v)$ of $N^\cdot_\omega(X_\sigma)$, so $\tau \in \mathcal{P}(X_\sigma)$ and $v \in V((X_\sigma)_\tau)$. Notice that

$$\hat{\eta}_{X,\sigma}(R_\omega((X_\sigma)_\tau, v)) = \sum_{a \in (X_\sigma)_\tau(v)} s_\omega(a) X_{\sigma \cup \tau \cup \{\underline{a}\}} = R_\omega(X_{\sigma \cup \tau}, v).$$

From this it follows that for all $\omega \in \Omega$ and $\sigma \in \mathcal{P}_j(X)$, $\hat{\eta}_{X,\sigma}(N^{\cdot}_{\omega}(X_\sigma)(-j)) \subseteq N^{\cdot}_{\omega}(X)$, and so (1.1) induces a family of homomorphisms

$$\eta^{\omega}_{X,\sigma} : K^{\cdot}_{\omega}(X_\sigma)(-j) \to K^{\cdot}_{\omega}(X) \tag{1.6}$$

which are compatible with Ψ and the maps κ_ω. Hence, these induce a natural homomorphism

$$\eta_{X,\sigma} : K^{\cdot}(X_\sigma)(-j) \to K^{\cdot}(X).$$

Theorem 1.3 has important consequences for the structure of Kirchhoff groups and circulation algebras, as we see in Sections 3 and 4.

Theorem 1.3. *Let X be a graph and let $\sigma \in \mathcal{P}_j(X)$.*
(a) *If σ contains a cut-edge of X then $\eta_{X,\sigma} = 0$.*
(b) *If σ does not contain any cut-edges of X then there is a homomorphism $\pi : K^{\cdot}(X) \to K^{\cdot}(X_\sigma)(-j)$ such that $\pi \circ \eta_{X,\sigma} = 1$. In particular, $\eta_{X,\sigma}$ is injective.*

Proof. Fix an orientation $\omega \in \Omega$. Identify $F^{\cdot}(X_\sigma)(-j)$ with its image under $\hat{\eta}_{X,\sigma}$. From (1.1), (1.6), and the kernel-cokernel exact sequence (see, *e.g.*, Lemma 1.10.8 of Borceux [6]) , we see that

$$0 \longrightarrow \ker(\eta^{\omega}_{X,\sigma}) \longrightarrow \frac{N^{\cdot}_{\omega}(X)}{N^{\cdot}_{\omega}(X_\sigma)(-j)} \xrightarrow{\phi} \frac{F^{\cdot}(X)}{F^{\cdot}(X_\sigma)(-j)} \longrightarrow \operatorname{coker}(\eta^{\omega}_{X,\sigma}) \longrightarrow 0$$

is exact, and ϕ is induced by the inclusion $N^{\cdot}_{\omega}(X) \hookrightarrow F^{\cdot}(X)$. Hence,

$$\ker(\eta^{\omega}_{X,\sigma}) \simeq \ker(\phi) = \frac{N^{\cdot}_{\omega}(X) \cap F^{\cdot}(X_\sigma)(-j)}{N^{\cdot}_{\omega}(X_\sigma)(-j)}. \tag{1.7}$$

For part (a), assume that $e \in \sigma$ is a cut-edge of X, and let $\rho := \sigma \smallsetminus \{e\}$. Then e is a cut-edge of X_τ for every $e \notin \tau \in \mathcal{P}(X)$. To prove that $\eta^{\omega}_{X,\sigma} = 0$, it suffices to show that $F^{\cdot}(X_\sigma)(-j) \subseteq N^{\cdot}_{\omega}(X)$. Consider any $\tau \in \mathcal{P}(X_\sigma)$, and the corresponding element $X_{\sigma\cup\tau}$ of $F^{\cdot}(X_\sigma)(-j)$. Now e is a cut-edge of $X_{\rho\cup\tau}$. Let U be the set of vertices of $X_{\rho\cup\tau}$ in the same connected component of $X_{\rho\cup\tau} \smallsetminus \{e\}$ as $t(\omega(e))$. By Lemma 1.1 we have $\sum_{v\in U} R_\omega(X_{\rho\cup\tau}, v) = X_{\sigma\cup\tau} \in N^{\cdot}_{\omega}(X)$. Since $F^{\cdot}(X_\sigma)(-j)$ is generated by elements of this form, $\eta^{\omega}_{X,\sigma} = 0$. The isomorphism κ_ω shows that $\eta_{X,\sigma} = 0$.

It suffices to prove part (b) in the special case $j = 1$, because of (1.2). Accordingly, $\sigma = \{e\}$ and e is not a cut-edge of X. Let $C \subseteq X$ be a cycle in X which contains e, and let $\zeta := E(C)$. We may choose $\omega \in \Omega$ so that for each $v \in V(C)$ there is exactly one $c \in \zeta$ such that $v = t(\omega(c))$. Define $\hat{\pi} : F^{\cdot}(X) \to F^{\cdot}(X_e)(-1)$ by

$$\hat{\pi}(X_\tau) := \begin{cases} X_\tau & \text{if } e \in \tau, \\ \sum_{c\in\tau\cap\zeta} X_{\tau\cup\{e\}\smallsetminus\{c\}} & \text{if } e \notin \tau, \end{cases}$$

for all $\tau \in \mathcal{P}(X)$. It is clear that $\hat{\pi} : F^{\cdot}(X) \to F^{\cdot}(X_e)(-1)$ and that $\hat{\pi} \circ \hat{\eta}_{X,e} = 1$. To finish the proof it remains to show that $\hat{\pi}(N^{\cdot}_{\omega}(X)) \subseteq N^{\cdot}_{\omega}(X_e)(-1)$, since then there is an induced homomorphism $\pi_\omega : K^{\cdot}_{\omega}(X) \to K^{\cdot}_{\omega}(X_e)(-1)$. Having done this, it is straightforward to check compatibility with Ψ to obtain π as claimed.

Consider a relation $R_\omega(X_\tau, v)$ in $N_\omega^{\cdot}(X)$, so $\tau \in \mathcal{P}(X)$ and $v \in V(X_\tau)$. If $e \in \tau$ then $\hat{\pi}(R_\omega(X_\tau, v)) = R_\omega(X_\tau, v)$, as is clear. If $e \notin \tau$ then

$$\begin{aligned}\hat{\pi}(R_\omega(X_\tau, v)) &= (\mathtt{tv}[t(e)=v] - \mathtt{tv}[o(e)=v])\, X_{\tau\cup\{e\}} \\ &\quad + \sum_{a\in X_\tau(v)\smallsetminus e} s_\omega(a) \sum_{c\in(\tau\cup\{\underline{a}\})\cap\zeta} X_{\tau\cup\{\underline{a}\}\cup\{e\}\smallsetminus\{c\}} \\ &= \sum_{c\in\tau\cap\zeta}\sum_{a\in X_\tau(v)} s_\omega(a) X_{\tau\cup\{e\}\cup\{\underline{a}\}\smallsetminus\{c\}} + \sum_{a\in X_\tau(v):\ \underline{a}\in\zeta} s_\omega(a) X_{\tau\cup\{e\}}.\end{aligned}$$

The second sum is zero, since it is either empty (if v is not a vertex of the image of C in X_τ) or it has an even number of terms, and these cancel in pairs because the orientation ω is directed consistently around C. For $c \in \tau\cap\zeta$, let $U(c)$ denote the set of vertices of $X_{\tau\cup\{e\}\smallsetminus\{c\}}$ which are contracted to the image of v in $X_{\tau\cup\{e\}}$; this $U(c)$ has either one or two elements. By Lemma 1.1,

$$\sum_{a\in X_\tau(v)} s_\omega(a) X_{\tau\cup\{e\}\cup\{\underline{a}\}\smallsetminus\{c\}} = \sum_{u\in U(c)} R_\omega(X_{\tau\cup\{e\}\smallsetminus\{c\}}, u),$$

and we conclude that $\hat{\pi}(R_\omega(X_\tau, v)) \in N_\omega^{\cdot}(X)$, as required. □

Notice that the homomorphism π constructed in part (b) is not natural, as it depends on arbitrary choices of cycles in X.

2. Functorial properties of Kirchhoff groups.

A *graph morphism* $f : Y \to X$ is a pair of functions $f_V : V(Y) \to V(X)$ and $f_A : A(Y) \to A(X)$ which are compatible with the functions o, t, $\bar{\cdot}$ defining Y and X. These induce a corresponding function $f_E : E(Y) \to E(X)$ as well. We construct, for each graph morphism $f : Y \to X$, an induced homomorphism $f^* : K^{\cdot}(X) \to K^{\cdot}(Y)$, in such a way that $K^{\cdot}$ becomes a contravariant functor from the category of finite graphs and graph morphisms to the category of finitely generated graded abelian groups and homomorphisms. Then we investigate the relationship between structural properties of f and algebraic properties of f^*.

For $\sigma \in \mathcal{P}(X)$, the set $T_f(\sigma)$ of *f-transversals of σ* is the set of all $\tau \in \mathcal{P}(Y)$ such that for each $e \in \sigma$ there is exactly one $c \in \tau$ for which $f_E(c) = e$, and for each $e \notin \sigma$ there is no $c \in \tau$ for which $f_E(c) = e$. In other words, $T_f(\sigma)$ may be naturally identified with $\prod\{f_E^{-1}(e) :\ e \in \sigma\}$. Define $f^\sharp : F^{\cdot}(X) \to F^{\cdot}(Y)$ by

$$f^\sharp(X_\sigma) := \sum_{\tau\in T_f(\sigma)} Y_\tau \tag{2.1}$$

for each $\sigma \in \mathcal{P}(X)$. Given any $\omega \in \Omega(X)$, the *pullback of ω along f* is the unique $\gamma \in \Omega(Y)$ making the following diagram commute:

$$\begin{array}{ccc} E(Y) & \xrightarrow{f_E} & E(X) \\ \gamma\downarrow & & \downarrow\omega \\ A(Y) & \xrightarrow{f_A} & A(X) \end{array}$$

Theorem 2.1. *Let $f : Y \to X$ be a graph morphism. Fix $\omega \in \Omega(X)$ and let γ be the pullback of ω along f. Then $f^\sharp(N_\omega^{\cdot}(X))$ is a subgroup of $N_\gamma^{\cdot}(Y)$.*

Proof. Consider one of the relations $R_\omega(X_\sigma, v)$ of $N^{\cdot}_\omega(X)$, so $\sigma \in \mathcal{P}(X)$ and $v \in V(X_\sigma)$. Let $U' \subseteq V(X)$ be the set of vertices of X which are contracted to v in X_σ, and let $U := f_V^{-1}(U') \subseteq V(Y)$ be the preimage of U' in Y. For each $\rho \in T_f(\sigma)$ let $U(\rho) \subseteq V(Y_\rho)$ be the set of vertices of Y_ρ to which some vertex in U is contracted. Let $\zeta := f_E^{-1}(\sigma)$ be the set of edges of Y mapped to some edge of σ. Notice that if $\rho \in T_f(\sigma)$ then $\rho \subseteq \zeta$. Now

$$\begin{aligned} f^\sharp(R_\omega(X_\sigma, v)) &= \sum_{a \in X_\sigma(v)} s_\omega(a) \sum_{\tau \in T_f(\sigma \cup \{\underline{a}\})} Y_\tau \\ &= \sum_{\rho \in T_f(\sigma)} \sum_{a \in X_\sigma(v)} s_\omega(a) \sum_{c \in f_E^{-1}(\underline{a})} Y_{\rho \cup \{c\}}. \end{aligned}$$

As a subset of $A(X)$, $X_\sigma(v)$ is the set of arcs a such that $\underline{a} \notin \sigma$ and $t(a) \in U'$. Thus, $f_A^{-1}(X_\sigma(v))$ is the set of arcs $a \in A(Y)$ such that $\underline{a} \notin \zeta$ and $t(a) \in U$. Fix any $\rho \in T_f(\sigma)$, and let $A(\rho)$ denote the image of $f_A^{-1}(X_\sigma(v))$ in Y_ρ. We have

$$\sum_{a \in X_\sigma(v)} s_\omega(a) \sum_{c \in f_E^{-1}(\underline{a})} Y_{\rho \cup \{c\}} = \sum_{a \in A(\rho)} s_\gamma(a) Y_{\rho \cup \{\underline{a}\}}.$$

But $A(\rho)$ is the set of arcs $a \in A(Y_\rho)$ such that $\underline{a} \notin \zeta$ and $t(a) \in U(\rho)$. If $\underline{a} \in \zeta$ then both $t(a) \in U(\rho)$ and $t(\overline{a}) \in U(\rho)$. Since $s_\gamma(a) = -s_\gamma(\overline{a})$, the inclusion of such arcs into the summation will cancel to zero. Thus, with $Y_\rho(U(\rho))$ denoting the set of arcs $a \in A(Y_\rho)$ such that $t(a) \in U(\rho)$ and $o(a) \notin U(\rho)$,

$$\begin{aligned} \sum_{a \in A(\rho)} s_\gamma(a) Y_{\rho \cup \{\underline{a}\}} &= \sum_{a \in Y_\rho(U(\rho))} s_\gamma(a) Y_{\rho \cup \{\underline{a}\}} \\ &= \sum_{u \in U(\rho)} R_\gamma(Y_\rho, u), \end{aligned}$$

by Lemma 1.1. Putting the pieces together, we have

$$f^\sharp(R_\omega(X_\sigma, v)) = \sum_{\rho \in T_f(\sigma)} \sum_{u \in U(\rho)} R_\gamma(Y_\rho, u),$$

showing that $f^\sharp$ takes an arbitrary generator of $N^{\cdot}_\omega(X)$ into $N^{\cdot}_\gamma(Y)$. □

For a graph morphism $f : Y \to X$ and orientation $\omega \in \Omega(X)$, let $\gamma \in \Omega(Y)$ be the pullback of ω along f. By Theorem 2.1 there is a homomorphism

$$f^\omega : K^{\cdot}_\omega(X) \to K^{\cdot}_\gamma(Y)$$

which is well-defined by putting $f^\omega(T + N^{\cdot}_\omega(X)) := f^\sharp(T) + N^{\cdot}_\gamma(Y)$ for all $T \in F^{\cdot}(X)$. These are compatible with $\Psi(X)$ and the maps κ_γ for $\gamma \in \Omega(Y)$, and thus they define a homomorphism

$$f^* : K^{\cdot}(X) \to K^{\cdot}(Y).$$

If $g : Z \to Y$ is another graph morphism then it is clear from the construction that $(f \circ g)^\sharp = g^\sharp \circ f^\sharp$, and it follows that $(f \circ g)^* = g^* \circ f^*$. This establishes that $K^{\cdot}$ is a contravariant functor, as claimed.

