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ROBINSON-SCHENSTED-KNUTH INSERTION AND CHARACTERS OF CYCLOTOMIC HECKE ALGEBRAS

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Dedicated to Robert Gaines, 1941–2001.

0. Introduction

The cyclotomic Hecke algebras $H_{n,r} = H_n(u_1, \dots, u_r; q)$ were defined by Ariki and Koike in [AK] as Iwahori-Hecke algebras of the complex reflection group $G_{n,r} = S_n \wr (\mathbb{Z}/r\mathbb{Z})^n$ where S_n is the symmetric group. If ζ is a primitive complex rth root of unity, then when $q \to 1$ and $u_i \to \zeta^i$, the algebra $H_{n,r}$ specializes to the group algebra $\mathbb{C}[G_{n,r}]$. The irreducible representations of $H_{n,r}$ are constructed in [AK]. They are indexed by the set of all r-tuples of partitions with a total of n boxes, called r-partitions.

For each r-partition $\boldsymbol{\mu}$, T. Shoji [Sho] defines a symmetric function $q_{\boldsymbol{\mu}}$ and proves that

$$q_{\boldsymbol{\mu}} = \sum_{\boldsymbol{\lambda}} \chi_q^{\boldsymbol{\lambda}}(a_{\boldsymbol{\mu}}) s_{\boldsymbol{\lambda}},$$

where $s_{\boldsymbol{\lambda}}$ is the Schur function associated to the r-partition $\boldsymbol{\lambda}$ and $\chi_q^{\boldsymbol{\lambda}}(a_{\boldsymbol{\mu}})$ is the irreducible $H_{n,r}$-character associated to $\boldsymbol{\lambda}$ and evaluated at an element $a_{\boldsymbol{\mu}}$. The function $q_{\boldsymbol{\mu}}$ is a deformation of the power sum symmetric function, and Shoji's formula is analogous to the Frobenius formula for symmetric group characters. Shoji proves it using the Schur-Weyl duality for $H_{n,r}$ found in [SS].

In this paper we derive the formula

$$q_{\boldsymbol{\mu}} = \sum_{\boldsymbol{\lambda}} \left(\sum_{Q_{\boldsymbol{\lambda}}} \mathrm{wt}_{\boldsymbol{\mu}}(Q_{\boldsymbol{\lambda}}) \right) s_{\boldsymbol{\lambda}},$$

where $Q_{\boldsymbol{\lambda}}$ ranges over the set of "standard tableaux" of shape $\boldsymbol{\lambda}$, and where $\mathrm{wt}_{\boldsymbol{\mu}}$ is a weight on standard tableaux that depends on the parameters q and u_i and that is computed combinatorially. By comparing coefficients of $s_{\boldsymbol{\lambda}}$ in these two formulas we obtain the expression

$$\chi_q^{\boldsymbol{\lambda}}(a_{\boldsymbol{\mu}}) = \sum_{Q_{\boldsymbol{\lambda}}} \mathrm{wt}_{\boldsymbol{\mu}}(Q_{\boldsymbol{\lambda}})$$

which computes the irreducible $H_{n,r}$-characters as a sum over standard tableaux. When $q = 1$ and $u_i = \zeta^i$ our character formula specializes to a character formula for the complex reflection group $G_{n,r}$.

In the special case where $r = 1$, the cyclotomic Hecke algebra $H_{n,1}$ is the Iwahori-Hecke algebra $H_n(q)$ of type A_{n-1} associated with the symmetric group S_n. Shoji's Frobenius formula specializes, in this case, to the Frobenius formula of A. Ram [Ra1] for $H_n(q)$ and our character formula is a generalization of the Roichman formula [Ro] for irreducible characters of $H_n(q)$ and S_n.

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Our method is to follow the work of Ram [Ra2] who gives a new proof of the Roichman formula for $H_n(q)$ using Robinson-Schensted-Knuth insertion. We write the function $q_{\boldsymbol{\mu}}$ as a sum over $\boldsymbol{\mu}$-weighted integer sequences. We then use RSK insertion, modified for r-partitions, to turn this into a sum over pairs (P, Q) where P is a column-strict tableau, Q is a standard tableau, and P and Q have the same shape $\boldsymbol{\lambda}$ for some r-partition $\boldsymbol{\lambda}$. As a special case of our insertion rule we obtain a bijective proof of the formula

$$n!r^n = \sum_{\boldsymbol{\lambda}} f_{\boldsymbol{\lambda}}^2$$

where $n!r^n = |G_{n,r}|$ and $f_{\boldsymbol{\lambda}}$ is the number of standard tableau whose shape is the r-partition $\boldsymbol{\lambda}$. This fact can be proved algebraically by decomponsing the regular representation of $G_{n,r}$ into irreducibles and comparing dimensions.

A Murnaghan-Nakayama type rule for the characters of $H_{n,r}$ is found in [HR]. It gives the irreducible characters of $H_{n,r}$ as weighted sums over broken-border-strip tableaux. The characters $\chi_q^{\boldsymbol{\lambda}}(a_{\boldsymbol{\mu}})$ found in Shoji's frobenius formula and in this paper are evaluated on a set $\{a_{\boldsymbol{\mu}}\}$ of elements in $H_{n,r}$ for which characters are completely determined. The character values found in [HR] are evaluated on different elements $T_{\boldsymbol{\mu}}$.

1. Cyclotomic Hecke Algebras

Let $u_1, \ldots, u_r$ and q be indeterminates. The *cyclotomic Hecke algebra* $H_{n,r} = H_n(u_1, \ldots, u_r; q)$ is the algebra over $\mathbb{C}(q, u_1, \ldots, u_r)$ defined by generators X_1, $T_1, \ldots, T_{n-1}$, and relations

$$\begin{array}{lll}
(1) & T_i^2 = (q - q^{-1})T_i + 1, & 1 \leq i \leq n-1, \\
(2) & T_iT_j = T_jT_i, & |i-j| > 1, \\
(3) & T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, & 1 \leq i \leq n-2, \\
(4) & X_1T_1X_1T_1 = T_1X_1T_1X_1, & \\
(5) & (X_1 - u_1)(X_1 - u_2)\cdots(X_1 - u_r) = 0. &
\end{array}$$

These algebras were introduced by Ariki and Koike [AK], and they are semisimple over $\mathbb{C}(q, u_1, \ldots, u_r)$.

