Quasicommuting families of quantum Plücker coordinates

Bernard Leclerc and Andrei Zelevinsky [1]

To A.A. Kirillov on his sixtieth birthday

1. Introduction

Let $\mathbf{Q}_q[\mathcal{F}]$ be the q-deformation of the coordinate ring of the flag variety of type A_r, as defined in [**LR**], [**TT**]. This is the algebra with unit over the field of rational functions $\mathbf{Q}(q)$ generated by $2^{r+1}-1$ generators $[J]$ labeled by nonempty subsets $J \subset [1, r+1] := \{1, 2, \ldots, r+1\}$, subject to the q-deformed Plücker relations (see Section 2 below). We refer to the generators $[J]$ as *quantum flag minors* (they can be identified with q-minors of a generic q-matrix whose row set consists of several initial rows).

We say that two quantum flag minors $[I]$ and $[J]$ *quasicommute* if $[J][I] = q^n[I][J]$ for some integer n. In this paper, we deal with the following problem.

Problem A: describe all families of quasicommuting quantum flag minors.

The motivation for Problem A comes from the study of canonical bases for quantum groups of type A_r (see [**BZ**], [**K**]). It was proved in [**BZ**] that the *dual canonical basis* (= Kashiwara's *upper global crystal basis*) in $\mathbf{Q}_q[\mathcal{F}]$ contains all generators $[J]$. The main conjecture of [**BZ**] claims that the dual canonical basis is stable under taking quasicommutative products. This makes Problem A quite natural.

Our first main result is a combinatorial criterion for quasicommutativity of two quantum flag minors $[I]$ and $[J]$. We will write $I \prec J$ if $i < j$ for any $i \in I$, $j \in J$ (in particular, if one or both of I, J are empty). We write $I - J := \{i : i \in I, i \notin J\}$, and write $|I|$ for the size of I. We say that I and J are *weakly separated* if at least one of the following two conditions holds:

(1) $|I| \geq |J|$, and $J - I$ can be partitioned into a disjoint union $J - I = J' \cup J''$ so that $J' \prec I - J \prec J''$;

(2) $|J| \geq |I|$, and $I - J$ can be partitioned into a disjoint union $I - J = I' \cup I''$ so that $I' \prec J - I \prec I''$.

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Theorem 1.1. *Two quantum flag minors $[I]$ and $[J]$ quasicommute if and only if the sets I and J are weakly separated. In the case (1) above, we have*

$$[J][I] = q^{|J''|-|J'|}[I][J]. \tag{1.1}$$

The proof will be given in Section 2. The "if" part was essentially proved in [**KL**]. Our proof of the "only if" part uses a method similar to that in [**BZ**].

In view of Theorem 1.1, Problem A becomes equivalent to a combinatorial problem of describing all collections of weakly separated subsets of $[1, r+1]$. Our first result in this direction gives the maximal possible size of such a collection.

Theorem 1.2. *The maximal possible size of a collection of weakly separated subsets of $[1, r+1]$ is $\binom{r+2}{2} + 1$.*

This theorem will be proved in Section 3. Note that $\binom{r+2}{2} = \binom{r+1}{2} + r + 1$ is the dimension of the affine cone over the flag variety $\mathcal{F}$ of $G = SL_{r+1}$ in its Plücker embedding. It follows that Theorem 1.2 would be a natural consequence of the above mentioned conjecture that the dual canonical basis in $\mathbf{Q}_q[\mathcal{F}]$ contains all quasi-commuting monomials in quantum flag minors. (The appearance of the extra summand 1 in Theorem 1.2 is due to the fact that the empty subset is allowed to be a member of a weakly separated collection.)

An example of a weakly separated collection of maximal possible size is given by the collection $\mathcal{C}_{\min}$ consisting of the empty set and all intervals $[a, b]$ $(1 \le a \le b \le r+1)$; another example is the collection $\mathcal{C}_{\max}$ consisting of complements of subsets from $\mathcal{C}_{\min}$. (The notation $\mathcal{C}_{\min}$ and $\mathcal{C}_{\max}$ refers to a natural partial order on the set of all weakly separated collections that will be discussed later.)

We will generalize Theorem 1.2 in two different directions. First, we impose some restrictions on the size of subsets in a weakly separated collection.