Proposition 2.2. *Every graph morphism $f : Y \to X$ can be factored as*

$$Y \xrightarrow{g} Z \xrightarrow{h} W \xrightarrow{i} X$$

such that: g_V is surjective and g_A is bijective; h_V is bijective and h_A is surjective; i_V and i_A are both injective.

Proof. Let W be the image $f(Y)$ of Y in X and let $i : W \to X$ be the inclusion of W as a subgraph of X. There is a graph morphism $p : Y \to W$ such that $f = i \circ p$, and both p_V and p_A are surjective. Let Z be the graph defined by $V(Z) := V(W)$, $A(Z) := A(Y)$, $o_Z := p_V \circ o_Y$, $t_Z := p_V \circ t_Y$, and with the same arc-reversal as Y. The morphism g is defined by $g_V := p_V$ and $g_A := \mathrm{id}_{A(Y)}$. The morphism h is defined by $h_V := \mathrm{id}_{V(W)}$ and $h_A := p_A$. □

(In fact, this factorization is unique up to isomorphism, but we make no use of this fact.)

An *elementary injection* is an injective function $f : U \to W$ between finite sets such that $\#W = 1 + \#U$. An *elementary surjection* is a surjective function $f : U \to W$ between finite sets such that $\#U = 1 + \#W$. Let $f : Y \to X$ be a graph morphism, factored $f = i \circ h \circ g$ as in Proposition 2.2. We may further factor g into a sequence of morphisms g' for each of which g'_V is an elementary surjection and g'_A is bijective. We may also factor h into a sequence of morphisms h' for each of which h'_V is bijective and h'_E is an elementary surjection. Finally, we may factor i into a sequence of morphisms i' for each of which exactly one of i'_V and i'_E is an elementary injection, and the other is a bijection. By functoriality of $K^{\cdot}$, it suffices (in principle) to describe f^* for graph morphisms f having one of these special forms.

Proposition 2.3. *Let $i : Y \to X$ be a graph morphism such that i_V is an elementary injection and i_A is bijective. Then i^* is an isomorphism.*

Proof. The graph X is the disjoint union of Y and a graph $\bullet$ with one vertex and no edges. By Theorem 1.2, $K^{\cdot}(X) = K^{\cdot}(Y) \otimes K^{\cdot}(\bullet) = K^{\cdot}(Y) \otimes \mathbb{Z} = K^{\cdot}(Y)$. This isomorphism is induced by i_E, so it is i^*. □

Proposition 2.4 is an algebraic analogue of the deletion/contraction algorithm of graph theory, and is a key component of several inductive proofs in what follows. In the following propositions it is convenient to use the notation $f^\flat$ for the restriction of $f^\sharp$ to $N^{\cdot}_\omega(X)$.

Proposition 2.4. *Let $i : Y \to X$ be a graph morphism such that i_V is bijective and i_E is an elementary injection. Let e be the unique edge of X not in $i_E(E(Y))$. Then*

$$K^{\cdot}(X_e)(-1) \xrightarrow{\eta_{X,e}} K^{\cdot}(X) \xrightarrow{i^*} K^{\cdot}(Y) \longrightarrow 0$$

is exact.

Proof. Identify $V(X)$ and $V(Y)$ via i_V, and consider $E(Y)$ as a subset of $E(X)$ via i_E. For all $\sigma \in \mathcal{P}(X)$, $i^\sharp(X_\sigma) := Y_\sigma$ if $e \notin \sigma$, and $i^\sharp(X_\sigma) := 0$ if $e \in \sigma$. Hence $i^\sharp : F^{\cdot}(X) \to F^{\cdot}(Y)$ is surjective, and $\ker(i^\sharp) = \mathrm{span}_{\mathbb{Z}}\{X_\sigma : e \in \sigma \in \mathcal{P}(X)\} = F^{\cdot}(X_e)(-1)$. Fix an $\omega \in \Omega(X)$, and denote by γ its pullback along i. Consider a generator $R_\gamma(Y_\sigma, v)$ of $N^{\cdot}_\gamma(Y)$, so $\sigma \in \mathcal{P}(Y)$ and $v \in V(Y_\sigma)$. Then

$$\begin{aligned} i^\sharp(R_\omega(X_\sigma, v)) &= \sum_{a \in X_\sigma(v)} s_\omega(a) i^\sharp(X_{\sigma \cup \{\underline{a}\}}) \\ &= \sum_{a \in Y_\sigma(v)} s_\gamma(a) Y_{\sigma \cup \{\underline{a}\}} = R_\gamma(Y_\sigma, v), \end{aligned}$$

which shows that $i^\flat : N^\cdot_\omega(X) \to N^\cdot_\gamma(Y)$ is surjective. From the kernel-cokernel exact sequence, we see that $i^\omega : K^\cdot_\omega(X) \to K^\cdot_\gamma(Y)$ is surjective and

$$0 \longrightarrow N^\cdot_\omega(X) \cap F^\cdot(X_e)(-1) \longrightarrow F^\cdot(X_e)(-1) \longrightarrow \ker(i^\omega) \longrightarrow 0$$

is exact. If e is a cut-edge then, as in Theorem 1.3(a), $F^\cdot(X_e)(-1) \subseteq N^\cdot_\omega(X)$, so $\ker(i^\omega) = 0$; since $\eta^\omega_{X,e} = 0$ from Theorem 1.3(a), this shows that

$$K^\cdot_\omega(X_e)(-1) \xrightarrow{\eta^\omega_{X,e}} K^\cdot_\omega(X) \xrightarrow{i^\omega} K^\cdot_\gamma(Y) \longrightarrow 0 \tag{2.2}$$

is exact in this case. If e is not a cut-edge then from Theorem 1.3(b), $\ker(\eta^\omega_{X,e}) = 0$, and so from (1.7) we find that $N^\cdot_\omega(X) \cap F^\cdot(X_e)(-1) = N^\cdot_\omega(X_e)(-1)$. Injectivity of $\eta^\omega_{X,e}$ implies that (2.2) is exact in this case as well. The result follows since everything is compatible with $\Psi(X)$ and the maps κ_γ for $\gamma \in \Omega(Y)$. □

Proposition 2.5. *Let $h : Y \to X$ be a graph morphism such that h_V is bijective and h_E is an elementary surjection. Let e be the unique edge of X such that $\#h_E^{-1}(e) = 2$. Then*

$$0 \longrightarrow K^\cdot(X) \xrightarrow{h^*} K^\cdot(Y) \longrightarrow K^\cdot(X_e)(-1) \oplus K^\cdot(X_e)(-2) \longrightarrow 0$$

is exact.

Proof. Identify $V(X)$ and $V(Y)$ via h_V, and consider $E(X)$ to be a subset of $E(Y)$ by identifying each edge of X with one member of its preimage under h_E. Let c be the unique edge of Y not in X, and for $\sigma \in \mathcal{P}(X)$ with $e \in \sigma$, let $\sigma' := \sigma \cup \{c\} \setminus \{e\}$. For all $\sigma \in \mathcal{P}(X)$, $h^\sharp(X_\sigma) := Y_\sigma$ if $e \notin \sigma$, and $h^\sharp(X_\sigma) := Y_\sigma + Y_{\sigma'}$ if $e \in \sigma$. Hence $h^\sharp$ is injective and

$$\begin{aligned} \operatorname{coker}(h^\sharp) &= \frac{\mathbb{Z}\{Y_\sigma, Y_{\sigma'} :\ e \in \sigma \in \mathcal{P}(X)\}}{\mathbb{Z}\{Y_\sigma + Y_{\sigma'} :\ e \in \sigma \in \mathcal{P}(X)\}} \oplus \mathbb{Z}\{Y_\rho : \{c, e\} \subseteq \rho \in \mathcal{P}(Y)\} \\ &\simeq F^\cdot(X_e)(-1) \oplus F^\cdot(X_e)(-2). \end{aligned}$$

Fix an $\omega \in \Omega(X)$, and denote by γ its pullback along h. Our next task is to determine the cokernel of $h^\flat : N^\cdot_\omega(X) \to N^\cdot_\gamma(Y)$. Consider any $\sigma \in \mathcal{P}(X)$ and $v \in V(X_\sigma)$; there are two cases. If $e \notin \sigma$ then $h^\flat(R_\omega(X_\sigma, v)) = R_\gamma(Y_\sigma, v)$. If $e \in \sigma$ then $h^\flat(R_\omega(X_\sigma, v)) = R_\gamma(Y_\sigma, v) + R_\gamma(Y_{\sigma'}, v)$. (In each case, the details of the calculation differ slightly depending on whether v is or is not incident with c in Y_σ.) Therefore

$$\begin{aligned} \operatorname{coker}(h^\flat) &= \frac{\operatorname{span}_{\mathbb{Z}}\{R_\gamma(Y_\sigma, v), :\ \sigma \in \mathcal{P}(Y) \text{ and } \{e, c\} \cap \sigma \not\emptyset\}}{\operatorname{span}_{\mathbb{Z}}\{R_\gamma(Y_\sigma, v) + R_\gamma(Y_{\sigma'}, v) :\ e \in \sigma \in \mathcal{P}(Y) \text{ and } c \notin \sigma\}} \\ &\simeq N^\cdot_\omega(X_e)(-1) \oplus N^\cdot_\omega(X_e)(-2), \end{aligned}$$

an isomorphism compatible with the one above for $\operatorname{coker}(h^\sharp)$. From the kernel-cokernel exact sequence we see that

$$0 \longrightarrow \ker(h^\omega) \longrightarrow \operatorname{coker}(h^\flat) \xrightarrow{\phi} \operatorname{coker}(h^\sharp) \longrightarrow \operatorname{coker}(h^\omega) \longrightarrow 0$$

is exact. One checks that ϕ is injective, and it follows that

$$0 \longrightarrow K^\cdot_\omega(X) \xrightarrow{h^\omega} K^\cdot_\gamma(Y) \longrightarrow K^\cdot_\omega(X_e)(-1) \oplus K^\cdot_\omega(X_e)(-2) \longrightarrow 0$$

is exact. The result follows for the usual reasons of compatibility. □

Proposition 2.6. *Let $g : Y \to X$ be a graph morphism such that g_V is an elementary surjection and g_A is bijective. Let v be the unique vertex of X such that $\#g_V^{-1}(v) = 2$, and let $g_V^{-1}(v) = \{u, w\}$. Then*

$$0 \longrightarrow \ker(g^*) \longrightarrow K^{\cdot}(X) \xrightarrow{g^*} K^{\cdot}(Y) \longrightarrow 0$$

is exact, in which $\ker(g^*) \simeq J^{\cdot}(g) := \mathrm{span}_{\mathbb{Z}}\{R_\gamma(Y_\sigma, u) : \sigma \in \mathcal{Q}(g)\}$ *and* $\mathcal{Q}(g)$ *is the set of all* $\sigma \in \mathcal{P}(Y)$ *such that the images of* u *and* w *in* Y_σ *do not coincide.*

Proof. Identify $V(X) \smallsetminus \{v\}$ with $V(Y) \smallsetminus \{u, w\}$ via g_V. Identifying $E(X)$ with $E(Y)$ via g_E, we see that $g^\sharp : F^{\cdot}(X) \to F^{\cdot}(Y)$ is the isomorphism defined by $g^\sharp(X_\sigma) := Y_\sigma$ for all $\sigma \in \mathcal{P}(X)$. Fix an $\omega \in \Omega(X)$ and let γ be the pullback of ω along g. To describe the cokernel of $g^\flat : N^{\cdot}_\omega(X) \to N^{\cdot}_\gamma(Y)$, consider any $\sigma \in \mathcal{P}(X)$ and $x \in V(X_\sigma)$. If x is not the image of v in $V(X_\sigma)$ then $g^\flat(R_\omega(X_\sigma, x)) = R_\gamma(Y_\sigma, x)$. If x is the image of v in X_σ and the images of u and w in σ coincide, then $g^\flat(R_\omega(X_\sigma, x)) = R_\gamma(Y_\sigma, u)$. In the remaining case, $g^\flat(R_\omega(X_\sigma, x)) = R_\gamma(Y_\sigma, u) + R_\gamma(Y_\sigma, w)$. Therefore,

$$\mathrm{coker}(g^\flat) = \frac{\mathrm{span}_{\mathbb{Z}}\{R_\gamma(Y_\sigma, u), R_\gamma(Y_\sigma, w) : \sigma \in \mathcal{Q}(g)\}}{\mathrm{span}_{\mathbb{Z}}\{R_\gamma(Y_\sigma, u) + R_\gamma(Y_\sigma, w) : \sigma \in \mathcal{Q}(g)\}} \simeq J^{\cdot}(g).$$

From the kernel-cokernel exact sequence we deduce that

$$0 \longrightarrow \ker(g^\omega) \longrightarrow K^{\cdot}_\omega(X) \xrightarrow{g^\omega} K^{\cdot}_\gamma(Y) \longrightarrow 0$$

is exact, in which $\ker(g^\omega) \simeq \mathrm{coker}(g^\flat)$. The proof is completed by compatibility of the maps, as usual. □

3. Structure of Kirchhoff groups.

For a graph X and nonnegative integer j, let $d_j(X) := \mathrm{rank}\, K^j(X)$, and collect these numbers as the coefficients of the *Poincaré polynomial* of X: $D_X(t) := d_0(X) + d_1(X)t + \cdots + d_m(X)t^m$. We see in Theorem 3.1(a) that Kirchhoff groups are torsion-free, and so the Poincaré polynomial of X completely determines the structure of $K^{\cdot}(X)$ as a graded abelian group. However, the functoriality of Kirchhoff groups induces a coalgebra structure on $K^{\cdot}(X)$, which allows for the definition of more subtle algebraic invariants of graphs.