Let S_n be the symmetric group on n letters, and let $G_{n,r} = S_n \wr (\mathbb{Z}/r\mathbb{Z})^n$. The group $G_{n,r}$ has a presentation on generators $t_1, s_1, \ldots, s_{n-1}$ where $t_1^r = 1$ and $s_1, \ldots, s_{n-1}$ are the simple transpositions in S_n. If we let

$$q \to 1, \quad u_i \to \zeta^i \ (1 \leq i \leq r), \quad T_i \to s_i \ (1 \leq i \leq n-1), \quad \text{and} \quad X_1 \to t_1,$$

where

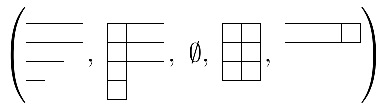
$$\zeta = \text{ a primitive } r\text{th root of unity in } \mathbb{C},$$

then the presentation for $H_{n,r}$ above becomes a presentation for $\mathbb{C}[G_{n,r}]$.

1.1. *r*-partitions.

We use the usual notation for partitions found in [Mac]. We identify a a partition with its Young diagram, let $\ell(\lambda)$ denote the number of rows of λ, and $|\lambda|$ denote the number of boxes in λ. For example, $\lambda = (5, 5, 3, 1, 1)$ has $\ell(\lambda) = 5$ and $|\lambda| = 15$.

An r-tuple of partitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ is called an *r-partition*. We refer to the $\lambda^{(k)}$ as the components of $\boldsymbol{\lambda}$. We let $|\boldsymbol{\lambda}| = \sum_{k=1}^r |\lambda^{(k)}|$ denote the total number of boxes in $\boldsymbol{\lambda}$, and we let $\ell(\boldsymbol{\lambda}) = \sum_{k=1}^{r} \ell(\lambda^{(k)})$ denote the total number of rows in $\boldsymbol{\lambda}$. If $|\boldsymbol{\lambda}| = n$, then we say that $\boldsymbol{\lambda}$ is an r-partition of n. For example, if $r = 5$, then

$\boldsymbol{\lambda} =$  has $\ell(\boldsymbol{\lambda}) = 11$ and $|\boldsymbol{\lambda}| = 24$,

and, for example, $\lambda^{(2)} = (3,3,1,1)$. We let $\mathcal{P}_{n,r}$ denote the set of all r-partitions of n.

1.2. **Irreducible Representations and Characters.**

It is known by [AK] that the irreducible representations of $H_{n,r}$ are indexed by $\mathcal{P}_{n,r}$. We let $V_q^{\boldsymbol{\lambda}}$ denote the irreducible $H_{n,r}$-module corresponding to $\boldsymbol{\lambda} \in \mathcal{P}_{n,r}$, and we let $\chi_q^{\boldsymbol{\lambda}}$ denote the corresponding irreducible character. The irreducible representations and characters of $G_{n,r}$ are also indexed by $\mathcal{P}_{n,r}$. We denote them by $V_1^{\boldsymbol{\lambda}}$ and $\chi_1^{\boldsymbol{\lambda}}$. The construction of $V_q^{\boldsymbol{\lambda}}$ in [AK] is such that when $q = 1$ and $u_i = \xi^i$, $V_q^{\boldsymbol{\lambda}}$ becomes $V_1^{\boldsymbol{\lambda}}$ and $\chi_q^{\boldsymbol{\lambda}}$ becomes $\chi_1^{\boldsymbol{\lambda}}$.

1.3. **Standard Elements.**

The conjugacy classes of $G_{n,r}$ are also parameterized by $\mathcal{P}_{n,r}$. Define $t_k = s_{k-1} \cdots s_1 t_1 s_1 \cdots s_{k-1}$ for $2 \le k \le n$, and define

$$w(1,i) = t_1^i \qquad \text{and} \qquad w(k,i) = t_k^i s_{k-1} \cdots s_1, \qquad 2 \le k \le n.$$

For a partition $\mu = (\mu_1, \dots, \mu_\ell)$ with $|\mu| = n$, define

$$w(\mu, i) = w(\mu_1, i) \times \cdots \times w(\mu_\ell, i)$$

with respect to the embedding $G_{\mu_1,r} \times \cdots \times G_{\mu_\ell,r} \subseteq G_{n,r}$. For $\boldsymbol{\mu} \in \mathcal{P}_{n,r}$, define

$$w_{\boldsymbol{\mu}} = w(\mu^{(1)}, 1) w(\mu^{(2)}, 2) \cdots w(\mu^{(r)}, r). \tag{1.1}$$

Then $\{w_{\boldsymbol{\mu}} | \boldsymbol{\mu} \in \mathcal{P}_{n,r}\}$ is a set of conjugacy class representatives for $G_{n,r}$.

Shoji ([Sho], §3.6) defines elements $\xi_1, \dots, \xi_n \in H_{n,r}$ and shows that $H_{n,r}$ is isomorphic to the algebra generated by $T_1, \dots, T_{n-1}, \xi_1, \dots, \xi_n$ subject to

(1) $T_i^2 = (q - q^{-1})T_i + 1$, $\quad 1 \le i \le n-1$,
(2) $T_i T_j = T_j T_i$, $\quad |i-j| > 1$,
(3) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, $\quad 1 \le i \le n-2$,
(4) $\xi_i \xi_j = \xi_j \xi_i$, $\quad 1 \le i, j \le n$,
(5) $(\xi_i - u_1)(\xi_i - u_2) \cdots (\xi_i - u_r) = 0$, $\quad 1 \le i \le n$,
(6) $T_j \xi_j = \xi_{j-1} T_j + \Delta^{-2} \sum_{k<\ell} (u_\ell - u_k)(q - q^{-1}) F_k(\xi_{j-1}) F_\ell(\xi_j)$,
(7) $T_j \xi_{j-1} = \xi_j T_j - \Delta^{-2} \sum_{k<\ell} (u_\ell - u_k)(q - q^{-1}) F_k(\xi_{j-1}) F_\ell(\xi_j)$,
(8) $T_i \xi_j = \xi_j T_i$, $\quad j \ne i-1, i$,

where $\Delta = \prod_{k<\ell} (u_\ell - u_k)$ is the determinant of the $r \times r$ Vandermonde matrix A, whose ℓ, k-entry is u_k^ℓ for $0 \le \ell \le r-1, 1 \le k \le r$, and

$$F_k(\xi_j) = \sum_{i=0}^{r-1} h_{ki}(u_1, \dots, u_r) \xi_j^i,$$

where $h_{ki}(u_1, \dots, u_r)$ is the k, i-entry of the matrix B determined by $A^{-1} = \Delta^{-1} B$. Unfortunately, it appears that the relation between the ξ_i and the X_j is complicated.