Theorem 1.3. *Let k and l be any two indices such that $0 \le k \le l \le r+1$. The maximal possible size of a collection $\mathcal{C}$ of weakly separated subsets of $[1, r+1]$ such that $k \le |I| \le l$ for all $I \in \mathcal{C}$ is $\binom{r+2}{2} - \binom{r+2-l}{2} - \binom{k+1}{2} + 1$.*

To formulate the second generalization, consider any permutation $w \in S_{r+1}$. As usual, $l(w)$ will denote the length of w (i.e., the number of inversions). Following [**BFZ**], Section 5.3, we will say that a subset $I \subset [1, r+1]$ is a *w-chamber set* if for each $j \in I$ the set I also contains all indices i such that $i < j$ and $w(i) < w(j)$ (this terminology will be explained in Section 4).

Theorem 1.4. *The maximal possible size of a weakly separated collection of w-chamber subsets of $[1, r+1]$ is $l(w) + r + 2$.*

This theorem includes Theorem 1.2 as a special case corresponding to $w = w_0$, the longest permutation (clearly, every subset of $[1, r+1]$ is a w_0-chamber set).

Both Theorems 1.3 and 1.4 will also be proved in Section 3. Geometrically, in Theorem 1.3 we think of the flag variety $\mathcal{F}$ as a member of the family of incomplete flag varieties, while in Theorem 1.4 we include $\mathcal{F}$ into the family of Schubert varieties.

We conjecture that Theorem 1.4 can be sharpened as follows.

Conjecture 1.5. All maximal (by inclusion) weakly separated collections of w-chamber subsets of $[1, r+1]$ are of the same size $l(w) + r + 2$.

The conjecture was computer checked for $r \leq 4$ and in some other special cases. In Section 4 we will prove it for a special class of *strongly separated collections* that are the second main object of study in this paper.

We say that subsets I and J of $[1, r+1]$ are *strongly separated* if either $I - J \prec J - I$ or $J - I \prec I - J$ (in particular, this is so if one of I and J contains the other). Clearly, any two strongly separated subsets are weakly separated. A collection of subsets of $[1, r+1]$ is *strongly separated* if so are any two members of this collection. An algebraic interpretation of strongly separated subsets is given by Corollary 2.10 below: two subsets I and J are strongly separated if and only if one of the products $[I][J]$ and $[J][I]$ is invariant under the natural involution $x \mapsto \overline{x}$ of $\mathbf{Q}_q[\mathcal{F}]$ that interchanges q and q^{-1} (see [**L**], [**K**]).

Strongly separated collections arise naturally in several interesting combinatorial and geometric contexts. The first such context is the study of *commutation classes* of reduced words for permutations. For a permutation $w \in S_{r+1}$ let $R(w)$ denote the set of *reduced words* for w, that is, the set of sequences $\mathbf{h} = (h_1, \ldots, h_l)$ of minimal length $l = l(w)$ such that $w = s_{h_1} \cdots s_{h_l}$ (as usual, s_i stands for the simple transposition $(i, i+1)$). Recall that a *2-move* for a reduced word $\mathbf{h} \in R(w)$ consists in interchanging two consecutive entries i and j in $\mathbf{h}$ provided $|i - j| \geq 2$. Two reduced words obtained from each other by a sequence of 2-moves are said to belong to the same *commutation class*. Let $R^c(w)$ denote the set of commutation classes in $R(w)$. These classes were studied recently in [**E**], [**F**], [**S**], but the following problem still remains open.

Problem B: describe the set $R^c(w)$ for all permutations $w \in S_{r+1}$.

The relationship between commutation classes and strongly separated collections is given by the following construction. Following [**BFZ**], we associate to any reduced word $\mathbf{h} = (h_1, \ldots, h_l) \in R(w)$ a collection $\mathcal{C}(\mathbf{h})$ of $l + r + 2$ non-empty subsets of $[1, r+1]$ called *chamber sets* for $\mathbf{h}$. The collection $\mathcal{C}(\mathbf{h})$ consists of the empty set, the $r+1$ intervals $[1, a]$, $a = 1, \ldots, r+1$ and the l subsets $s_{h_l} s_{h_{l-1}} \cdots s_{h_k}([1, h_k])$, $k = 1, \ldots, l$. For example, if $r = 3$ and $\mathbf{h} = (3, 2, 1, 3)$, then

$$\mathcal{C}(\mathbf{h}) = \{\emptyset, 1, 2, 12, 24, 123, 124, 234, 1234\}.$$

(The members of $\mathcal{C}(\mathbf{h})$ are called chamber sets because in [**BFZ**] they were defined in terms of the geometry of the *pseudo-line arrangement* naturally associated with $\mathbf{h}$; see

Section 4 below. The connection between pseudo-line arrangements, commutation classes and other combinatorial structures like sorting networks or CC systems is discussed in the monograph [**Kn**].) By [**BFZ**], Proposition 5.3.1, any chamber set for some $\mathbf{h} \in R(w)$ is a w-chamber set, as defined above. It is also easy to see that $\mathcal{C}(\mathbf{h}) = \mathcal{C}(\mathbf{h}')$ if and only if reduced words $\mathbf{h}$ and $\mathbf{h}'$ belong to the same commutation class. In Section 4 we will prove the following.