Theorem 3.1. *Let X be a graph.*
(a) *The Kirchhoff group $K^{\cdot}(X)$ is torsion-free.*
(b) *Let $i : Y \to X$ be as in Proposition 2.4, and such that e is not a cut-edge. Then*

$$D_X(t) = D_Y(t) + t\, D_{X_e}(t).$$

(c) *Let $h : Y \to X$ be as in Proposition 2.5. Then*

$$D_Y(t) = D_X(t) + (t + t^2) D_{X_e}(t).$$

(d) *If $T_X(x, y)$ denotes the Tutte polynomial of X then*

$$D_X(t) = t^{\#V(X)-1} T_X\left(\frac{1}{t}, 1 + t\right).$$

Proof. For part (a), we show that $K^{\cdot}(X)$ is torsion-free by induction on $\#E$. For the basis, notice that $K^{\cdot}(X) = \mathbb{Z}$ for every graph X with no edges. For a connected graph •—• with two vertices and one cut-edge, Proposition 2.4 and Theorem 1.3(a) imply that $K^{\cdot}(\bullet\!-\!\bullet) = K^{\cdot}(\bullet\ \bullet) = \mathbb{Z}$. Theorem 1.2 now implies that if every edge of X is a cut-edge (*i.e.* X is a forest) then $K^{\cdot}(X) = \mathbb{Z}$. For the induction step we may assume that $e \in E$ is not a cut-edge. Theorem 1.3(b) and Proposition 2.4 show that

$$0 \longrightarrow K^{\cdot}(X_e)(-1) \xrightarrow{\eta_{X,e}} K^{\cdot}(X) \longrightarrow K^{\cdot}(X \smallsetminus \{e\}) \longrightarrow 0 \tag{3.1}$$

is exact. The homomorphism π from Theorem 1.3(b) shows that this sequence is split, so that $K^{\cdot}(X) \simeq K^{\cdot}(X_e)(-1) \oplus K^{\cdot}(X \smallsetminus \{e\})$. By induction, both $K^{\cdot}(X_e)(-1)$ and $K^{\cdot}(X \smallsetminus \{e\})$ are torsion-free, and so $K^{\cdot}(X)$ is torsion-free.

Parts (b) and (c) follow directly from (3.1) and Proposition 2.5, respectively.

For part (d), first notice that $D_{\circ}(t) = 1 + t$ and $D_{\bullet\!-\!\bullet}(t) = 1$. The function $\widetilde{D}_X(t) := t^{1-\#V(X)} D_X(t)$ has the property that if $e \in E$ is neither a loop nor a cut-edge, then

$$\widetilde{D}_X(t) = \widetilde{D}_{X \smallsetminus \{e\}}(t) + \widetilde{D}_{X_e}(t),$$

as follows from part (b). Since $T_X(x, y)$ is the universal Tutte-Grothendieck invariant of the category of graphs (see Tutte [19] or Theorem 6.2.2 of Brylawski and Oxley [8]), it follows that $\widetilde{D}_X(t) = T_X(\widetilde{D}_{\bullet\!-\!\bullet}(t), \widetilde{D}_{\circ}(t))$. The result now follows easily. $\square$

Corollary 3.2. *If X and Y are graphs with isomorphic graphic matroids, then $K^{\cdot}(X)$ and $K^{\cdot}(Y)$ are isomorphic (but not naturally so).*

First proof. If X and Y have isomorphic graphic matroids then $T_X(x, y) = T_Y(x, y)$ (since the Tutte polynomial is really a matroid invariant; see [8]). If L and M are finitely generated torsion-free abelian groups of the same rank then $L \simeq M$. $\square$

Sketch of second proof. Whitney [26] (see also Theorem 6.3.1 of Oxley [16]) proves that two graphs have isomorphic graphic matroids if and only if they are "2-isomorphic" (see the above references for the definition). If X' is obtained from X by splitting at a cut-vertex or by merging two components to produce a cut-vertex then Theorem 1.2 shows that $K^{\cdot}(X') = K^{\cdot}(X)$. If X' is obtained from X by twisting one component relative to a 2-vertex-cut then fix an $\omega \in \Omega(X)$ and let $\gamma \in \Omega(X')$ be obtained from ω by changing the orientation on all edges of the twisted component. It is not difficult to verify that $N^{\cdot}_{\gamma}(X') = N^{\cdot}_{\omega}(X)$, and so $K^{\cdot}_{\omega}(X) = K^{\cdot}_{\gamma}(Y)$. These equalities are compatible with $\Psi(X)$ and $\Psi(X')$, and so $K^{\cdot}(X') = K^{\cdot}(X)$. Now X and Y are 2-isomorphic if and only if there is a graph X'', obtained from X by a finite sequence of operations $X \mapsto X'$ as above, such that $X'' \simeq Y$. The above remarks show that $K^{\cdot}(X'') \simeq K^{\cdot}(X)$, but the isomorphism depends nontrivially on a choice of a sequence of operations $X \mapsto X'$ and thus is not natural. Finally, an isomorphism $f : Y \to X''$ induces an isomorphism $f^* : K^{\cdot}(X'') \to K^{\cdot}(Y)$; since this depends upon a choice for f it is also not natural. $\square$

For a graph $X = (V, A, o, t, \bar{\cdot})$ and a positive integer r, define a graph $X^{(r)} := (V', A', o', t', \bar{\cdot})$ as follows: it has vertices $V' := V \times \{1, 2, ..., r\}$, arcs $A' := A \times \{1, 2, ..., r\}$, origin $o'(a, i) := (o(a), i)$, terminus $t'(a, i) := (t(a), i)$, and reversal $\overline{(a, i)} := (\overline{a}, i)$. This is a disjoint union of r copies of X, and is equipped with a natural graph morphism $p_{(r)} : X^{(r)} \to X$ defined by projection onto the first coordinate. By Theorem 1.2, $K^{\cdot}(X^{(2)}) = K^{\cdot}(X) \otimes K^{\cdot}(X)$. By functoriality of $K^{\cdot}$, there is a natural homomorphism $\Delta := p^*_{(2)}$

$$\Delta : K^{\cdot}(X) \longrightarrow K^{\cdot}(X) \otimes K^{\cdot}(X). \tag{3.2}$$

The automorphism of $X^{(2)}$ which exchanges the indices in the second coordinate shows that Δ is cocommutative. The two natural factorizations of $p_{(3)}$ through $p_{(2)}$ show that Δ is coassociative. With the projection $K^{\cdot}(X) \to K^0(X) = \mathbb{Z}$ for counit, this gives $K^{\cdot}(X)$ the structure of a coalgebra over $\mathbb{Z}$.

Proposition 3.3. *Let $f : Y \to X$ be a graph morphism. Then $f^* : K^{\cdot}(X) \to K^{\cdot}(Y)$ is coalgebra homomorphism.*

Proof. Define $f^{(2)} : Y^{(2)} \to X^{(2)}$ by $f_V^{(2)} := f_V \times \mathrm{id}_{\{1,2\}}$ and $f_A^{(2)} := f_A \times \mathrm{id}_{\{1,2\}}$. The diagram of graph morphisms

$$\begin{array}{ccc} Y^{(2)} & \xrightarrow{f^{(2)}} & X^{(2)} \\ p_{(2)} \downarrow & & \downarrow p_{(2)} \\ Y & \xrightarrow{f} & X \end{array}$$

commutes. Functoriality of $K^{\cdot}$ now implies the result, since $(f^{(2)})^* = f^* \otimes f^*$. $\square$

We remark that the abelian group isomorphisms constructed in the second proof of Corollary 3.2 are in fact coalgebra isomorphisms.

4. Circulation algebras of graphs.

Next, we dualize the coalgebra structure on $K^{\cdot}(X)$ to obtain functorial algebra structures on $\mathrm{Hom}(K^{\cdot}(X), B)$ for any ring B, give an explicit description of these algebras when B is commutative, and use this to derive the results on the ranks $d_j(X)$ presented in the Introduction.

For a ring B with unit 1, let $\Phi.(X,B) := \mathrm{Hom}(K^{\cdot}(X), (B,+))$. The ring structure of B is a map $B \otimes B \to B$, which induces a natural homomorphism

$$\begin{aligned} \Phi.(X,B) \otimes \Phi.(X,B) &\xrightarrow{\sim} \mathrm{Hom}(K^{\cdot}(X) \otimes K^{\cdot}(X), B \otimes B) \\ &\longrightarrow \mathrm{Hom}(K^{\cdot}(X) \otimes K^{\cdot}(X), B). \end{aligned}$$

(The first map is an isomorphism.) This factors through $\Phi.(X,B) \otimes_B \Phi.(X,B)$, giving

$$\Phi.(X,B) \otimes_B \Phi.(X,B) \longrightarrow \mathrm{Hom}(K^{\cdot}(X) \otimes K^{\cdot}(X), B).$$

Dualizing (3.2), we obtain

$$\mathrm{Hom}(K^{\cdot}(X) \otimes K^{\cdot}(X), B) \longrightarrow \Phi(X,B).$$

The composition of these gives a natural homomorphism

$$\Phi.(X,B) \otimes_B \Phi.(X,B) \longrightarrow \Phi.(X,B),$$

and this defines a B-algebra structure on $\Phi.(X,B)$. Since Δ is cocommutative, $\Phi.(X,B)$ is commutative if and only if B is commutative. From the functoriality of Hom and $K^{\cdot}$ and Proposition 3.3, it follows that $\Phi.$ is a functor which is covariant in both of its arguments. In particular, a graph morphism $f : Y \to X$ induces a B-algebra homomorphism $f_* : \Phi.(Y,B) \to \Phi.(X,B)$.

The *B-circulation algebra* of a graph X is $\Phi.(X,B)$; it is a graded B-algebra which is a finitely generated B-module. As mentioned above, for any $\varphi \in \Phi.(X,B)$ the j-th graded component of φ is a coherent family of B-valued flows on $\{X_\sigma : \sigma \in \mathcal{P}_{j-1}(X)\}$. An element of $\Phi.(X,B)$ will be called a *B-circulation on X*; the *B-flows in X* are precisely the B-circulations on X which are in $\Phi_1(X,B)$. Each B-circulation on X is a homomorphism $\varphi : F^{\cdot}_{\Omega}(X) \to (B,+)$ which annihilates

$N^{\cdot}_{\Omega}(X)$. Thus $\Phi.(X,B)$ is naturally a B-submodule of $\mathrm{Hom}(F^{\cdot}_{\Omega}(X),B)$. Fix an $\omega \in \Omega$. For each $\varphi \in \Phi.(X,B)$, the composition

$$F^{\cdot}(X) \longrightarrow F^{\cdot}(X)/N^{\cdot}_{\omega}(X) \xrightarrow{\kappa_\omega} K^{\cdot}(X) \xrightarrow{\varphi} B$$

is an element of $\mathrm{Hom}(F^{\cdot}(X),B)$. This defines a monomorphism

$$\lambda_\omega : \Phi.(X,B) \to \mathrm{Hom}(F^{\cdot}(X),B)$$

which identifies $\Phi.(X,B)$ with the submodule of $\mathrm{Hom}(F^{\cdot}(X),B)$ annihilating $N^{\cdot}_{\omega}(X)$; it will be convenient to refer to this identification as *coordinatization of* $\Phi.(X,B)$ *by* ω. The basis of $\mathrm{Hom}(F^{\cdot}(X),B)$ dual to $\{X_\sigma : \sigma \in \mathcal{P}(X)\}$ will be denoted by $\{\xi_\sigma : \sigma \in \mathcal{P}(X)\}$.

There is a comultiplication $\hat{\Delta}$ defined on $F^{\cdot}(X)$ by

$$\hat{\Delta}(X_\sigma) := \sum_{\tau \subseteq \sigma} X_\tau \otimes X_{\sigma \smallsetminus \tau}$$

for each $\sigma \in \mathcal{P}(X)$. Dualizing gives a multiplication on $\mathrm{Hom}(F^{\cdot}(X),B)$: for any $\varphi, \theta \in \mathrm{Hom}(F^{\cdot}(X),B)$ and any $\sigma \in \mathcal{P}(X)$,

$$(\varphi \cdot \theta)(X_\sigma) = \sum_{\tau \subseteq \sigma} \varphi(X_\tau)\theta(X_{\sigma \smallsetminus \tau}). \tag{4.1}$$

A graph morphism $f : Y \to X$ induces a B-algebra homomorphism $f_\sharp : \mathrm{Hom}(F^{\cdot}(Y),B) \to \mathrm{Hom}(F^{\cdot}(X),B)$ by $f_\sharp(\varphi) := \varphi \circ f^\sharp$, and this is functorial. Restriction of the multiplication on $\mathrm{Hom}(F^{\cdot}(X),B)$ to the annihilator of $N^{\cdot}_{\omega}(X)$ coincides with the multiplication on $\Phi.(X,B)$ via the map λ_ω.

The *positive ideal* of $\mathrm{Hom}(F^{\cdot}(X),B)$ is $\mathrm{Hom}(F^{+}(X),B) = \bigoplus_{j>0} \mathrm{Hom}(F^{j}(X),B)$, and for circulation algebras $\Phi_+(X,B) := \bigoplus_{j>0} \Phi_j(X,B)$. For each $\varphi \in \mathrm{Hom}(F^{+}(X),B)$ there is a nonnegative integer r such that $\varphi^r = 0$; the *nilpotence* $\mathrm{np}(\varphi)$ of such a φ is the greatest integer n such that $\varphi^n \neq 0$. For $\varphi \in \mathrm{Hom}(F^{\cdot}(X),B)$, define the *support of* φ to be $\mathrm{supp}(\varphi) := \{\sigma \in \mathcal{P}(X) : \varphi(X_\sigma) \neq 0\}$. For $\omega, \varepsilon \in \Omega$, $\varphi \in \Phi.(X,B)$, and $\sigma \in \mathcal{P}(X)$, notice that $\lambda_\varepsilon(\varphi)(X_\sigma) = (-1)^{m_\sigma(\varepsilon,\omega)}\lambda_\omega(\varphi)(X_\sigma)$; that is, $\lambda_\varepsilon(\varphi) = \lambda_\omega(\varphi) \circ \psi^{\varepsilon}_{\omega}$. Thus, the set $\mathrm{supp}(\lambda_\omega(\varphi))$ does not depend on the choice of $\omega \in \Omega$, so each B-circulation on X also has a well-defined support.

Lemma 4.1. *Let X be a graph, let B be a commutative ring, and let $\varphi \in \mathrm{Hom}(F^{+}(X),B)$.*
(a) *For every $\sigma \in \mathcal{P}(X)$ and nonnegative integer r,*

$$\varphi^r(X_\sigma) = r! \sum_{\{\tau_1,\ldots,\tau_r\}}^{\sigma} \prod_{i=1}^{r} \varphi(X_{\tau_i}),$$

in which the sum is over all partitions of the set σ into exactly r pairwise disjoint nonempty blocks.
(b) *If $\varphi \in \mathrm{Hom}(F^{1}(X),B)$ then* $\mathrm{np}(\varphi) \leq \#\mathrm{supp}(\varphi)$.
(c) *If $\varphi \in \mathrm{Hom}(F^{1}(X),B)$ and B is an integral domain with* $\mathrm{char}(B) = 0$, *then* $\mathrm{np}(\varphi) = \#\mathrm{supp}(\varphi)$.
(d) *If $\varphi \in \mathrm{Hom}(F^{1}(X),B)$ and B is an integral domain with* $\mathrm{char}(B) = p > 0$, *then* $\mathrm{np}(\varphi) = \min\{p-1, \#\mathrm{supp}(\varphi)\}$.