Define

$$a(1,i)=\xi_1^i \qquad \text{and} \qquad a(k,i)=\xi_k^i T_{k-1}\cdots T_1, \qquad 2\le k\le n.$$

For a partition $\mu=(\mu_1,\ldots,\mu_\ell)$ with $|\mu|=n$, define

$$a(\mu,i)=a(\mu_1,i)\times\cdots\times a(\mu_\ell,i)$$

with respect to the embedding $H_{\mu_1,r}\otimes\cdots\otimes H_{\mu_\ell,r}\subseteq H_{n,r}$. For $\boldsymbol{\mu}\in\mathcal{P}_{n,r}$, define

$$a_{\boldsymbol{\mu}}=a(\mu^{(1)},1)a(\mu^{(2)},2)\cdots a(\mu^{(r)},r). \tag{1.2}$$

Shoji [Sho], Proposition 7.5, proves that any character of $H_{n,r}$ is completely determined by its value on the set $\{a_{\boldsymbol{\mu}}|\boldsymbol{\mu}\in\mathcal{P}_{n,r}\}$.

2. Symmetric Functions

In this section, we follow [Mac], Appendix B, and [Sho] and define symmetric functions indexed by r-partitions.

Let $m_1,\ldots,m_r$ be positive integers satisfying $m_k\ge n$ for each $1\le k\le r$, and let $m=\sum_{k=1}^r m_k$. We define a set $\mathbf{x}$ of m indeterminates as follows

$$\begin{aligned}\mathbf{x}^{(k)} &= \{x_1^{(k)},\ldots,x_{m_k}^{(k)}\}, \qquad 1\le k\le r,\\ \mathbf{x} &= \mathbf{x}^{(1)}\cup\cdots\cup\mathbf{x}^{(r)}.\end{aligned}$$

We say that the indeterminates in $\mathbf{x}^{(k)}$ are of *color* k, and we linearly order the indeterminates $\mathbf{x}=x_1^{(1)},\ldots,x_{m_r}^{(r)}$ by the rule,

$$x_i^{(k)}<x_j^{(\ell)} \qquad \text{if and only if} \qquad k<\ell \quad \text{or} \quad k=\ell \text{ and } i<j. \tag{2.1}$$

It is sometimes notationally convenient to identify the variables $\mathbf{x}=x_1^{(1)},\ldots,x_{m_r}^{(r)}$ with the variables $\mathbf{x}=x_1,\ldots,x_m$ as follows,

$$\begin{matrix} x_1, & x_2, & \ldots, & x_{m_1}, & x_{m_1+1}, & x_{m_1+2}, & \ldots\ldots, & x_m,\\ \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \updownarrow & & \updownarrow\\ x_1^{(1)}, & x_2^{(1)}, & \ldots, & x_{m_1}^{(1)}, & x_1^{(2)}, & x_2^{(2)}, & \ldots\ldots, & x_{m_r}^{(r)}.\end{matrix} \tag{2.2}$$

To do this explicitly, set $x_j=x_{j-d_j}^{(b(j))}$, with $d_j=\sum_{i=1}^{b(j)}m_i$, and we define a function

$$b(j)=k, \quad \text{where} \quad m_1+\ldots+m_k<j\le m_1+\ldots+m_{k+1}, \tag{2.3}$$

so that $b(j)$ gives the color of the indeterminate x_j. We will use these two notations interchangeably.

Recall from Section 1, that ζ is a primitive rth root of unity in $\mathbb{C}$. For integers $t\ge 1$ and $1\le i\le r$, let

$$p_t^{(i)}(\mathbf{x})=\sum_{j=1}^r \zeta^{ij}p_t(\mathbf{x}^{(j)}), \tag{2.4}$$

where $p_t(\mathbf{x}^{(j)})$ denotes the tth power sum symmetric function ([Mac], I§2) with respect to the variables $\mathbf{x}^{(j)}$. As a special case, we let $p_0^{(i)}(\mathbf{x})=1$ for each i. For $\boldsymbol{\mu}\in\mathcal{P}_{n,r}$ with $\boldsymbol{\mu}=(\mu^{(1)},\ldots,\mu^{(r)})$ and $\mu^{(k)}=(\mu_1^{(k)},\ldots,\mu_{\ell_k}^{(k)})$, define

$$p_{\boldsymbol{\mu}}(\mathbf{x})=\prod_{k=1}^r\prod_{j=1}^{\ell_k}p^{(k)}_{\mu_j^{(k)}}(\mathbf{x}). \tag{2.5}$$

Definition (2.5) is given in [Sho] and it is the complex conjugate of the definition of $p_{\boldsymbol{\mu}}$ given in [Mac].

Now we define the Schur function associated to $\boldsymbol{\lambda} \in \mathcal{P}_{n,r}$ by

$$s_{\boldsymbol{\lambda}}(\mathbf{x}) = \prod_{k=1}^{r} s_{\lambda^{(k)}}(\mathbf{x}^{(k)}), \tag{2.6}$$

where $s_{\lambda^{(k)}}(\mathbf{x}^{(k)})$ denotes the Schur function ([Mac], I§3) associated to the partition $\lambda^{(k)}$ with respect to the variables $\mathbf{x}^{(k)}$. If $\boldsymbol{\lambda} \in \mathcal{P}_{n,r}$, then a *column-strict tableau of shape* $\boldsymbol{\lambda}$ is a filling of the boxes of $\boldsymbol{\lambda}$ with integers such that for each k

1. $\lambda^{(k)}$ contains integers from the set $\{1, \ldots, m_k\}$,
2. the columns of $\lambda^{(k)}$ strictly increase from top to bottom, and
3. the rows of $\lambda^{(k)}$ weakly increase (do not decrease) from left to right.

For example,

$$\left(\begin{array}{|c|c|c|} \hline 1&1&1 \\ \hline 2&3& \\ \cline{1-2} 3&& \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|} \hline 1&1&2 \\ \hline 2&2&4 \\ \hline 3&& \\ \cline{1-1} 5&& \\ \cline{1-1} \end{array}, \emptyset, \begin{array}{|c|c|} \hline 1&4 \\ \hline 2&5 \\ \hline 3&6 \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline 1&3&3&3 \\ \hline \end{array} \right) \text{ is a column-strict tableau of shape } \boldsymbol{\lambda}.$$

For a column-strict tableau $P_{\boldsymbol{\lambda}}$ of shape $\boldsymbol{\lambda}$ we define

$$\mathbf{x}^{P_{\boldsymbol{\lambda}}} = \prod_{k=1}^{r} \prod_{j=1}^{m_k} (x_j^{(k)})^{m_{jk}(P_{\boldsymbol{\lambda}})}, \tag{2.7}$$

where $m_{jk}(P_{\boldsymbol{\lambda}})$ denotes the number of times that j appears in the kth component (i.e., $\lambda^{(k)}$) of $P_{\boldsymbol{\lambda}}$. It follows from [Mac] I.5.12 that

$$s_{\boldsymbol{\lambda}}(\mathbf{x}) = \sum_{P_{\boldsymbol{\lambda}}} \mathbf{x}^{P_{\boldsymbol{\lambda}}}, \tag{2.8}$$

where the sum is over all column-strict tableaux $P_{\boldsymbol{\lambda}}$ of shape $\boldsymbol{\lambda}$.