Theorem 1.6. *For any permutation $w \in S_{r+1}$, the correspondence $\mathbf{h} \mapsto \mathcal{C}(\mathbf{h})$ is a bijection between the set $R^c(w)$ of commutation classes and the set of all maximal (by inclusion) strongly separated collections of w-chamber sets.*

In particular, Theorem 1.6 implies Conjecture 1.5 for strongly separated collections.

The relationship between (commutation classes of) reduced words and strongly separated collections has an interesting geometric application that we learned from Peter Magyar. Namely, every $\mathbf{h} \in R(w)$ gives rise to a Bott-Samelson desingularization of the Schubert variety corresponding to w. As shown by Magyar, this Bott-Samelson variety has a nice explicit description in terms of the collection $\mathcal{C}(\mathbf{h})$.

The quasicommuting family of quantum Plücker coordinates $\{[I] : I \in \mathcal{C}(\mathbf{h})\}$ is closely related with the "quantization" of the corresponding Bott-Samelson variety, recently constructed by B. Feigin (this construction is discussed in [**IM**] (for type A_r) and [**J**] (for an arbitary semisimple Lie algebra); recently, A. Berenstein has extended it to any Kac-Moody Lie algebra). The relationship between quantum Plücker coordinates and quantum "Bott-Samelson coordinates" will be studied in a separate publication; in the classical case $q = 1$, this relationship is given by Theorem 1.4 in [**BFZ**] (for type A_r). Note that geometric meaning of weakly but not strongly separated collections has yet to be understood.

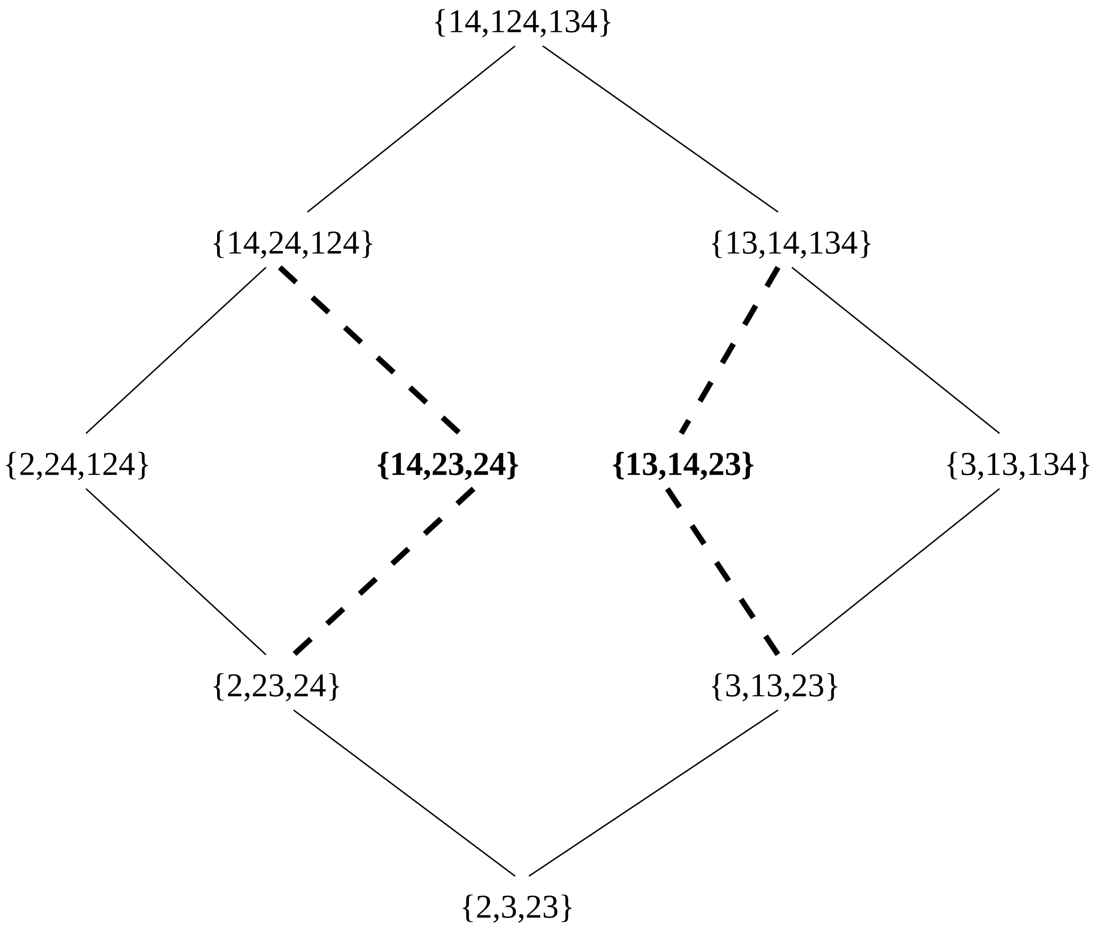
Let $\mathbf{W} = \mathbf{W}_r$ be the set of all weakly separated collections of subsets in $[1, r+1]$ that have maximal possible size $\binom{r+2}{2} + 1$. Let $\mathbf{S} = \mathbf{S}_r$ be the subset of $\mathbf{W}$ formed by strongly separated collections. Using Theorem 1.6, we can identify $\mathbf{S}$ with $R^c(w_0)$. The set $R^c(w_0)$ has an important poset structure via the *second Bruhat order* introduced in [**MS**]. This partial order is the transitive closure of the following relation on reduced words: $\mathbf{h}' \geq \mathbf{h}$ if $\mathbf{h}'$ is obtained from $\mathbf{h}$ by a *raising 3-move* replacing three consecutive indices $i, i+1, i$ by $i+1, i, i+1$. Using the above identification of $R^c(w_0)$ with $\mathbf{S}$, we can transfer the second Bruhat order to $\mathbf{S}$. The geometric interpretation of 3-moves in terms of pseudo-line arrangements given in [**BFZ**], Section 2, implies the following description of the transfered partial order on $\mathbf{S}$. If three indices $i < j < k$ and a subset $L \subset [1, r+1]$ disjoint from $\{i, j, k\}$ are such that a strongly separated collection $\mathcal{C}$ contains seven subsets L, Li, Lj, Lk, Lij, Ljk and $Lijk$ then a raising 3-move on $\mathcal{C}$ replaces Lj with Lik (here and in the sequel, we write $Li, Lj, \ldots$ for subsets $L \cup \{i\}, L \cup \{j\}, \ldots$). We have $\mathcal{C}' \geq \mathcal{C}$ if $\mathcal{C}'$ is obtained from $\mathcal{C}$ by a sequence of raising 3-moves. Under this partial order, the collection