Proof. From (4.1) we obtain

$$\varphi^r(X_\sigma) = \sum_{(\tau_1,\ldots,\tau_r)}^{\sigma} \varphi(X_{\tau_1})\varphi(X_{\tau_2})\cdots\varphi(X_{\tau_r}),$$

in which the sum is over all ordered partitions of the set σ into exactly r pairwise disjoint sets. If any of the $\tau_i = \varnothing$ then the corresponding term is zero, since $\varphi(X_\varnothing) = 0$. Since B is commutative, the $r!$ terms corresponding to the same unordered partition all have the same value, proving part (a).

If $\varphi \in \mathrm{Hom}(F^1(X), B)$ then $\varphi(X_\tau) = 0$ unless $\tau \in \mathcal{P}_1(X)$. It follows that for all $\sigma \in \mathcal{P}_j(X)$,

$$\varphi^r(X_\sigma) = r!\delta_{jr} \prod_{e\in\sigma} \varphi(X_e),$$

in which δ_{jr} is the Kronecker delta function. If $j > \#\mathrm{supp}(\varphi)$ then every $\sigma \in \mathcal{P}_j(X)$ contains some $e \notin \mathrm{supp}(\varphi)$, and it follows that $\varphi^j = 0$. This proves (b). If B is an integral domain and $\mathrm{supp}(\varphi) = \{e_1, ..., e_r\}$ then for all $0 \le j \le r$, $\varphi^j(X_{\{e_1,...,e_j\}}) \neq 0$ if and only if $j! \neq 0$ in B. Since $\mathrm{char}(B)$ is either zero or prime, parts (c) and (d) follow. □

When B is commutative, the *exponential function*

$$\exp : \mathrm{Hom}(F^+(X), B) \to \mathrm{Hom}(F^{\cdot}(X), B)$$

is defined for each $\varphi \in \mathrm{Hom}(F^+(X), B)$ by putting

$$\exp(\varphi)(X_\sigma) := \sum_{r=0}^{\infty} \sum_{\{\tau_1,...,\tau_r\}}^{\sigma} \prod_{i=1}^{r} \varphi(X_{\tau_i}) \tag{4.2}$$

for all $\sigma \in \mathcal{P}(X)$, with notation as in Lemma 4.1(a). If B is a $\mathbb{Q}$-algebra then one may also use the formal power series $\exp(\varphi) = \sum_{r=0}^{\infty} \varphi^r/r!$ because of Lemma 4.1(a). For $\varphi \in \mathrm{Hom}(F^1(X), B)$ we use the notation

$$\exp(\varphi) = \varphi^{<0>} + \varphi^{<1>} + \varphi^{<2>} + \cdots + \varphi^{<j>} + \cdots$$

in which $\varphi^{<j>}$ is the j-th graded component of $\exp(\varphi)$, so $\varphi^{<0>} = 1$ and $\varphi^{<1>} = \varphi$.

Lemma 4.2. *Let X be a graph and let B be a commutative ring.*
(a) *For $\varphi, \theta \in \mathrm{Hom}(F^+(X), B)$, we have* $\exp(\varphi + \theta) = \exp(\varphi) \cdot \exp(\theta)$.
(b) *If $f : Y \to X$ is a graph morphism then* $\exp \circ f_\sharp = f_\sharp \circ \exp$.
(c) *For any $\omega \in \Omega$, we have* $\exp(\lambda_\omega(\Phi_+(X, B))) \subseteq \lambda_\omega(\Phi.(X, B))$.

Proof. Part (a) is a straightforward calculation from (4.1) and (4.2). For (b), let $\varphi \in \mathrm{Hom}(F^{\cdot}(Y), B)$ and $\sigma \in \mathcal{P}(X)$, and calculate that

$$\begin{aligned}
\exp(f_\sharp(\varphi))(X_\sigma) &= \exp(\varphi \circ f^\sharp)(X_\sigma) \\
&= \sum_{r=0}^{\infty} \sum_{\{\tau_1,...,\tau_r\}}^{\sigma} \prod_{i=1}^{r} \sum_{\rho_i \in T_f(\tau_i)} \varphi(Y_{\rho_i}) \\
&= \sum_{\rho\in T_f(\sigma)} \sum_{r=0}^{\infty} \sum_{\{\tau_1,...,\tau_r\}}^{\rho} \prod_{i=1}^{r} \varphi(Y_{\tau_i}) \\
&= \sum_{\rho\in T_f(\sigma)} \exp(\varphi)(Y_\rho) \\
&= \exp(\varphi)(f^\sharp(X_\sigma)) = f_\sharp(\exp(\varphi))(X_\sigma).
\end{aligned}$$

For part (c) let $\varphi \in \lambda_\omega(\Phi_+(X,B))$, consider a relation $R_\omega(X_\sigma, v)$ in $N^{\cdot}_\omega(X)$, and calculate that

$$\begin{aligned}
&\exp(\varphi)(R_\omega(X_\sigma, v)) \\
=\ & \exp(\varphi)\left(\sum_{a\in X_\sigma(v)} s_\omega(a) X_{\sigma\cup\{\underline{a}\}}\right) = \sum_{a\in X_\sigma(v)} s_\omega(a) \sum_{r=0}^{\infty} \sum_{\{\tau_1,\dots,\tau_r\}}^{\sigma\cup\{\underline{a}\}} \prod_{i=1}^{r} \varphi(X_{\tau_i}) \\
=\ & \sum_{r=0}^{\infty} \sum_{\{\tau_1,\dots,\tau_r\}}^{\sigma} \left[\sum_{h=1}^{r}\left(\prod_{i\neq h} \varphi(X_{\tau_i})\right) \sum_{a\in X_\sigma(v)} s_\omega(a)\varphi(X_{\tau_h\cup\{\underline{a}\}})\right. \\
&+ \left.\left(\prod_{i=1}^{r}\varphi(X_{\tau_i})\right) \sum_{a\in X_\sigma(v)} s_\omega(a)\varphi(X_{\underline{a}})\right] \\
=\ & \sum_{r=0}^{\infty} \sum_{\{\tau_1,\dots,\tau_r\}}^{\sigma} \left[\sum_{h=1}^{r}\left(\prod_{i\neq h} \varphi(X_{\tau_i})\right) \varphi\left(\sum_{u\in U(h)} R_\omega(X_{\tau_h}, u)\right)\right. \\
&+ \left.\left(\prod_{i=1}^{r}\varphi(X_{\tau_i})\right) \varphi\left(\sum_{u\in U} R_\omega(X, u)\right)\right] = 0.
\end{aligned}$$

In the last expression, for each partition $\{\tau_1, \dots, \tau_r\}$ of σ, $U(h)$ denotes the set of vertices of X_{τ_h} which are contracted to the vertex v of X_σ (for each $1 \le h \le r$), and U denotes the set of vertices of X which are contracted to the vertex v of σ. (The penultimate equality follows from Lemma 1.1.) This proves part (c). $\square$

For a maximal forest Y of X and an edge e of X not in Y, the subgraph $Y \cup \{e\}$ contains a unique cycle C, the *fundamental cycle of e with respect to Y*. Let $a \in e$ be one of the arcs in e, let $\omega \in \Omega$ be such that $s_\omega(a) = 1$, and coordinatize $\Phi.(X,B)$ by ω. There is a unique flow $\beta \in \Phi_1(C,B) \subseteq \Phi_1(X,B)$ such that $\beta(X_e) = 1$, and it is independent of the choice of $\omega \in \Omega$ (subject to $s_\omega(a) = 1$). This is the *basic flow of a with respect to Y*.

It is convenient to standardize some notation which will be used repeatedly in what follows.

(*) Coordinatize $\Phi.(X,B)$ by some $\omega \in \Omega$, and fix a maximal forest Y contained in X. Let $\mathcal{E} := E(X) \smallsetminus E(Y)$, and for each $c \in \mathcal{E}$ let β_c be the basic flow of $\omega(c)$ with respect to Y. For $\mathbf{j} : \mathcal{E} \to \mathbb{N}$ define $\boldsymbol{\beta}^{<\mathbf{j}>} := \prod_{c\in\mathcal{E}} \beta_c^{<j(c)>}$.

(**) Continuing from (*), fix an $e \in \mathcal{E}$ which is not a loop. Then the image of Y in X_e contains a unique cycle C_e, and this is the image in X_e of the unique cycle C in $Y \cup \{e\}$. We may choose $\omega \in \Omega$ so that for every vertex $v \in V(C)$ there is a unique edge $c \in E(C)$ such that $v = t(\omega(e))$. (Changing the orientation on a subset of $E(Y)$ does not affect any of the basic flows $\{\beta_c : \ c \in \mathcal{E}\}$.) Let $\zeta := E(C_e)$, and let e' be any edge in ζ. Then $Y' := Y \smallsetminus \{e'\}$ is a maximal forest in X_e. Let $\mathcal{E}' := E(X_e) \smallsetminus E(Y')$ and for each $c \in \mathcal{E}'$ let α_c be the basic flow of $\omega(c)$ with respect to Y'. Let $\pi : K^{\cdot}(X) \to K^{\cdot}(X_e)(-1)$ be the map constructed from C in the proof of Theorem 1.3(b), and let $\pi^* : \Phi.(X_e,B)(-1) \to \Phi.(X,B)$ be the map induced from π by duality.

We require one more lemma for the proofs of the main results of this section. For $e \in E$ define $p_e : \mathrm{Hom}(F^{\cdot}(X), B) \to \mathrm{Hom}(F^{\cdot}(X_e), B)$ by putting $p_e(\theta)((X_e)_\sigma) := \theta(X_\sigma)$ for all $\theta \in \mathrm{Hom}(F^{\cdot}(X), B)$ and $\sigma \in \mathcal{P}(X_e)$.

Lemma 4.3. *Let X be a graph, let $e \in E$, and let B be a commutative ring.*
(a) *The map p_e is a B-algebra homomorphism.*
(b) *For any $\omega \in \Omega$, we have $p_e(\lambda_\omega(\Phi.(X,B))) \subseteq \lambda_\omega(\Phi.(X_e,B))$.*
(c) *With the notation (**), $p_e(\beta_e) = \alpha_{e'}$ and if $c \in \mathcal{E} \smallsetminus \{e\}$ then $p_e(\beta_c) = \alpha_c + s_c\alpha_{e'}$, in which $s_c := \beta_c(X_{e'}) \in \{-1,0,1\}$.*
(d) *With the notation (**), $(\pi^* \circ p_e)(\boldsymbol{\beta}^{<\mathbf{j}>}) = \boldsymbol{\beta}^{<\mathbf{j}+\delta_e>}$ for all $\mathbf{j} : \mathcal{E} \to \mathbb{N}$, in which $\delta_e : \mathcal{E} \to \mathbb{N}$ is defined by $\delta_e(c) := \mathtt{tv}[c = e]$ for all $c \in \mathcal{E}$.*

Proof. Parts (a) and (b) are straightforward. For part (c), if $c \in \mathcal{E}$ then let $C(c)$ be the fundamental cycle of c with respect to Y, and if $c \in \mathcal{E}'$ then let $C'(c)$ be the fundamental cycle of c with respect to Y'. Then $C'(e') = C_e$ as defined in (**), and for $c \in \mathcal{E}' \smallsetminus \{e'\}$, if $e' \notin E(C(c))$ then $C'(c) = C(c)$, and if $e' \in E(C(c))$ then $C'(c)$ has edge-set the symmetric difference of $E(C(c))$ and $E(C_e)$. Also, for $c, g \in \mathcal{E}$, $\beta_c(X_g) = 0$ if g is not an edge of $C(c)$, and otherwise $\beta_c(X_g) = \pm 1$ according to whether $\omega(c)$ and $\omega(g)$ are directed in the same or in opposite directions around $C(c)$. From this part (c) follows. For part (d), notice for $\mathbf{j} : \mathcal{E} \to \mathbb{N}$ and $\sigma \in \mathcal{P}(X)$ that

$$\boldsymbol{\beta}^{<\mathbf{j}>}(X_\sigma) = \sum_{(\tau(c):\ c\in\mathcal{E})}^{\sigma} \prod_{c\in\mathcal{E}} \prod_{g\in\tau(c)} \beta_c(X_g),$$

in which the sum is over the set of all $\mathcal{E}$-indexed partitions $(\tau(c) :\ c \in \mathcal{E})$ of σ into disjoint subsets such that $\#\tau(c) = j(c)$ for all $c \in \mathcal{E}$. Now for all $\mathbf{j} : \mathcal{E} \to \mathbb{N}$ and $\sigma \in \mathcal{P}(X)$,

$$\begin{aligned}
(\pi^* \circ p_e)(\boldsymbol{\beta}^{<\mathbf{j}>})(X_\sigma) &= p_e(\boldsymbol{\beta}^{<\mathbf{j}>})(\pi(X_\sigma)) \\
&= \begin{cases} p_e(\boldsymbol{\beta}^{<\mathbf{j}>})((X_e)_{\sigma\smallsetminus\{e\}}) & \text{if } e \in \sigma, \\ \sum_{c\in\sigma\cap\zeta} p_e(\boldsymbol{\beta}^{<\mathbf{j}>})((X_e)_{\sigma\smallsetminus\{c\}}) & \text{if } e \notin \sigma, \end{cases} \\
&= \begin{cases} \boldsymbol{\beta}^{<\mathbf{j}>}(X_{\sigma\smallsetminus\{e\}}) & \text{if } e \in \sigma, \\ \sum_{c\in\sigma\cap\zeta} \boldsymbol{\beta}^{<\mathbf{j}>}(X_{\sigma\smallsetminus\{c\}}) & \text{if } e \notin \sigma, \end{cases} \\
&= \begin{cases} \boldsymbol{\beta}^{<\mathbf{j}+\delta_e>}(X_\sigma) & \text{if } e \in \sigma, \\ \boldsymbol{\beta}^{<\mathbf{j}+\delta_e>}(X_\sigma) & \text{if } e \notin \sigma. \end{cases}
\end{aligned}$$

The last equality in case $e \notin \sigma$ follows from (4.1) and the fact that $\beta_e = \sum_{c\in\zeta} \xi_c$. Hence $(\pi^* \circ p_e)(\boldsymbol{\beta}^{<\mathbf{j}>}) = \boldsymbol{\beta}^{<\mathbf{j}+\delta_e>}$ for all $\mathbf{j} : \mathcal{E} \to \mathbb{N}$. □

Theorem 4.4. *Let X be a graph, let B be a commutative ring, and adopt the notation (*).*
(a) *Then $\{\boldsymbol{\beta}^{<\mathbf{j}>} :\ \mathbf{j} : \mathcal{E} \to \mathbb{N}\}$ spans $\Phi.(X,B)$ as a B-module.*
(b) *If $e \in \mathcal{E}$ is not a loop then p_e is surjective.*

Proof. We prove (a) and (b) together by induction on $\#E(X)$. As basis of induction we have the case that X is a forest; then $Y = X$ and $\mathcal{E} = \varnothing$ and $\Phi.(X,B) = B$ is spanned by $\{1\}$ as a B-module, conforming with the statement to be proved. For the induction step, let $e \in \mathcal{E}$, so e is not a cut-edge of X. If e is a loop then the dual of Theorem 1.2 implies that

$$\Phi.(X,B) = \Phi.(X \smallsetminus \{e\}, B) \otimes_B \Phi.(\text{\textbf{O}}, B), \tag{4.3}$$

and the induction argument is easy. In the remaining case, adopt the notation (**). Dualizing (3.1) we see that

$$0 \longrightarrow \Phi.(X \smallsetminus \{e\}, B) \xrightarrow{i_*} \Phi.(X,B) \xrightarrow{\eta^*_{X,e}} \Phi.(X_e,B)(-1) \longrightarrow 0$$

is exact, and the map π^* shows that the sequence is split. Thus, any $\varphi \in \Phi.(X, B)$ can be written in the form $\varphi = i_*(\varphi'') + \pi^*(\varphi')$ with $\varphi'' \in \Phi.(X \smallsetminus \{e\}, B)$ and $\varphi' \in \Phi.(X_e, B)(-1)$. By induction, $\Phi.(X \smallsetminus \{e\}, B)$ is spanned by $\{\boldsymbol{\beta}^{<\mathbf{j}>} : \mathbf{j} : \mathcal{E} \smallsetminus \{e\} \to \mathbb{N}\}$. Since $i_*(\beta_c) = \beta_c$ for all $c \in \mathcal{E} \smallsetminus \{e\}$ and i_* is a B-algebra homomorphism, the term $i_*(\varphi'')$ is of the required form.