We now define a deformation of $p_{\boldsymbol{\mu}}$. Let $\mathbf{u}$ denote the parameters $u_1, \ldots, u_r$. For integers $t \geq 1$ and $1 \leq i \leq r$, let

$$q_t^{(i)}(\mathbf{x}; q, \mathbf{u}) = \sum_{\substack{I=(i_1,\ldots,i_t) \\ 1 \leq i_1 \leq \cdots \leq i_t \leq m}} u^i_{b(\max(I))} q^{e(I)} (q - q^{-1})^{\ell(I)} \, x_{i_1} x_{i_2} \cdots x_{i_t}, \tag{2.9}$$

where $e(I)$ is the number of $i_j \in I$ such that $i_j = i_{j+1}$, $\ell(I)$ is the number of $i_j \in I$ such that $i_j < i_{j+1}$, $\max(I)$ is the maximum element of I, and b is the function defined in (2.3). This definition of $q_t^{(i)}$ is given in [Sho]. For $\boldsymbol{\mu} \in \mathcal{P}_{n,r}$ with $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(r)})$ and $\mu^{(k)} = (\mu_1^{(k)}, \ldots, \mu_{\ell_k}^{(k)})$, define

$$q_{\boldsymbol{\mu}}(\mathbf{x}; q, \mathbf{u}) = \prod_{k=1}^{r} \prod_{j=1}^{\ell_k} q^{(k)}_{\mu_j^{(k)}}(\mathbf{x}; q, \mathbf{u}). \tag{2.10}$$

Note that when $q = 1$ and $u_i = \zeta^i$, we have $q_{\boldsymbol{\mu}} = p_{\boldsymbol{\mu}}$.

In [Mac], Appendix B, (9.7), we find the following Frobenius formula for the irreducible characters of $G_{n,r}$,

$$p_{\boldsymbol{\mu}}(\mathbf{x}) = \sum_{\boldsymbol{\lambda} \in \mathcal{P}_{n,r}} \chi_1^{\boldsymbol{\lambda}}(w_{\boldsymbol{\mu}}) s_{\boldsymbol{\lambda}}(\mathbf{x}), \tag{2.11}$$

for each $\boldsymbol{\mu} \in \mathcal{P}_{n,r}$. Shoji [Sho] extends this formula to a Frobenius formula for the irreducible characters of $H_{n,r}$,

$$q_{\boldsymbol{\mu}}(\mathbf{x}; q, \mathbf{u}) = \sum_{\boldsymbol{\lambda} \in \mathcal{P}_{n,r}} \chi_q^{\boldsymbol{\lambda}}(a_{\boldsymbol{\mu}}) s_{\boldsymbol{\lambda}}(\mathbf{x}), \tag{2.12}$$

for each $\boldsymbol{\mu} \in \mathcal{P}_{n,r}$.

We say that $I = (i_1, \ldots, i_t)$ is an *up-down sequence* if there exists an s, with $0 \le s \le t$, such that

$$i_1 < \cdots < i_s < i_{s+1} \ge \cdots \ge i_t, \qquad \text{for some } s, \text{ with } 0 \le s < t,$$

and we say that i_{s+1} is the *peak* of the up-down sequence I. Note that any of $i_1, \ldots, i_t$ can potentially be the peak of an up-down sequence $I = (i_1, \ldots, i_t)$. Following [Ra2], we define the weight

$$\text{wt}(i_1, \ldots, i_t) = \begin{cases} 0, & \text{if } i_1, \ldots, i_t \text{ is not an up-down sequence} \\ (-q)^{-s} q^{t-1-s}, & \text{if } \ i_1 < \cdots < i_s < i_{s+1} \ge \cdots \ge i_t. \end{cases} \tag{2.13}$$

If $t = 1$ the weight is $\text{wt}(i_1) = 1$.

Lemma 2.1. [Ra2] *Let $I = (i_1, \ldots, i_t)$ with $1 \le i_1 \le i_2 \le \cdots \le i_t \le m$, and let S_I denote the set of all distinct permutations of I. Then*

$$q^{e(I)}(q - q^{-1})^{\ell(I)} = \sum_{\sigma \in S_I} \text{wt}(\sigma I)$$

where $e(I)$ is the number of $i_j \in I$ such that $i_j = i_{j+1}$ and $\ell(I)$ is the number of $i_j \in I$ such that $i_j < i_{j+1}$.

Proof. In [Ra2], Lemma 1.5, Ram proves the first equality below

$$\begin{aligned} \sum_{\substack{I=(i_1,\ldots,i_t) \\ 1 \le i_1 \le \cdots \le i_t \le m}} q^{e(I)}(q - q^{-1})^{\ell(I)}\, x_{i_1} \cdots x_{i_t} &= \sum_{\substack{I=(i_1,\ldots,i_k) \\ 1 \le i_1,\ldots,i_t \le m}} \text{wt}(I)\, x_{i_1} \cdots x_{i_t} \\ &= \sum_{\substack{I=(i_1,\ldots,i_k) \\ 1 \le i_1 \le \cdots \le i_t \le m}} \sum_{\sigma \in S_I} \text{wt}(\sigma I)\, x_{i_1} \cdots x_{i_t}. \end{aligned}$$

The second equality follows from the fact that $x_{i_1} \cdots x_{i_k} = x_{i_{\sigma(1)}} \cdots x_{i_{\sigma(t)}}$ for all $\sigma \in S_I$. The result is obtained by comparing coefficients of $x_{i_1} \cdots x_{i_t}$. □

Proposition 2.2. *For integers $t \ge 1$ and $1 \le k \le r$, we have*

$$q_t^{(k)}(\mathbf{x}; q, \mathbf{u}) = \sum_{i_1,\ldots,i_t} \text{wt}(i_1, \ldots, i_t) u^k_{b(i_{s+1})}\ x_{i_1} \cdots x_{i_t},$$

where the sum is over all sequences $i_1, \ldots, i_t$ with $1 \le i_j \le m$ and $\text{wt}(i_1, \ldots, i_t)$ is given in (2.13). Note that wt is zero unless $i_1, \ldots, i_t$ is an up-down sequence.

Proof. As in Lemma 2.1, let S_I denote the set of distinct permutations of I. For all $\sigma \in S_I$ we have $\max(I) = \max(\sigma(I))$ and $x_{i_1} \cdots x_{i_t} = x_{i_{\sigma(1)}} \cdots x_{i_{\sigma(t)}}$. Furthermore, if I is an up-down sequence then its peak is $max(I) = i_{s+1}$.

We use Lemma 2.1 to write the sum over non-decreasing sequences

$$\begin{aligned}
\sum_{I=(i_1,\dots,i_t)} \mathrm{wt}(I) u^k_{b(max(I))}\, x_{i_1}\cdots x_{i_t} \\
&= \sum_{\substack{I=(i_1,\dots,i_t)\\ 1\le i_1\le\dots\le i_t\le m}} \sum_{\sigma\in S_I} \mathrm{wt}(\sigma I)\; u^k_{b(\max(\sigma I))}\; x_{i_{\sigma(1)}}\cdots x_{i_{\sigma(t)}} \\
&= \sum_{\substack{I=(i_1,\dots,i_t)\\ 1\le i_1\le\dots\le i_t\le m}} u^k_{b(\max(I))}\; x_{i_1}\cdots x_{i_t} \sum_{\sigma\in S_I} \mathrm{wt}(\sigma I) \\
&= \sum_{\substack{I=(i_1,\dots,i_t)\\ 1\le i_1\le\dots\le i_t\le m}} q^{e(I)}(q-q^{-1})^{\ell(I)} u^k_{b(\max(I))}\; x_{i_1}\cdots x_{i_t}, \\
&= q_t^{(k)}(\mathbf{x};q,\mathbf{u})
\end{aligned}$$

by the definition (2.9) of $q_t^{(k)}$. □

Let $\boldsymbol{\mu}\in\mathcal{P}_{n,r}$. The *row reading tableau* $R_{\boldsymbol{\mu}}$ of shape $\boldsymbol{\mu}$ is the r-partition $\boldsymbol{\mu}$ with the boxes filled in with the numbers $1,\dots,n$ so that $\mu^{(1)}$ contains the numbers $1,\dots,|\mu^{(1)}|$ in order from left-to-right and top-to-bottom, $\mu^{(2)}$ contains the numbers $|\mu^{(1)}|+1,\dots,|\mu^{(1)}|+|\mu^{(2)}|$ in order from left-to-right and top-to-bottom, and so on. For $1\le i\le n$ we define the component function $c_{R_{\boldsymbol{\mu}}}(i)$ by

$$c_{R_{\boldsymbol{\mu}}}(i)=k, \qquad \text{if } i \text{ is in the } k\text{th component of } R_{\boldsymbol{\mu}}. \tag{2.14}$$

We say that $I=(i_1,\dots,i_n)$ is a ***μ**-up-down sequence* if it satisfies the following property

$$\begin{array}{l}
\text{if } k,k+1,\dots,k+t \text{ is a row of } R_{\boldsymbol{\mu}}, \text{ then} \\
\text{the subsequence } i_k,i_{k+1},\dots,i_{k+t} \text{ is an up-down sequence,} \\
\text{i.e, } \; i_k<i_{k+1}<\dots<i_p\ge\dots\ge i_{k+t}.
\end{array} \tag{2.15}$$

The index i_p, shown above, is the peak of the row. When I is a $\boldsymbol{\mu}$-up-down sequence, we let $P_I^{\boldsymbol{\mu}}$ denote the set of peaks i_p in I, one for each row of $R_{\boldsymbol{\mu}}$. We define the $\boldsymbol{\mu}$-weight of a sequence $I=(i_1,\dots,i_n)$ by

$$\mathrm{wt}_{\boldsymbol{\mu}}(I)=\begin{cases}0, & \text{if } I \text{ is not a } \boldsymbol{\mu}\text{-up-down sequence,}\\ (-q^{-1})^{\ell(I)}q^{\gamma(I)}\displaystyle\prod_{i_p\in P_I^{\boldsymbol{\mu}}} u_{b(i_p)}^{c_{R_{\boldsymbol{\mu}}}(p)}, & \text{if } I \text{ is a } \boldsymbol{\mu}\text{-up-down sequence,}\end{cases} \tag{2.16}$$

where $\gamma(I)$ is the number of $i_j\ge i_{j+1}$ with j and $j+1$ in the same row of $R_{\boldsymbol{\mu}}$ and $\ell(I)$ is the number of $i_j<i_{j+1}$ with j and $j+1$ in the same row of $R_{\boldsymbol{\mu}}$. The functions $c_{R_{\boldsymbol{\mu}}}$ and b are defined in (2.15) and (2.3), respectively.

Example 2.3. Let $n=24, r=5, m_1=m_2=m_4=m_5=24, m=120$, and $\boldsymbol{\mu}=((5,1),(3,3,1,1),\emptyset,(2,2,2),(4))$. The row reading tableau of shape $\boldsymbol{\mu}$ is

$$R_{\boldsymbol{\mu}}=\left(\begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\\hline 6\\\cline{1-1}\end{array}\;,\;\begin{array}{|c|c|c|}\hline 7&8&9\\\hline 10&11&12\\\hline 13\\\cline{1-1} 14\\\cline{1-1}\end{array}\;,\;\emptyset,\;\begin{array}{|c|c|}\hline 15&16\\\hline 17&18\\\hline 19&20\\\hline\end{array}\;,\;\begin{array}{|c|c|c|c|}\hline 21&22&23&24\\\hline\end{array}\right).$$

The following squence is a $\boldsymbol{\mu}$-up-down sequence

$$I=[7,11,\underline{12},12,4][\underline{110}]\,\Big|\,[48,70,\underline{75}][\underline{75},75,30][\underline{1}][\underline{50}]\,\Big|\,\Big|\,[\underline{72},25][16,\underline{18}][\underline{119},97]\,\Big|\,[5,\underline{80},79,25].$$

The braces group the components elements according to the rows of $R_{\boldsymbol{\mu}}$, the vertical bars indicate the separation between the components of $R_{\boldsymbol{\mu}}$, and the peaks are underlined. The $\boldsymbol{\mu}$-weight of I is

$$\mathrm{wt}_{\boldsymbol{\mu}}(I) = ((-q^{-1})^2 q^2 u_1) u_5 ((-q^{-1})^2 u_4^2)(q^2 u_4^2) u_1^2 u_3^2 (q u_3^4)(-q^{-1} u_1^4)(q u_5^4)(-q^{-1} q^2 u_4^5).$$

□

The definition (2.10) of $q_{\boldsymbol{\mu}}$ can be thought of as a product of $q_t^{(k)}$ over the rows of $R_{\boldsymbol{\mu}}$ where t is the length of the row and k is the component of the row. Thus the following collollary is immediate from Proposition 2.2.