To understand the term $\pi^*(\varphi')$, the induction hypothesis implies that $\{\boldsymbol{\alpha}^{<\mathbf{h}>} : \mathbf{h} : \mathcal{E}' \to \mathbb{N}\}$ spans $\Phi.(X_e, B)$ as a B-module. From Lemma 4.3(c) it follows for all $h \in \mathbb{N}$ that $\alpha_{e'}{}^{<h>} = p_e(\beta_e^{<h>})$ and if $c \in \mathcal{E}' \smallsetminus \{e'\}$ then $\alpha_c{}^{<h>} = p_e\left((\beta_c - s_c\beta_e)^{<h>}\right) = p_e\left(\sum_{j=0}^{h} \beta_c^{<h-j>} \cdot (-s_c)^j \beta_e^{<j>}\right)$. Since $\beta_e^{<i>} \cdot \beta_e^{<j>} = \binom{i+j}{j}\beta_e^{<i+j>}$ it follows for all $\mathbf{h} : \mathcal{E}' \to \mathbb{N}$ that $\boldsymbol{\alpha}^{<\mathbf{h}>}$ is in the span over B of the elements $p_e(\boldsymbol{\beta}^{<\mathbf{j}>})$ for $\mathbf{j} : \mathcal{E} \to \mathbb{N}$. This proves (b), and so there is a (finite) expression

$$\varphi' = \sum_{\mathbf{j}:\mathcal{E}\to\mathbb{N}} b(\mathbf{j}) p_e(\boldsymbol{\beta}^{<\mathbf{j}>})$$

with coefficients $b(\mathbf{j})$ in B. Therefore, from Lemma 4.3(d) we conclude that

$$\pi^*(\varphi') = \sum_{\mathbf{j}:\mathcal{E}\to\mathbb{N}} b(\mathbf{j}) \boldsymbol{\beta}^{<\mathbf{j}+\delta_e>},$$

completing the induction step, and the proof of (a). $\square$

For example, applying Theorem 4.4 to the n-vertex cycle C_n and $B = \mathbb{Z}$ we see that

$$\Phi.(\mathsf{C}_n, \mathbb{Z}) = \mathrm{span}_{\mathbb{Z}}\{1, \beta, \beta^{<2>}, ..., \beta^{<n>}\},$$

in which β is the basic flow with respect to some arc and maximal tree in C_n (since $\mathrm{np}(\beta) = n$, by Lemma 4.1(c)). This is isomorphic to the quotient of the *divided power algebra* $\mathbb{Z}_![x] := \mathbb{Z}[x^j/j! : j \in \mathbb{N}]$ (a non-Noetherian subring of $\mathbb{Q}[x]$) by the ideal $(x^j/j! : j > n)$. The notation $x^{<j>} := x^j/j!$ for $j \in \mathbb{N}$ will be convenient. Notice that d/dx and $\int dx$ are endomorphisms of $\mathbb{Z}_![x]$ which are surjective and injective, respectively.

Corollary 4.5. *Let X be a graph, let B be a commutative $\mathbb{Q}$-algebra, and adopt the notation* (*)*. Then $\{\beta_c : c \in \mathcal{E}\}$ generates $\Phi.(X, B)$ as a B-algebra.*

Proof. This follows immediately from Theorem 4.4, since $\beta_c^{<j>} = \beta_c^j/j!$ for all $c \in \mathcal{E}$ and $j \in \mathbb{N}$. $\square$

Corollary 4.6. *Let X be a graph with n vertices, m edges, k connected components, and ℓ cut-edges, and let $D_X(t) = d_0 + d_1 t + \cdots + d_m t^m$. Then $d_0 = 1$, $d_1 = m - n + k$, $d_{m-\ell} = 1$, $d_j \neq 0$ if and only if $0 \leq j \leq m - \ell$, and with the notation of* (0.2)*, if $1 \leq j \leq m - \ell - 1$ then $d_{j+1} \leq \psi_j(d_j)$.*

Proof. Adopt the notation (*) with $B = \mathbb{Q}$. We have $d_j = \dim_{\mathbb{Q}} \Phi_j(X, \mathbb{Q})$ for all $0 \leq j \leq m$. The equality $d_0 = 1$ is clear, since $\Phi_0(X, \mathbb{Q}) = \mathbb{Q}$. From Theorem 4.4, the basic flows $\{\beta_c : c \in \mathcal{E}\}$ span $\Phi_1(X, \mathbb{Q})$, and since $\beta_c(X_g) = \delta_c(g)$ for all $c, g \in \mathcal{E}$, these flows are linearly independent. Hence $d_1 = \#\mathcal{E} = m - n + k$, as is well-known. Since $\Phi.(\bullet\!\!-\!\!\bullet, \mathbb{Q}) = \mathbb{Q}$, Theorem 1.2 (dualized) allows us to reduce to the case that X has no cut-edges, and implies in general that $d_j = 0$ for $m - \ell < j \leq m$. When X has no cut-edges, $F^m(X)$ is spanned by X_E and for any $\sigma \in \mathcal{P}_{m-1}(X)$ the graph X_σ is isomorphic with $\circlearrowleft$; thus, for the unique $v \in V(X_\sigma)$ we have $R_\omega(X_\sigma, v) = 0$. Hence $N^m(X) = 0$, and it follows that $K^m(X) \simeq \mathbb{Z}$, and so $d_m = 1$ (in case X has no cut-edges). Since $\Phi_1(X, \mathbb{Q})$ generates $\Phi.(X, \mathbb{Q})$ as a $\mathbb{Q}$-algebra, $d_j \neq 0$ for all

$0 \leq j \leq m-\ell$, and the inequalities $d_{j+1} \leq \psi_j(d_j)$ follow from Macaulay's Theorem (see Theorems II.2.2 and II.2.3 of Stanley [18]). □

Corollary 4.7. *Let X be a graph with m edges and ℓ cut-edges, and let $D_X(t) := d_0 + d_1 t + \cdots + d_{m-\ell} t^{m-\ell}$. Adopt the notation (*) with $B = \mathbb{Q}$, and let $r(c) := \#\mathrm{supp}(\beta_c)$ for all $c \in \mathcal{E}$. Then, for $0 \leq j \leq m-\ell$,*

$$d_j \leq [t^j] \prod_{c \in \mathcal{E}} \left(1 + t + t^2 + \cdots + t^{r(c)}\right).$$

Proof. For $c \in \mathcal{E}$, Lemma 4.1(c) shows that $\beta_c^{1+r(c)} = 0$, and so $\{\boldsymbol{\beta}^{\mathbf{j}} : 0 \leq j(c) \leq r(c)$ for all $c \in \mathcal{E}\}$ spans $\Phi.(X, \mathbb{Q})$ over $\mathbb{Q}$. If these monomials are linearly independent then the Poincaré polynomial on the right side of the inequality is attained; in any case the dimensions d_j are bounded above as described. □

In fact, a common generalization of Corollaries 4.6 and 4.7 can be obtained by applying the Clements-Lindström Theorem [9], but we will not state the resulting inequalities explicitly here.

Next, for a commutative ring B, we present the algebra $\Phi.(X, B)$ explicitly as a quotient of a (multivariate) divided power algebra. Adopt the notation (*), let

$$\mathbb{Z}_!{[\mathbf{x}]} := \bigotimes_{c \in \mathcal{E}} \mathbb{Z}_!{[x_e]},$$

in which $\mathbf{x} := \{x_c : c \in \mathcal{E}\}$ are indeterminates algebraically independent over $\mathbb{Q}$, and let $B_![\mathbf{x}] := B \otimes \mathbb{Z}_![\mathbf{x}]$. For $\mathbf{j} : \mathcal{E} \to \mathbb{N}$ we use the notation $\mathbf{x}^{<\mathbf{j}>}$ for $\prod_{c \in \mathcal{E}} x_c^{<j(c)>}$; these elements form a basis for $B_![\mathbf{x}]$ as a free B-module. For a homogeneous linear polynomial $P(\mathbf{x}) = \sum_{c \in \mathcal{E}} b_c x_c \in B_![\mathbf{x}]$ and any $r \in \mathbb{N}$, the multinomial theorem shows that we may define

$$P^{<r>}(\mathbf{x}) := \sum_{\mathbf{j}} \prod_{c \in \mathcal{E}} b_c^{j(c)} x_c^{<j(c)>},$$

with the sum over all $\mathbf{j} : \mathcal{E} \to \mathbb{N}$ such that $\sum_{c \in \mathcal{E}} j(c) = r$, so that $P^{<r>}(\mathbf{x}) \in B_![\mathbf{x}]$. Define a B-linear homomorphism $\phi : B_![\mathbf{x}] \to \Phi.(X, B)$ by putting $\phi(\mathbf{x}^{<\mathbf{j}>}) := \boldsymbol{\beta}^{<\mathbf{j}>}$ for all $\mathbf{j} : \mathcal{E} \to \mathbb{N}$. By Theorem 4.4, ϕ is surjective, and it is not difficult to verify that it is a ring homomorphism. Consider any $\theta \in \Phi_1(X, B)$, and let $r \geq \#\mathrm{supp}(\theta)$. The linear polynomial

$$P_\theta(\mathbf{x}) := \sum_{c \in \mathcal{E}} \theta(X_c) x_c$$

is such that $P_\theta^{<1+r>}(\mathbf{x})$ is in the kernel of ϕ, as follows from (4.2) and the fact that $\theta = \sum_{c \in \mathcal{E}} \theta(X_c) \beta_c$. Let $\mathcal{C}$ be the subset of $\Phi_1(X, B)$ consisting of those flows θ such that $\mathrm{supp}(\theta)$ is a cycle and for each $c \in E$, $\theta(X_c) \in \{-1, 0, 1\}$. Define an ideal in $B_![\mathbf{x}]$ by

$$I(X, Y, \omega) := (P_\theta^{<1+r>}(\mathbf{x}) : \theta \in \mathcal{C} \text{ and } r \geq \#\mathrm{supp}(\theta)).$$

Theorem 4.8. *Let X be a graph, let B be a commutative ring, and adopt the notation (*). Then*

$$\Phi.(X, B) \simeq \frac{B_![\mathbf{x}]}{I(X, Y, \omega)}.$$

Proof. We proceed by induction on $\#E(X)$, using as basis the case that X is a forest, in which case the claim is trivial. For the induction step, let $e \in \mathcal{E}$. If e is a loop of X then (4.3) holds and the induction step is easy. Otherwise, adopt the notation (**). Since e is not a cut-edge of X, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_![x_c :\ c \in \mathcal{E} \setminus \{e\}] & \xrightarrow{\tilde{\imath}} & B_![\mathbf{x}] & \xrightarrow{\tilde{\eta}} & B_![\mathbf{y}](-1) & \longrightarrow & 0 \\ & & \downarrow \phi'' & & \downarrow \phi & & \downarrow \phi' & & \\ 0 & \longrightarrow & \Phi.(X \setminus \{e\}, B) & \xrightarrow{i_*} & \Phi.(X, B) & \xrightarrow{\eta^*} & \Phi.(X_e, B)(-1) & \longrightarrow & 0 \end{array}$$

with exact rows, in which the bottom row is split by π^* (and η^* is an abbreviation for $\eta^*_{X,e}$). The map ϕ'' is just the restriction of ϕ, and $\phi'(y_c) := \alpha_c$ for all $c \in \mathcal{E}'$. The map $\tilde{\imath}$ is the natural inclusion, and to describe $\tilde{\eta}$ we first define a B-algebra isomorphism $\tilde{p} : B_![\mathbf{x}] \to B_![\mathbf{y}]$ by

$$\tilde{p}(x_c^{<j>}) := \begin{cases} y_{e'}^{<j>} & \text{if } c = e, \\ (y_c + s_c y_{e'})^{<j>} & \text{if } c \in \mathcal{E} \setminus \{e\}, \end{cases}$$

for all $c \in \mathcal{E}$ and $j \in \mathbb{N}$; here, as in Lemma 4.3(c), $s_c := \beta_c(X_{e'})$. Now, with $\tilde{\eta} := \tilde{p} \circ \partial/\partial x_e$ it is clear that the top row is exact. We also define $\tilde{\pi} : B_![\mathbf{y}](-1) \to B_![\mathbf{x}]$ by putting $\tilde{\pi}(P(\mathbf{y})) := \int \tilde{p}^{-1}(P(\mathbf{y}))dx_e$ for all $P(\mathbf{y}) \in B_![\mathbf{y}]$. From the definitions it is clear that $(\tilde{\pi} \circ \tilde{p})(P(\mathbf{x})) = \int P(\mathbf{x})dx_e$ for every $P(\mathbf{x}) \in B_![\mathbf{x}]$. It is also clear that $\tilde{\eta} \circ \tilde{\pi} = 1$, and so the top row of the diagram is also split. It remains to check commutativity of the diagram, which we do in the following three claims.