Corollary 2.4. *For $\boldsymbol{\mu} \in \mathcal{P}_{n,r}$,*

$$q_{\boldsymbol{\mu}}(\mathbf{x}; q, \mathbf{u}) = \sum_{i_1,\dots,i_n} \mathrm{wt}_{\boldsymbol{\mu}}(i_1, \dots, i_n)\, x_{i_1} \cdots x_{i_n},$$

where the sum is over all $\boldsymbol{\mu}$-up-down sequences $i_1, \dots, i_n$ and $\mathrm{wt}_{\boldsymbol{\mu}}$ is defined in (2.16). Note that wt *is zero unless $i_1, \dots, i_n$ is a $\boldsymbol{\mu}$-up-down sequence.*

3. RSK Insertion and Roichman Weights

If $\boldsymbol{\lambda} \in \mathcal{P}_{n,r}$, then a *standard tableau* $Q_{\boldsymbol{\lambda}}$ of shape $\boldsymbol{\lambda}$ is a filling of the boxes of $\boldsymbol{\lambda}$ with integers from $\{1, 2, \dots, n\}$ such that each integer from $\{1, 2, \dots, n\}$ appears in $Q_{\boldsymbol{\lambda}}$ exactly once, and for each $1 \leq k \leq r$

(1) the columns of $\lambda^{(k)}$ strictly increase from top to bottom, and
(2) the rows of $\lambda^{(k)}$ strictly increase from left to right.

The Robinson-Schensted-Knuth (RSK) insertion scheme (see [Sag]) is an algorithm which gives a bijection between sequences $x_{i_1}, \dots, x_{i_n}$, with $1 \leq i_j \leq m$, and pairs (P, Q) where P is a column-strict tableaux, Q is a standard tableau, and P and Q have shape λ for some partition λ with n boxes. The RSK insertion algorithm constructs the pair of tableaux (P, Q) iteratively,

$$(\emptyset, \emptyset) = (P_0, Q_0), (P_1, Q_1), \dots, (P_n, Q_n) = (P, Q),$$

in such a way that

(1) P_j is a column strict tableau that contains j boxes, and Q_j is a standard tableau that has the same shape as P_j,
(2) P_j is obtained from P_{j-1} by *column* inserting i_j into P_{j-1}, denoted $P_j = P_{j-1} \leftarrow i_j$, as follows
    (a) Insert i_j into the first column of P_{j-1} by displacing the smallest number $\geq i$; if every number is $< i$, add i to the bottom of the first column.
    (b) If i displaces x from the first column, insert x into the second column using the rules of (a).
    (c) Repeat for each subsequent column, until a number is added to the bottom of some (possibly empty) column.
(3) Q_j is obtained from Q_{j-1} by putting j in the newly added box (i.e., the box created in going from P_{j-1} to P_j).

The standard tableau Q is called the recording tableau.

We extend the RSK algorithm to work for tableaux whose shape are r-partitions. Given a sequence $x_{i_1}^{(k_1)}, x_{i_2}^{(k_2)}, \dots, x_{i_n}^{(k_n)}$, with $1 \leq k_j \leq r$ and $1 \leq i_j \leq m_{k_j}$, we construct a sequence $(\emptyset, \emptyset) = (P_0, Q_0), \dots, (P_n, Q_n) = (P, Q)$, where P_i is a column-strict tableau, Q_i is a standard tableau, and P_i and Q_i have the same r-partition shape. We insert $x_i^{(k)}$ into a semistandard tableau P_{j-1} having r-partition shape as follows

$$(P_{j-1}^{(1)}, \dots, P_{j-1}^{(r)}) \leftarrow x_i^{(k)} = (P_{j-1}^{(1)}, \dots, P_{j-1}^{(k)} \leftarrow x_i^{(k)}, \dots, P_{j-1}^{(r)}), \tag{3.1}$$

where we use usual column insertion to insert variables of type k into the kth component of P_{j-1}.

For example, if $r = 3$ the result of inserting $x_2^{(1)}, x_1^{(2)}, x_4^{(2)}, x_1^{(2)}, x_1^{(3)}$ is

$$P_i : (\emptyset,\emptyset,\emptyset), \left(\boxed{2}, \emptyset, \emptyset\right), \left(\boxed{2}, \boxed{1}, \emptyset\right), \left(\boxed{2}, \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}, \emptyset\right), \left(\boxed{2}, \begin{array}{|c|c|}\hline 1&1\\\hline 4\\\cline{1-1}\end{array}, \emptyset\right), \left(\boxed{2}, \begin{array}{|c|c|}\hline 1&1\\\hline 4\\\cline{1-1}\end{array}, \boxed{1}\right),$$

$$Q_i : (\emptyset,\emptyset,\emptyset), \left(\boxed{1}, \emptyset, \emptyset\right), \left(\boxed{1}, \boxed{2}, \emptyset\right), \left(\boxed{1}, \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}, \emptyset\right), \left(\boxed{1}, \begin{array}{|c|c|}\hline 2&4\\\hline 3\\\cline{1-1}\end{array}, \emptyset\right), \left(\boxed{1}, \begin{array}{|c|c|}\hline 2&4\\\hline 3\\\cline{1-1}\end{array}, \boxed{5}\right).$$

To see that this insertion provides a bijection, we can construct the inverse algorithm by using usual column uninsertion, in the reverse order of the entries of Q, and using the component of P to tell us the type of the uninserted variable. We denote this bijection by

$$(P, Q) \quad \overset{\text{RSK}}{\longleftrightarrow} \quad x_{i_1}^{(k_1)}, \dots, x_{i_n}^{(k_n)}$$

Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathcal{P}_{n,r}$, and let $Q_{\boldsymbol{\lambda}}$ be a standard tableau of shape $\boldsymbol{\lambda} \in \mathcal{P}_{n,r}$. If a and b are entries of $Q_{\boldsymbol{\lambda}}$, define

$$a \overset{\text{SW}}{\longrightarrow} b \quad \text{if} \quad \begin{cases} b \in \lambda^{(k)}, a \in \lambda^{(\ell)}, \text{ and } k > \ell, \\ \textit{or} \\ b \text{ is south (below) and/or west (left) of } a \text{ in } \lambda^{(k)}, \end{cases}$$

$$a \overset{\text{NE}}{\longrightarrow} b \quad \text{if} \quad \begin{cases} b \in \lambda^{(k)}, a \in \lambda^{(\ell)}, \text{ and } k < \ell, \\ \textit{or} \\ b \text{ is north (above) and/or east (right) of } a \text{ in } \lambda^{(k)}. \end{cases}$$

In the ordering on our indeterminates, we have $x_i^{(k)} < x_j^{(\ell)}$ if $k < \ell$ or $k = \ell$ and $i < j$. The following proposition is an immediate consequence of this fact and well-known facts about RSK insertion (see [Ra2], Proposition 2.1).