Claim 1: $\phi' \circ \tilde{p} = p_e \circ \phi$. Since these maps are B-algebra homomorphisms, the following verification suffices. For each $c \in \mathcal{E}$ and $j \in \mathbb{N}$, using Lemma 4.3(c), we have

$$\begin{aligned} (p_e \circ \phi)(x_c^{<j>}) &= p_e(\beta_c^{<j>}) = \begin{cases} \alpha_{e'}^{<j>} & \text{if } c = e', \\ (\alpha_c + s_c \alpha_{e'})^{<j>} & \text{if } c \neq e', \end{cases} \\ &= \begin{cases} \phi'(y_{e'}^{<j>}) & \text{if } c = e', \\ \phi'((y_c + s_c y_{e'})^{<j>}) & \text{if } c \neq e', \end{cases} \\ &= (\phi' \circ \tilde{p})(x_c^{<j>}). \end{aligned}$$

Claim 2: $\phi \circ \tilde{\pi} = \pi^* \circ \phi'$. Consider any $Q(\mathbf{y}) \in B_![\mathbf{y}]$. Since $\tilde{p}$ is surjective, there is an $S(\mathbf{x}) \in B_![\mathbf{x}]$ such that $\tilde{p}(S(\mathbf{x})) = Q(\mathbf{y})$. Since the maps are B-linear it suffices to verify that $(\phi \circ \tilde{\pi} \circ \tilde{p})(\mathbf{x}^{<\mathbf{j}>}) = (\pi^* \circ \phi' \circ \tilde{p})(\mathbf{x}^{<\mathbf{j}>})$ for all $\mathbf{j} : \mathcal{E} \to \mathbb{N}$. Now

$$(\phi \circ \tilde{\pi} \circ \tilde{p})(\mathbf{x}^{<\mathbf{j}>}) = \phi\left(\int \mathbf{x}^{<\mathbf{j}>} dx_e\right) = \boldsymbol{\beta}^{<\mathbf{j}+\delta_e>}$$

and

$$\begin{aligned} & (\pi^* \circ \phi' \circ \tilde{p})(\mathbf{x}^{<\mathbf{j}>}) = (\pi^* \circ p_e \circ \phi)(\mathbf{x}^{<\mathbf{j}>}) \\ = \;& (\pi^* \circ p_e)(\boldsymbol{\beta}^{<\mathbf{j}>}) = \boldsymbol{\beta}^{<\mathbf{j}+\delta_e>}, \end{aligned}$$

using Claim 1 and Lemma 4.3(d).

Claim 3: $\phi' \circ \tilde{\eta} = \eta^* \circ \phi$. Since the maps are B-linear it suffices to verify the identity applied to each $\mathbf{x}^{<\mathbf{j}>}$, for $\mathbf{j} : \mathcal{E} \to \mathbb{N}$, so fix any such $\mathbf{j}$. First notice that

$$(\tilde{\pi} \circ \tilde{\eta})(\mathbf{x}^{<\mathbf{j}>}) = \int \left(\frac{\partial \mathbf{x}^{<\mathbf{j}>}}{\partial x_e}\right) dx_e = \mathtt{tv}[j(e) > 0]\mathbf{x}^{<\mathbf{j}>}.$$

Now, since $\eta^* \circ \pi^* = 1$ on $\Phi.(X_e, B)(-1)$, and since $\eta^*(\boldsymbol{\beta}^{<\mathbf{j}>}) = 0$ if $j(e) = 0$, we calculate (using Claim 2) that

$$\begin{aligned}(\phi' \circ \tilde{\eta})(\mathbf{x}^{<\mathbf{j}>}) &= (\eta^* \circ \pi^* \circ \phi' \circ \tilde{\eta})(\mathbf{x}^{<\mathbf{j}>}) \\ &= (\eta^* \circ \phi \circ \tilde{\pi} \circ \tilde{\eta})(\mathbf{x}^{<\mathbf{j}>}) \\ &= (\eta^* \circ \phi)(\mathtt{tv}[j(e) > 0]\mathbf{x}^{<\mathbf{j}>}) \\ &= (\eta^* \circ \phi)(\mathbf{x}^{<\mathbf{j}>}),\end{aligned}$$

verifying Claim 3.

Since $\phi \circ \tilde{\imath} = i_* \circ \phi''$ is clear, we see that the diagram is commutative, and since ϕ'' is surjective it follows from the kernel-cokernel exact sequence and the induction hypothesis that

$$\ker(\phi) = \tilde{\imath}(I(X \smallsetminus \{e\}, Y, \omega)) \oplus \tilde{\pi}(I(X_e, Y', \omega)(-1)).$$

It is clear from the definition that $I(X, Y, \omega) \subseteq \ker(\phi)$. For the converse, let $Q(\mathbf{x})$ be an arbitrary element of $\ker(\phi)$. From the above, we may write

$$Q(\mathbf{x}) = \tilde{\imath}(Q''(\mathbf{x})) + \tilde{\pi}(Q'(\mathbf{y}))$$

with $Q''(\mathbf{x}) \in I(X \smallsetminus \{e\}, Y, \omega)$ and $Q'(\mathbf{y}) \in I(X_e, Y', \omega)(-1)$. The first term is easily dealt with, since $\tilde{\imath}$ includes $I(X \smallsetminus \{e\}, Y, \omega)$ into $I(X, Y, \omega)$. For the second term, $Q'(\mathbf{y})$ may be written as a B-linear combination of elements of the form $\mathbf{y}^{<\mathbf{j}>} P_{\theta'}^{<1+r>}(\mathbf{y})$ with $\mathbf{j} : \mathcal{E}' \to \mathbb{N}$, $\theta' \in \mathcal{C}(X_e)$, and $r \geq \#\mathrm{supp}(\theta')$, so it suffices to consider just one term of this form. Since p_e is surjective, there is some $\theta \in \Phi_1(X, B)$ such that $p_e(\theta) = \theta'$. In fact, since $\theta(X_c) = \theta'((X_e)_c)$ for all $c \in E(X) \smallsetminus \{e\}$, and then $\theta(X_e)$ is determined by the condition that θ is a B-flow in X, this θ is unique. Moreover, it follows that since $\theta' \in \mathcal{C}(X_e)$ we have $\theta \in \mathcal{C}(X)$, that $\#\mathrm{supp}(\theta') \leq \#\mathrm{supp}(\theta) \leq \#\mathrm{supp}(\theta') + 1$, and that $\tilde{p}(P_\theta(\mathbf{x})) = P_{\theta'}(\mathbf{y})$. Now let $L(\mathbf{x}) := \tilde{p}^{-1}(\mathbf{y}^{<\mathbf{j}>})$. If $\theta(X_e) = 0$ then $\#\mathrm{supp}(\theta) = \#\mathrm{supp}(\theta')$ and

$$\begin{aligned}\tilde{\pi}(\mathbf{y}^{<\mathbf{j}>} P_{\theta'}^{<1+r>}(\mathbf{y})) &= (\tilde{\pi} \circ \tilde{p})(L(\mathbf{x}) P_\theta^{<1+r>}(\mathbf{x})) \\ &= \int L(\mathbf{x}) P_\theta^{<1+r>}(\mathbf{x}) dx_e \\ &= P_\theta^{<1+r>}(\mathbf{x}) \int L(\mathbf{x}) dx_e \in I(X, Y, \omega),\end{aligned}$$

since $B_!{[\mathbf{x}]}$ is closed under $\int dx_e$. If $\theta(X_e) = b \in \{-1, 1\}$ then $\#\mathrm{supp}(\theta) = \#\mathrm{supp}(\theta') + 1$, and using integration by parts repeatedly we find that

$$\begin{aligned}\tilde{\pi}(\mathbf{y}^{<\mathbf{j}>} P_{\theta'}^{<1+r>}(\mathbf{y})) &= (\tilde{\pi} \circ \tilde{p})(L(\mathbf{x}) P_\theta^{<1+r>}(\mathbf{x})) = \int L(\mathbf{x}) P_\theta^{<1+r>}(\mathbf{x}) dx_e \\ &= L(\mathbf{x}) P_\theta^{<2+r>}(\mathbf{x}) b^{-1} - \int \left(\frac{\partial L(\mathbf{x})}{\partial x_e}\right) P_\theta^{<2+r>}(\mathbf{x}) b^{-1} dx_e \\ &= L(\mathbf{x}) P_\theta^{<2+r>}(\mathbf{x}) b^{-1} - \left(\frac{\partial L(\mathbf{x})}{\partial x_e}\right) P_\theta^{<3+r>}(\mathbf{x}) b^{-2} \\ &\quad + \int \left(\frac{\partial^2 L(\mathbf{x})}{\partial^2 x_e}\right) P_\theta^{<3+r>}(\mathbf{x}) b^{-2} dx_e \\ &= \ldots = \sum_{i=2}^{g} M_i(\mathbf{x}) P_\theta^{<i+r>}(\mathbf{x}),\end{aligned}$$

for finitely many polynomials $M_i(\mathbf{x}) \in B_![\mathbf{x}]$. Since $P_\theta^{<i+r>}(\mathbf{x})$ is in $I(X,Y,\omega)$ for all $i \geq 2$, this suffices to complete the proof. $\square$

Notice that if B is a $\mathbb{Q}$-algebra then $B_![\mathbf{x}] = B[\mathbf{x}]$, so that Theorem 4.8 presents $\Phi.(X,B)$ as a quotient of a polynomial algebra in this case. For example, $\Phi.(\mathsf{K}_4,\mathbb{Q})$ is isomorphic to the quotient of $\mathbb{Q}[x,y,z]$ by the ideal

$$(x^4, y^4, z^4, (x+y)^5, (x+z)^5, (y+z)^5, (x+y+z)^4)$$

(and has Poincaré polynomial $D_{\mathsf{K}_4}(t) = 1 + 3t + 6t^2 + 10t^3 + 11t^4 + 6t^5 + t^6$).

Corollary 4.9. *Let X be a graph with n vertices, m edges, ℓ cut-edges, k connected components, and with shortest cycle of length g, and let $D_X(t) = d_0 + d_1 t + \cdots + d_{m-\ell}t^{m-\ell}$. Then for all $0 \leq j \leq g$,*

$$d_j = \binom{m-n+k+j-1}{j}.$$

Proof. Adopt the notation (*) with $B = \mathbb{Q}$. The ideal $I(X,Y,\omega)$ of $\mathbb{Q}[\mathbf{x}]$ is zero in the graded components of degree $0 \leq j \leq g$, and so by Theorem 4.8, d_j is the dimension of the j-th graded component of $\mathbb{Q}[\mathbf{x}]$ for all $0 \leq j \leq g$, proving the result. $\square$

Theorem 4.10. *Let X be a graph with m edges and ℓ cut-edges, and let B be a commutative ring. Coordinatize $\Phi.(X,B)$ by some $\omega \in \Omega$. Let $\varphi \in \Phi_1(X,B)$ be such that $\varphi(X_e)$ is a non-zerodivisor in B for each non-cut-edge e of X. For any integer $0 \leq j \leq (m-\ell)/2$, if $\mathrm{char}(B)$ is zero or coprime to $\binom{m-\ell-j-i}{j-i}$ for all $0 \leq i \leq j$ then the homomorphism $\Phi_j(X,B) \to \Phi_{m-\ell-j}(X,B)$ defined by $\theta \mapsto \theta \cdot \varphi^{<m-\ell-2j>}$ is injective.*

Proof. Because of (the dual of) Theorem 1.2 and the fact that $\Phi.(\bullet\!\!-\!\!\bullet, B) = B$, it suffices to consider the case of a graph X with no cut-edges. Consider an integer $0 \leq j \leq m/2$. Let M be the matrix with rows indexed by $\mathcal{P}_{m-j}(X)$ and columns indexed by $\mathcal{P}_j(X)$, with $M_{\sigma,\rho} := 1$ if $\rho \subseteq \sigma$ and $M_{\sigma,\rho} := 0$ otherwise. Let P be the matrix which represents the homomorphism $\cdot\varphi^{<m-2j>} : \mathrm{Hom}(F^j(X),B) \to \mathrm{Hom}(F^{m-j}(X),B)$ with respect to the bases $\{\xi_\rho : \rho \in \mathcal{P}_j(X)\}$ and $\{\xi_\sigma : \sigma \in \mathcal{P}_{m-j}(X)\}$. That is, $P_{\sigma,\rho} := \prod_{e\in\sigma\smallsetminus\rho}\varphi(X_e)$ if $\rho \subseteq \sigma$, and $P_{\sigma,\rho} := 0$ otherwise. Let R be the diagonal matrix with entries $R_{\rho,\rho} := \prod_{e\in\rho}\varphi(X_e)$, and let S be the diagonal matrix with entries $S_{\sigma,\sigma} := \prod_{e\in\sigma}\varphi(X_e)$. One verifies that the matrix equation $PR = SM$ holds, and hence $\det(P)\det(R) = \det(S)\det(M)$. By the hypothesis on φ, both $\det(R)$ and $\det(S)$ are non-zerodivisors in B, and hence $\det(P)$ is a non-zerodivisor if and only if $\det(M)$ is. Thus, if $\det(M)$ is coprime to $\mathrm{char}(B)$ then $\cdot\varphi^{<m-2j>}$ is injective, and so its restriction to $\Phi_j(X,B)$ is also injective. From a theorem of Wilson [25] it follows that

$$\det(M) = \prod_{i=0}^{j}\binom{m-j-i}{j-i}^{\binom{m}{i}-\binom{m}{i-1}}$$

which suffices to complete the proof. $\square$

Corollary 4.11. *Let X be a graph with m edges and ℓ cut-edges, and let $D_X(t) = d_0 + d_1 t + \cdots + d_{m-\ell}t^{m-\ell}$. Then $1 = d_0 \leq d_1 \leq \cdots \leq d_{\lfloor (m-\ell)/2\rfloor}$ and $d_j \leq d_{m-\ell-j}$ for each $0 \leq j \leq (m-\ell)/2$.*

Proof. Apply Theorem 4.10 with $B = \mathbb{Q}$. With the notation (*), if $\mathcal{E} = \{e_1, ..., e_d\}$ then $\varphi := \sum_{i=1}^{d} 3^i \beta_{e_i}$ is a $\mathbb{Z}$-flow in X satisfying the hypothesis of Theorem 4.10. The inequalities $d_j \leq d_{m-\ell-j}$ are immediate. Since $\varphi^{<m-2j>} = \varphi^{m-2j}/(m-2j)!$ for all $0 \leq j \leq m/2$, it follows that $\cdot\varphi : \Phi_j(X, B) \to \Phi_{j+1}(X, B)$ is injective for all $0 \leq j \leq (m-\ell)/2$, implying the remaining inequalities. □

5. Integer flows and theta functions.

Finally, we determine the determinant and theta function of the lattice $\Phi_1(X, \mathbb{Z})$ of integer-valued flows on the graph X in terms of combinatorial data of X. Other results about this lattice (including the determinant) have been obtained by Bacher, de la Harpe, and Nagnibeda [2]; in particular, it is interesting to compare their description of the facets of the Voronoi polytope of $\Phi_1(X, \mathbb{Z})$ with the generators of the ideal $I(X, Y, \omega)$ described in Section 4.