Proposition 3.1. *Let $P_{j+1} = (P_{j-1} \leftarrow x_{i_j}^{(k_j)}) \leftarrow x_{i_{j+1}}^{(k_{j+1})}$, where P_{j-1} is a column-strict tableau, and let Q_{j+1} be the associated recording tableau.*

(1) *If $x_{i_j}^{(k_j)} < x_{i_{j+1}}^{(k_{j+1})}$ then $j \overset{\text{SW}}{\longrightarrow} (j+1)$ in Q_{j+1}.*
(2) *If $x_{i_j}^{(k_j)} \geq x_{i_{j+1}}^{(k_{j+1})}$ then $j \overset{\text{NE}}{\longrightarrow} (j+1)$ in Q_{j+1}.*

Let $\boldsymbol{\mu}, \boldsymbol{\lambda} \in \mathcal{P}_{n,r}$. We say that a standard tableau $Q_{\boldsymbol{\lambda}}$ of shape $\boldsymbol{\lambda}$ is a *$\boldsymbol{\mu}$-SW-NE tableau* if it satisfies the following property

$$\begin{array}{l} \text{if } k, k+1, \dots, k+t \text{ is a row of } R_{\boldsymbol{\mu}}, \text{ then} \\ k \overset{\text{SW}}{\longrightarrow} (k+1) \overset{\text{SW}}{\longrightarrow} \cdots \overset{\text{SW}}{\longrightarrow} p \overset{\text{NE}}{\longrightarrow} \cdots \overset{\text{NE}}{\longrightarrow} (k+t) \text{ in } Q_{\boldsymbol{\lambda}} \end{array} \tag{3.2}$$

The number p, shown above, is called the peak of the row. When $Q_{\boldsymbol{\lambda}}$ is a $\boldsymbol{\mu}$-SW-NE tableau, we let $P_{Q_{\boldsymbol{\lambda}}}^{\boldsymbol{\mu}}$ denote the set of peaks p in $Q_{\boldsymbol{\lambda}}$, one for each row of $R_{\boldsymbol{\mu}}$.

We define the $\boldsymbol{\mu}$-weight of a standard tableau Q_λ by

(3.3)
$$\mathrm{wt}_{\boldsymbol{\mu}}(Q_{\boldsymbol{\lambda}}) = \begin{cases} 0, & \text{if } Q_{\boldsymbol{\lambda}} \text{ is not a } \boldsymbol{\mu}\text{-SW-NE tableau,} \\ (-q^{-1})^{\ell(Q_{\boldsymbol{\lambda}})} q^{\gamma(Q_{\boldsymbol{\lambda}})} \displaystyle\prod_{i_p \in P_{Q_{\boldsymbol{\lambda}}}} u_{b(i_p)}^{c_{R_{\boldsymbol{\mu}}}(p)}, & \text{if } Q_{\boldsymbol{\lambda}} \text{ is a } \boldsymbol{\mu}\text{-SW-NE tableau,} \end{cases}$$

where $\gamma(Q_{\boldsymbol{\lambda}})$ is the number of $j \xrightarrow{\mathrm{NE}} (j+1)$ in $Q_{\boldsymbol{\lambda}}$ with j and $j+1$ in the same row of $R_{\boldsymbol{\mu}}$ and $\ell(Q_{\boldsymbol{\lambda}})$ is the number of $j \xrightarrow{\mathrm{SW}} (j+1)$ In $Q_{\boldsymbol{\lambda}}$ with j and $j+1$ in the same row of $R_{\boldsymbol{\mu}}$. The functions $c_{R_{\boldsymbol{\mu}}}$ and b are defined in (2.15) and (2.3), respectively.

Example 3.2. Let $n = 24, r = 5, m_1 = m_2 = m_4 = m_5 = 24, m = 120$, and $\boldsymbol{\mu} = ((5,1),(3,3,1,1),\emptyset,(2,2,2),(4))$. We will insert the up-down sequence of Example 2.3. First we apply the bijection (2.2) to give the variables their color superscript thereby converting

$$I = [7,11,12,12,4][110]\Big|[48,70,75][75,75,30][1][50]\Big|\Big|[72,25][16,18][119,97]\Big|[5,80,79,25].$$

to

$$[7^{(1)},11^{(1)},12^{(1)},12^{(1)},4^{(1)}][14^{(5)}]\Big|[24^{(2)},22^{(3)},3^{(4)}][3^{(4)},3^{(4)},6^{(2)}][1^{(1)}][2^{(3)}]\Big|\Big|$$
$$[24^{(3)},1^{(2)}][16^{(1)},18^{(1)}][23^{(5)},1^{(5)}]\Big|[5^{(1)},8^{(4)},7^{(4)},1^{(2)}]$$

Upon inserting these variables we get

$$Q_{\boldsymbol{\lambda}} = \left(\begin{array}{|c|c|c|c|} \hline 1 & 4 & 5 & 13 \\ \hline 2 & 21 \\ \cline{1-2} 3 \\ \cline{1-1} 17 \\ \cline{1-1} 18 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|c|} \hline 7 & 12 & 16 & 24 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 8 & 14 \\ \hline 15 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|} \hline 9 & 10 & 11 \\ \hline 22 & 23 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|} \hline 6 & 20 \\ \hline 19 \\ \cline{1-1} \end{array} \right),$$

and

$$P_{\boldsymbol{\lambda}} = \left(\begin{array}{|c|c|c|c|} \hline 1 & 4 & 7 & 12 \\ \hline 5 & 11 \\ \cline{1-2} 12 \\ \cline{1-1} 16 \\ \cline{1-1} 18 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 1 & 6 & 24 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & 22 \\ \hline 24 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|} \hline 3 & 3 & 3 \\ \hline 7 & 8 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|} \hline 1 & 14 \\ \hline 23 \\ \cline{1-1} \end{array} \right).$$

The weight $\mathrm{wt}_{\boldsymbol{\mu}}(Q_{\boldsymbol{\lambda}})$ is computed using the row reading tableaux $R_{\boldsymbol{\mu}}$ in Example 2.3 and is the same as the $\boldsymbol{\mu}$-weight of the sequence I,

$$\mathrm{wt}_{\boldsymbol{\mu}}(Q_{\boldsymbol{\lambda}}) = ((-q^{-1})^2 q^2 u_1) u_5 ((-q^{-1})^2 u_4^2)(q^2 u_4^2) u_1^2 u_3^2 (q u_3^4)(-q^{-1} u_1^4)(q u_5^4)(-q^{-1} q^2 u_4^5).$$