For a graph X, define a nondegenerate symmetric positive definite bilinear form $\langle \cdot, \cdot \rangle$ on $\mathrm{Hom}(F^{\cdot}(X), \mathbb{R})$ by requiring that the basis $\{\xi_\sigma : \sigma \in \mathcal{P}(X)\}$ be orthonormal. That is,

$$\langle \varphi, \theta \rangle := \sum_{\sigma \in \mathcal{P}(X)} \varphi(X_\sigma)\theta(X_\sigma)$$

for all $\varphi, \theta \in \mathrm{Hom}(F^{\cdot}(X), \mathbb{R})$. This makes $\mathrm{Hom}(F^{\cdot}(X), \mathbb{R})$ into a Euclidean space. Since $\langle \cdot, \cdot \rangle$ is invariant under the automorphisms Ψ, it induces a well-defined Euclidean form on $\Phi.(X, \mathbb{R})$, also denoted by $\langle \cdot, \cdot \rangle$.

Lemma 5.1. *Let X be a graph.*
(a) *For any nonempty affine subspace U of* $\mathrm{Hom}(F^{\cdot}(X), \mathbb{R})$, *there is a unique $\varphi \in U$ with $\langle \varphi, \varphi \rangle$ minimum.*
(b) *If U is defined by equations with coefficients in $\mathbb{Q}$ (relative to the basis $\{\xi_\sigma : \sigma \in \mathcal{P}(X)\}$) then the φ from part* (a) *is in* $\mathrm{Hom}(F^{\cdot}(X), \mathbb{Q})$.
(c) *Coordinatize $\Phi.(X, \mathbb{R})$ by some $\omega \in \Omega$. Let $\sigma \in \mathcal{P}(X)$ contain no cut-edges of X. Then there is a unique $\varphi \in \Phi.(X, \mathbb{Q})$ such that $\varphi(X_\sigma) = 1$ and $\langle \varphi, \varphi \rangle$ is minimum subject to this condition.*
(d) *With notation as in* (c), *if $\#\sigma = j$ then $\varphi \in \Phi_j(X, \mathbb{Q})$.*

Proof. Part (a) is the fact that an affine subspace U of a Euclidean space contains a unique vector φ which is closest to the origin; indeed, if U_0 is the translate of U passing through $\mathbf{0}$ and $U_0^\perp$ is the orthogonal complement of U_0 then φ is the unique vector in $U_0^\perp \cap U$. If U is defined over $\mathbb{Q}$ then so is $U_0^\perp$, and it follows that φ has rational coordinates, proving (b). Part (c) follows from (a) and (b) since if σ contains no cut-edges of X then $U := \lambda_\omega(\Phi.(X, \mathbb{Q})) \cap \{\varphi \in \mathrm{Hom}(F^{\cdot}(X), \mathbb{Q}) : \varphi(X_\sigma) = 1\}$ is not empty, and part (d) is evident. □

Consider an arc a such that $\underline{a}$ is not a cut-edge of X, coordinatize $\Phi.(X, \mathbb{R})$ by some $\omega \in \Omega$ such that $s_\omega(a) = 1$, and let χ_a^ω be the element of $\Phi_1(X, \mathbb{Q})$ guaranteed by Lemma 5.1(c,d). One checks that if $\varepsilon \in \Omega$ is such that $s_\varepsilon(a) = 1$ then $\chi_a^\varepsilon = \chi_a^\omega \circ \psi_\omega^\varepsilon$. It follows that $\chi_a := \lambda_\omega^{-1}(\chi_a^\omega)$ does not depend on the choice of ω; this is the *characteristic flow of a in X*. It is easy to see that $\chi_{\overline{a}} = -\chi_a$.

Lemma 5.2. *Let X be a graph, let $a \in A$ be such that $e := \underline{a}$ is not a cut-edge, and let χ_a be the characteristic flow of a in X. Coordinatize $\Phi.(X, \mathbb{R})$ by some $\omega \in \Omega$ such that $s_\omega(a) = 1$.*
(a) *For any $\varphi \in \Phi_1(X, \mathbb{R})$, one has $\langle \varphi, \chi_a \rangle = \varphi(X_e)\langle \chi_a, \chi_a \rangle$.*

(b) *There is a "potential function"* $\nu_a : V \to \mathbb{Q}$ *such that* $\nu_a(t(a)) = 0$ *and for all* $c \in E \setminus \{e\}$, $\chi_a(X_c) = \nu_a(t(\omega(c))) - \nu_a(o(\omega(c)))$.
(c) *With* ν_a *from part* (b), *we have* $\langle \chi_a, \chi_a \rangle = 1 + \nu_a(o(a))$.

Proof. For part (a), first consider the case that $\varphi(X_e) = 0$, and suppose that $\langle \varphi, \chi_a \rangle \neq 0$. Replacing φ by $-\varphi$ if necessary, we may assume that $\langle \varphi, \chi_a \rangle > 0$. Let $\theta := \chi_a - \varepsilon\varphi$, with $\varepsilon > 0$ small. Now

$$\langle \theta, \theta \rangle = \langle \chi_a, \chi_a \rangle - 2\varepsilon \langle \varphi, \chi_a \rangle + \varepsilon^2 \langle \varphi, \varphi \rangle,$$

and as $\varepsilon \to 0$ the last term is negligible. Thus, for some $\varepsilon > 0$ we have $\langle \theta, \theta \rangle < \langle \chi_a, \chi_a \rangle$, but since $\theta \in \Phi_1(X, \mathbb{R})$ and $\theta(X_e) = 1$, this contradicts the definition of χ_a. Thus, $\langle \varphi, \chi_a \rangle = 0$, as claimed. For the general case of part (a), consider any $\varphi \in \Phi_1(X, \mathbb{R})$ and let $\varphi' := \varphi - \varphi(X_e)\chi_a$. Then $\varphi'(X_e) = 0$, so $\langle \varphi', \chi_a \rangle = 0$, so $\langle \varphi, \chi_a \rangle = \varphi(X_e) \langle \chi_a, \chi_a \rangle$, as claimed.

For part (b), consider any $v \in V$, and let $P := (a_1, ..., a_h)$ be a sequence of arcs in $A \setminus \{a, \overline{a}\}$ such that $o(a_1) = t(a)$, $t(a_h) = v$, and $t(a_i) = o(a_{i+1})$ for all $1 \leq i \leq h-1$. Define $\psi_P \in \mathrm{Hom}(F^{\cdot}(X), \mathbb{Q})$ by $\psi_P := \sum_{i=1}^{h} s_\omega(a_i)\xi_{\underline{a}_i}$, and let $\nu_a(v) := \langle \psi_P, \chi_a \rangle$. To check that this is well-defined, let $Q := (b_1, ..., b_g)$ be another walk in $X \setminus \{e\}$ from $t(a)$ to v, and define ψ_Q as for ψ_P. Then $\psi_P - \psi_Q$ is in $\Phi_1(X, \mathbb{Q})$ and $(\psi_P - \psi_Q)(X_e) = 0$, so $\langle \psi_P, \chi_a \rangle = \langle \psi_Q, \chi_a \rangle$ from part (a), as required. It is clear that $\nu_a(t(a)) = 0$. Now, for any edge $c \in E \setminus \{e\}$, let P be a walk in $X \setminus \{e\}$ from $t(a)$ to $o(\omega(c))$, and let $Q := (P, \omega(c))$ be the concatenation of P with $\omega(c)$. Then $\chi_a(X_c) = \langle (\psi_Q - \psi_P), \chi_a \rangle = \nu_a(t(\omega(c))) - \nu_a(o(\omega(c)))$.

For part (c), let C be any cycle in X containing e, and let $\theta \in \Phi_1(C, \mathbb{Q}) \subseteq \Phi_1(X, \mathbb{Q})$ be such that $\theta(X_e) = 1$. Then $\langle \theta, \chi_a \rangle = \langle \chi_a, \chi_a \rangle$, by part (a). But since the support of θ is the cycle C containing e, it is easy to see that $\langle \theta, \chi_a \rangle = 1 + \nu_a(o(a))$, from part (b). ☐

Lemma 5.2 can be interpreted physically by imagining the graph X as an electrical network in which each edge has unit resistance. When a unit current is forced through the edge e in the direction of a, the principle of least action ensures that the current flowing in each edge of X is given by χ_a. The electric potential at each vertex is given by $-\nu_a$.

For a graph X let $\varkappa(X)$ denote the number of maximal forests of X, often called the *complexity* of X.

Proposition 5.3. *Let* X *be a graph, let* $a \in A$ *be such that* $e := \underline{a}$ *is not a cut-edge, and coordinatize* $\Phi_\cdot(X, \mathbb{R})$ *by some* $\omega \in \Omega$ *such that* $s_\omega(a) = 1$. *Then* $\langle \chi_a, \chi_a \rangle = \varkappa(X)/\varkappa(X \setminus \{e\})$.

Proof. Let $T(X)$ denote the set of maximal forests of X, and notice that $T(X \setminus \{e\}) \subset T(X)$ since e is not a cut-edge of X. Let $\mathbb{Q}T(X)$ denote the $\mathbb{Q}$-vector space with basis $T(X)$. Define $L : \mathbb{Q}T(X \setminus \{e\}) \to \mathbb{Q}T(X)$ as follows. For a given $Y \in T(X \setminus \{e\})$ there is a unique directed path $P = (b_1, ..., b_h)$ in Y from $t(a)$ to $o(a)$. Define

$$L(Y) := Y + \sum_{i=1}^{h} s_\omega(b_i)\chi_a(X_{\underline{b}_i})(Y \cup \{e\} \setminus \{\underline{b}_i\})$$

(and extend the definition $\mathbb{Q}$-linearly). Now consider

$$\sum_{Y \in T(X \setminus \{e\})} L(Y) = \sum_{Z \in T(X)} c_Z Z,$$

for some coefficients $c_Z \in \mathbb{Q}$. If $Z \in T(X \smallsetminus \{e\})$ then clearly $c_Z = 1$. On the other hand, if $e \in E(Z)$ then $c_Z = \sum_b s_\omega(b)\chi_a(X_{\underline{b}})$, in which the sum is over all arcs $b \notin e$ such that $o(b)$ is in the component of $Z \smallsetminus \{e\}$ containing $t(a)$, and $t(b)$ is in the component of $Z \smallsetminus \{e\}$ containing $o(a)$. By Lemma 1.1, and since χ_a is a flow, it follows that $c_Z - \chi_a(X_e) = 0$. Thus we have

$$\sum_{Y \in T(X \smallsetminus \{e\})} L(Y) = \sum_{Z \in T(X)} Z.$$

Now apply the linear functional on $\mathbb{Q}T(X)$ defined by $Z \mapsto 1$ for all $Z \in T(X)$. For $Y \in T(X \smallsetminus \{e\})$ with path $P = (b_1, ..., b_h)$ as above, Lemma 5.2 implies that

$$\begin{aligned} L(Y) \quad &\mapsto \quad 1 + \sum_{i=1}^{h} s_\omega(b_i)\chi_a(X_{\underline{b}_i}) \\ &= \quad 1 + \sum_{i=1}^{h} s_\omega(b_i)(\nu_a(t(b_i)) - \nu_a(o(b_i))) \\ &= \quad 1 + \nu_a(o(a)) = \langle \chi_a, \chi_a \rangle. \end{aligned}$$

Therefore $\langle \chi_a, \chi_a \rangle \varkappa(X \smallsetminus \{e\}) = \varkappa(X)$, completing the proof. $\square$

An (integral) *lattice* in a Euclidean space $(W, \langle \cdot, \cdot \rangle)$ is a subgroup Λ of $(W, +)$ such that $\mathrm{span}_{\mathbb{R}}\Lambda = W$ and $\langle \cdot, \cdot \rangle$ is integer-valued when restricted to Λ; Conway and Sloane [10] is the canonical reference. For an ordered basis $(\lambda_1, ..., \lambda_d)$ of Λ, its *Gram matrix* is the d-by-d matrix G with entries $G_{i,j} := \langle \lambda_i, \lambda_j \rangle$. The *determinant of* Λ is $\det(\Lambda) := \det(G)$, and does not depend on the choice of ordered basis. (In fact, $\det(\Lambda)$ is the square of the volume of the compact group W/Λ with respect to the metric induced from $\langle \cdot, \cdot \rangle$.) The *theta function* of Λ is

$$\boldsymbol{\vartheta}(\Lambda|z) := \sum_{\lambda \in \Lambda} e^{\pi i \langle \lambda, \lambda \rangle z} = \sum_{\lambda \in \Lambda} q^{\langle \lambda, \lambda \rangle},$$

in which $q := e^{\pi i z}$. We also need theta functions of translates of a lattice Λ by a vector γ such that $\langle \gamma, \lambda \rangle \in \mathbb{Q}$ for all $\lambda \in \Lambda$, which merely requires the use of rational exponents of q (with a bounded denominator for any given γ). It is convenient to express the theta function of $\Phi_1(X, \mathbb{Z})$ in terms of the series

$$\boldsymbol{\psi}(\alpha|z) := \sum_{n \in \mathbb{Z}} q^{(n+\alpha)^2}$$

(which converge absolutely for $\alpha \in \mathbb{R}$ and $z \in \mathbb{C}$ with $\mathrm{Im}(z) > 0$, uniformly on any compact subset). In terms of the Jacobi theta functions

$$\vartheta_3(\xi|z) := \sum_{n \in \mathbb{Z}} e^{2i\xi n + \pi i n^2 z}$$

we have $\boldsymbol{\psi}(\alpha|z) = e^{\pi i \alpha^2 z}\vartheta_3(\pi\alpha z|z)$. The Jacobi triple product formula (see equation (32) in Section 4.4 of [10], or Theorem 2.8 of Andrews [1]) yields

$$\boldsymbol{\psi}(\alpha|z) = q^{\alpha^2} \prod_{n=1}^{\infty} \left(1 - q^{2n}\right) \left(1 + q^{2n-1+2\alpha}\right) \left(1 + q^{2n-1-2\alpha}\right).$$

For a graph X and edge $e \in E$, we define the *index of* e *in* X to be the least positive integer r such that $r\chi_a \in \Phi_1(X, \mathbb{Z})$, in which χ_a is the characteristic flow of $a \in e$ in X (since $\chi_{\overline{a}} = -\chi_a$, the choice of $a \in e$ does not matter).