□

Theorem 3.3. *Let $\boldsymbol{\mu} \in \mathcal{P}_{n,r}$, then*

$$q_{\boldsymbol{\mu}}(\mathbf{x}; q, \mathbf{u}) = \sum_{\boldsymbol{\lambda} \in \mathcal{P}_{n,r}} \left(\sum_{Q_{\boldsymbol{\lambda}}} \mathrm{wt}_{\boldsymbol{\mu}}(Q_{\boldsymbol{\lambda}}) \right) s_{\boldsymbol{\lambda}}(\mathbf{x}),$$

where the inner sum is over all standard tableaux $Q_{\boldsymbol{\lambda}}$ of shape $\boldsymbol{\lambda}$.

Proof. Comparing (2.15) and (2.16) with (3.2) and (3.3), we see that our insertion satisfies

$$\text{if} \quad (P_{\boldsymbol{\lambda}}, Q_{\boldsymbol{\lambda}}) \overset{\mathrm{RSK}}{\longleftrightarrow} x_{i_1}, \ldots, x_{i_n}, \quad \text{then} \quad \mathrm{wt}_{\boldsymbol{\mu}}(i_1, \ldots, i_n) = \mathrm{wt}_{\boldsymbol{\mu}}(Q_{\boldsymbol{\lambda}}).$$

We now apply RSK insertion to the formula for $q_{\boldsymbol{\mu}}$ found in Corollary 2.4:

$$\begin{aligned} q_{\boldsymbol{\mu}}(\mathbf{x};q,\mathbf{u}) &= \sum_{i_1,\dots,i_n} \mathrm{wt}_{\boldsymbol{\mu}}(i_1,\dots,i_n)\, x_{i_1}\cdots x_{i_n} \\ &= \sum_{\boldsymbol{\lambda}\in\mathcal{P}_{n,r}} \sum_{(P_{\boldsymbol{\lambda}},Q_{\boldsymbol{\lambda}})} \mathrm{wt}_{\boldsymbol{\mu}}(Q_{\boldsymbol{\lambda}})\, \mathbf{x}^{P_{\boldsymbol{\lambda}}} \\ &= \sum_{\boldsymbol{\lambda}\in\mathcal{P}_{n,r}} \sum_{Q_{\boldsymbol{\lambda}}} \mathrm{wt}_{\boldsymbol{\mu}}(Q_{\boldsymbol{\lambda}}) \sum_{P_{\boldsymbol{\lambda}}} \mathbf{x}^{P_{\boldsymbol{\lambda}}} \\ &= \sum_{\boldsymbol{\lambda}\in\mathcal{P}_{n,r}} \sum_{Q_{\boldsymbol{\lambda}}} \mathrm{wt}_{\boldsymbol{\mu}}(Q_{\boldsymbol{\lambda}}) s_{\boldsymbol{\lambda}}(\mathbf{x}), \end{aligned}$$

where $P_{\boldsymbol{\lambda}}$ varies over all column-strict tableaux of shape $\boldsymbol{\lambda}$ and $Q_{\boldsymbol{\lambda}}$ varies over all standard tableaux of shape $\boldsymbol{\lambda}$. □

The Schur functions $s_{\boldsymbol{\lambda}}$ are linearly independent [Mac], Appendix B (7.4), so comparing coefficients of $s_{\boldsymbol{\lambda}}$ in (2.12) and Theorem 3.3 gives

Corollary 3.4. *For $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathcal{P}_{n,r}$, we have*

$$\chi_q^{\boldsymbol{\lambda}}(a_{\boldsymbol{\mu}}) = \sum_{Q_{\boldsymbol{\lambda}}} \mathrm{wt}_{\boldsymbol{\mu}}(Q_{\boldsymbol{\lambda}}),$$

where $\chi_q^{\boldsymbol{\lambda}}(a_{\boldsymbol{\mu}})$ is the irreducible character of $H_{n,r}$ indexed by $\boldsymbol{\lambda}$ and evaluated at $a_{\boldsymbol{\mu}}$ and the sum is over all standard tableaux $Q_{\boldsymbol{\lambda}}$ of shape $\boldsymbol{\lambda}$.

Remark 3.5. Upon setting $q=1$ and $u_i=\zeta^i$, the formulas in Theorem 3.3 and Corollary 3.4 become a symmetric function identity

$$p_{\boldsymbol{\mu}}(\mathbf{x}) = \sum_{\boldsymbol{\lambda}\in\mathcal{P}_{n,r}} \left(\sum_{Q_{\boldsymbol{\lambda}}} \mathrm{wt}_{\boldsymbol{\mu}}(Q_{\boldsymbol{\lambda}}) \Big|_{\substack{q=1\\ u_i=\zeta^i}} \right) s_{\boldsymbol{\lambda}}(\mathbf{x}), \tag{3.4}$$

and a character formula

$$\chi_1^{\boldsymbol{\lambda}}(w_{\boldsymbol{\mu}}) = \sum_{Q_{\boldsymbol{\lambda}}} \mathrm{wt}_{\boldsymbol{\mu}}(Q_{\boldsymbol{\lambda}}) \Big|_{\substack{q=1\\ u_i=\zeta^i}}, \tag{3.5}$$

for the complex reflection group $G_{n,r}$.

Remark 3.6. Let $f_{\boldsymbol{\lambda}} = \dim(V_1^{\boldsymbol{\lambda}}) = \chi_1^{\boldsymbol{\lambda}}(1)$. This dimension is equal to the number of standard tableaux of shape $\boldsymbol{\lambda}$. As a special case of our insertion, we can restrict to sequences $x_{i_1}^{(k_1)},\dots,x_{i_n}^{(k_n)}$ where $i_1,\dots,i_n$ is a permutation of $1,\dots,n$ and $1\le k_i\le r$. There are $n!r^n$ such sequences. Furthermore, when we insert these special sequences, we get a pair (P,Q) of standard tableaux (the column-strict tableau P is standard because all the subscripts are unique). Thus, our modified RSK insertion gives a bijective proof of the identity

$$n!r^n = \sum_{\boldsymbol{\lambda}\in\mathcal{P}_{n,r}} f_{\boldsymbol{\lambda}}^2, \tag{3.6}$$

which also follows by decomposing the regular representation of $G_{n,r}$ into irreducibles and comparing dimensions.

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