Theorem 5.4. *Let X be a graph, and adopt the notation (*) with $B = \mathbb{Z}$. Let $\mathcal{E} = \{e_1, ..., e_d\}$, and for $1 \leq i \leq d$ let $X_i := X \smallsetminus \{e_1, ..., e_{i-1}\}$ and let χ_i be the characteristic flow of $\omega(e_i)$ in $X_i \subseteq X$.*

(a) *Then $\{\chi_1, ..., \chi_d\}$ are pairwise orthogonal in $\Phi_1(X, \mathbb{Q})$.*

(b) *The determinant of $\Phi_1(X, \mathbb{Z})$ is $\varkappa(X)$.*

(c) *For $1 \leq i \leq d$ let r_i be the index of e_i in X_i, and let $\varphi_i := r_i\chi_i$. Then*

$$\mathcal{S} := \left\{ \sum_{i=1}^{d} g_i\beta_i : \ 0 \leq g_i < r_i, \ g_i \in \mathbb{Z} \right\}$$

is a system of representatives for the cosets of the subgroup Γ of $\Phi_1(X, \mathbb{Z})$ generated by $\{\varphi_1, ..., \varphi_d\}$.

(d) *For $1 \leq i \leq d$ let $w_i := \langle \varphi_i, \varphi_i \rangle$. The theta function of $\Phi_1(X, \mathbb{Z})$ is*

$$\boldsymbol{\vartheta}(\Phi_1(X, \mathbb{Z})|z) = \sum_{\lambda \in \mathcal{S}} \prod_{i=1}^{d} \boldsymbol{\psi} \left(\frac{\langle \lambda, \varphi_i \rangle}{w_i} \middle| w_i z \right).$$

Proof. For $1 \leq h < i \leq d$ we have $\chi_i \in \Phi_1(X_h, \mathbb{Z})$ and $\chi_i((X_h)_{e_h}) = 0$, so by Lemma 5.2(a), $\langle \chi_i, \chi_h \rangle = 0$, proving part (a).

For part (b), let P be the m-by-d matrix in which the i-th column is the coordinate vector of χ_i with respect to the basis $\{\xi_e : e \in \mathcal{P}_1(X)\}$ of $\mathrm{Hom}(F^1(X), \mathbb{Q})$. Let Q be the m-by-d matrix in which the i-th column is the coordinate vector of β_i with respect to the same basis. By Theorem 4.4, $(\beta_1, ..., \beta_d)$ is an ordered basis for $\Phi_1(X, \mathbb{Z})$, with Gram matrix $Q^\top Q$. For each $1 \leq i \leq d$ we have $\chi_i = \sum_{h=1}^{d} \chi_i(X_{e_h})\beta_h$; denoting by M the d-by-d matrix such that $M_{h,i} := \chi_i(X_{e_h})$ we have $P = QM$. By construction of $(\chi_1, ..., \chi_d)$, M is lower triangular and $M_{i,i} = 1$ for all $1 \leq i \leq d$. Therefore, $\det(M) = 1$ and hence

$$\det(\Phi_1(X, \mathbb{Z})) = \det(Q^\top Q) = \det((M^{-1})^\top P^\top P M^{-1}) = \det(P^\top P).$$

But part (a) shows that $P^\top P$ is diagonal, and by Proposition 5.3,

$$\det(P^\top P) = \prod_{i=1}^{d} \frac{\varkappa(X_i)}{\varkappa(X_{i+1})} = \varkappa(X),$$

in which we have defined $X_{d+1} := Y$, so that $\varkappa(X_{d+1}) = 1$.

For part (c), every $\theta \in \Phi_1(X, \mathbb{Z})$ may be written as $\theta = \sum_{i=1}^{d} f_i\beta_i$ for unique integers $f_1, ..., f_d$ (in fact, $f_i = \theta(X_{e_i})$). By the division algorithm we write $f_1 = h_1r_1 + g_1$ with $0 \leq g_1 < r_1$, and consider $\theta_2 := \theta - h_1\varphi_1$. Inductively, for $2 \leq i \leq d$ we write $f_i = h_ir_i + g_i$ with $0 \leq g_i < r_i$, and consider $\theta_{i+1} := \theta_i - h_i\varphi_i$. Since $\varphi_i(X_{e_h}) = 0$ if $h < i$, it follows that $\theta_{d+1} \in \mathcal{S}$, the candidate set of representatives. Since $\theta = h_1\varphi_1 + \cdots + h_d\varphi_d + \theta_{d+1}$, this shows that every $\mathbb{Z}$-flow in X is congruent modulo Γ to some flow in $\mathcal{S}$. Conversely, assume that $\theta, \theta' \in \mathcal{S}$ are congruent modulo Γ, and suppose that $\theta \neq \theta'$. Let i be the least index such that $\theta(X_{e_i}) \neq \theta'(X_{e_i})$. From the definition of $\mathcal{S}$ it follows that $1 - r_i \leq (\theta - \theta')(X_{e_i}) \leq r_i - 1$. But since $(\theta - \theta')(X_{e_h}) = 0$ for all $0 \leq h < i$ and $\theta - \theta'$ is assumed to be in Γ, it follows that $(\theta - \theta')(X_{e_i})$ is a multiple of r_i. Therefore, $\theta(X_{e_i}) = \theta'(X_{e_i})$, a contradiction. This proves part (c).

For part (d), first notice that for $\alpha \in \mathbb{Q}$ and $c \in \mathbb{R}$ such that $c^2 \in \mathbb{Z}$, the theta function of $c\alpha + c\mathbb{Z}$ is $\boldsymbol{\psi}(\alpha|c^2 z)$. Secondly, for translates of lattices $\lambda_1 + \Lambda_1$ and

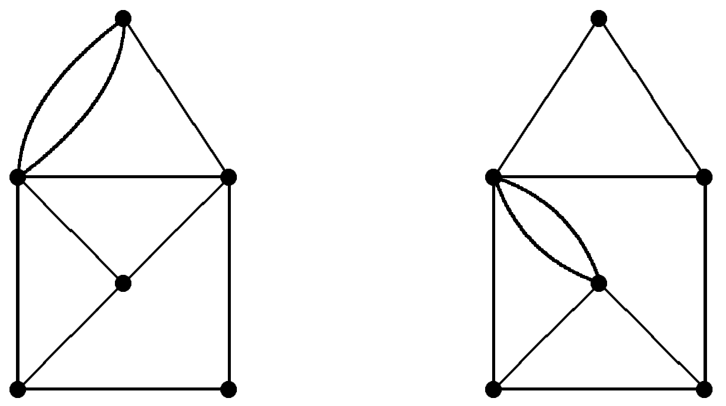

FIGURE 1. A pair of nontrivially codichromatic graphs.

$\lambda_2 + \Lambda_2$ in orthogonal Euclidean spaces, the Pythagorean theorem implies that

$$\boldsymbol{\vartheta}((\lambda_1 \oplus \lambda_2) + (\Lambda_1 \oplus \Lambda_2)|z) = \boldsymbol{\vartheta}(\lambda_1 + \Lambda_1|z)\boldsymbol{\vartheta}(\lambda_2 + \Lambda_2|z).$$

Now, since the cosets of Γ partition $\Phi_1(X, \mathbb{Z})$ we have

$$\boldsymbol{\vartheta}(\Phi_1(X, \mathbb{Z})|z) = \sum_{\lambda \in S} \boldsymbol{\vartheta}(\lambda + \Gamma|z).$$

Since $\{\varphi_1, ..., \varphi_d\}$ are pairwise orthogonal, for each $\lambda \in S$ we have an orthogonal sum

$$\lambda + \Gamma = \bigoplus_{i=1}^{d} \left(\frac{\langle \lambda, \varphi_i \rangle}{w_i} + \mathbb{Z} \right) \varphi_i,$$

since $w_i^{-1}\langle \lambda, \varphi_i \rangle \varphi_i$ is the projection of λ onto the line spanned by φ_i. From the above remarks it is clear that

$$\boldsymbol{\vartheta}(\lambda + \Gamma|z) = \prod_{i=1}^{d} \boldsymbol{\psi} \left(\left. \frac{\langle \lambda, \varphi_i \rangle}{w_i} \right| w_i z \right),$$

completing the proof. □

Theorem 5.4(b) follows from Theorems 2 and 3 of Biggs [4], and also appears as Proposition 1(iii) of Bacher, de la Harpe, and Nagnibeda [2].

The two graphs depicted in Figure 1 are *nontrivially codichromatic*: they have identical Tutte polynomials but nonisomorphic graphic matroids. (This example is due to M.C. Gray; see Tutte [21].) Thus, $\Phi.(X_1, \mathbb{Z})$ and $\Phi.(X_2, \mathbb{Z})$ are isomorphic as graded abelian groups. However, using the Euclidean forms $\langle \cdot, \cdot \rangle$ these groups are not isometric. Indeed, an easy case analysis shows that X_1 (on the left) has 20 integer-valued flows of squared-norm 7 while X_2 has 22 such flows; therefore $\boldsymbol{\vartheta}(\Phi_1(X_1, \mathbb{Z})|z) \neq \boldsymbol{\vartheta}(\Phi_1(X_2, \mathbb{Z})|z)$.

6. Open Problems.

Problem 6.1. Can one identify a basis for $\Phi_j(X, \mathbb{Q})$ when $j > 1$? With the notation of Theorem 4.4 it would be especially interesting to find a set $\mathcal{J}$ of functions $\mathbf{j} : \mathcal{E} \to \mathbb{N}$ such that the monomials $\{\beta^{\mathbf{j}} : \mathbf{j} \in \mathcal{J}\}$ form a basis for $\Phi.(X, \mathbb{Q})$ which is closed under the relation of divisibility. One hint is that the dimension of $\Phi.(X, \mathbb{Q})$ as a $\mathbb{Q}$-vector space is $D_X(1) = T_X(1, 2)$, which is the number of spanning subgraphs of X with the same number of connected components as X (see, *e.g.* Proposition 6.2.11(iii) of Brylawski and Oxley [8]). If such a basis can be used to obtain results analogous to those in Section 5 for $\Phi_j(X, \mathbb{Z})$ with $j > 1$ then so much the better.

Problem 6.2. For $i, j \geq 0$ with $i + j \leq m - \ell$, $\Phi_i(X, \mathbb{Z}) \cdot \Phi_j(X, \mathbb{Z})$ generates a subgroup of $\Phi_{i+j}(X, \mathbb{Z})$; denote the quotient by $T_{i,j}(X)$. Corollary 4.5 implies that $T_{i,j}(X) \otimes \mathbb{Q} = 0$, so $T_{i,j}(X)$ is a finite abelian group. Can one determine the structure of $T_{i,j}(X)$, or even just its order? (For the n-cycle C_n we have $T_{i,j}(\mathsf{C}_n) = \mathbb{Z}/\binom{i+j}{i}\mathbb{Z}$ for all $i, j \geq 0$ with $i + j \leq n$.) Can these groups be used to show that the ring structure of $\Phi.(\cdot, \mathbb{Z})$ distinguishes between pairs of nontrivially codichromatic graphs as in the example at the end of Section 5?

Problem 6.3. Circulation algebras of graphs have some formal similarities with cohomology rings. Is there a contravariant functor $X \mapsto \mathcal{X}$ from the category of graphs to the category of topological spaces so that $\Phi.(X, \mathbb{Z})$ is the cohomology ring of $\mathcal{X}$? Alternatively, can $\Phi.(X, \mathbb{Q})$ be interpreted as the rational Chow ring of some algebraic variety associated to X? (This possibility is suggested by analogy with the application of Fulton and Sturmfels' presentation [13] of the Chow ring of a toric variety to the varieties associated with partial orders in [22].) It would even be interesting just to identify an abstract cochain complex with cohomology $\Phi.(X, \mathbb{Z})$.

Problem 6.4. Let X and Y be two graphs with nonempty intersection. Can one give an "excision" exact sequence (or spectral sequence) for computing $K^{\cdot}(X \cup Y)$ in terms of $K^{\cdot}(X)$, $K^{\cdot}(Y)$, and $K^{\cdot}(X \cap Y)$? A solution to Problem 6.3 would provide a big hint for this.

Problem 6.5. Quantum deformation of rational Chow rings of homogeneous spaces is a hot topic in algebraic geometry these days (see, *e.g.*, Fulton and Pandharipande [12]). Even without a solution to Problem 6.3, one could attempt to define a quantum deformation of a circulation algebra purely combinatorially. Are there combinatorial analogues of Gromov-Witten invariants in this context? If so, then what do they mean?

Problem 6.6. Can one describe the minimal free resolution of $\Phi.(X, \mathbb{Q})$ in terms of the combinatorial structure of X? In particular, is there a reasonable formula for the dimension of the j-th graded component of the i-th resolvent of $\Phi.(X, \mathbb{Q})$? As a small first step, can one identify a minimal set of generators for the ideal $I(X, Y, \omega)$ in Theorem 4.8?

Problem 6.7. Can one use circulation algebras to give a proof of the nowhere-zero 5-flow conjecture (see Seymour [17])? This is rather unlikely, as a detailed combinatorial argument seems inevitable, but here is a related question. Let $\varphi \in \Phi_1(X, \mathbb{Z})$ be such that $\mathrm{supp}(\varphi) = E$ and $\langle \varphi, \varphi \rangle$ is minimum with respect to this condition. Can one give an upper bound on $|\varphi(X_e)|$ independent of e and X? (We have tacitly coordinatized $\Phi.(X, \mathbb{Z})$ by some orientation.) In particular, can one prove that $|\varphi(X_e)| < 5$? This would imply the 5-flow conjecture.

Problem 6.8. How much of the theory of Kirchhoff groups and circulation algebras can be extended from graphs to more general matroids? It seems likely that it will all work for regular (unimodular) matroids, and perhaps even for all orientable matroids. For a matroid represented over a field, the rowspace of the representing matrix is a natural analogue of $N^1(X)$, but this depends on the representation. Can one find for a representable matroid an analogue of the Kirchhoff group which does not depend on a choice of representation? Can anything be said for nonrepresentable matroids?

Problem 6.9. The previous problem brings to mind the matroid-theoretic duality between contraction and deletion. Contraction is ubiquitous in the above theory, but deletion is used much less. Can one make use of deletion to enrich this theory? One natural approach is to associate to a graph X not just $\Phi.(X, B)$, but the whole family $\{\Phi.(Y, B) : Y \subseteq X\}$ as Y ranges over the spanning subgraphs of X. Because of the algebra homomorphisms induced by graph morphisms, this is a directed set of algebras and behaves functorially. Does it have any interesting or useful structure?

Problem 6.10. Many specializations of the Tutte polynomial of a matroid are conjectured to have a logarithmically concave sequence of coefficients. Computations of hundreds of examples suggest that the Poincaré polynomials of graphs also share this property. That is, for a graph X with m edges and ℓ cut-edges, can one prove that $d_j(X)^2 \geq d_{j-1}(X)d_{j+1}(X)$ for all $1 \leq j \leq m - \ell - 1$?

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