$\mathcal{C}_{\min}$ described above is the unique minimal element of $\mathbf{S}$, while $\mathcal{C}_{\max}$ is the unique maximal element.

In Section 5, we introduce a natural extension of raising 3-moves and the partial order from $\mathbf{S}$ to $\mathbf{W}$. It is based on the following result.

Theorem 1.7. *Let $i < j < k$, and let $L \subset [1, r+1]$ be a subset disjoint from $\{i, j, k\}$. Suppose that $\mathcal{C}$ is a weakly separated collection containing five subsets Li, Lj, Lk, Lij and Ljk. Then the collection obtained from $\mathcal{C}$ by replacing Lj with Lik is also weakly separated.*

The operation in Theorem 1.7, i.e., replacing Lj with Lik in the presence of four "witnesses" Li, Lk, Lij and Ljk, will be called a *weak raising flip.* We define the poset structure on $\mathbf{W}$ as follows: $\mathcal{C}' \geq \mathcal{C}$ if $\mathcal{C}'$ is obtained from $\mathcal{C}$ by a sequence of weak raising flips. It is not hard to show that the restriction of this partial order to $\mathbf{S}$ coincides with the transfered second Bruhat order.



For instance, the picture above shows the poset $\mathbf{W}_3$. (For convenience, the subsets of the form $[1, a]$ or $[a, 4]$, which belong to all collections, have been omitted.) Removing in it the two vertices printed in bold type and the four dashed edges, one gets the usual poset $\mathbf{S}_3$.

Conjecture 1.8. The collection $\mathcal{C}_{\min}$ is the unique minimal element of the poset $\mathbf{W}$, and $\mathcal{C}_{\max}$ is the unique maximal element.

In other words, we conjecture that every collection $\mathcal{C} \in \mathbf{W}$ different from $\mathcal{C}_{\max}$ admits a weak raising flip, and every $\mathcal{C} \in \mathbf{W}$ different from $\mathcal{C}_{\min}$ admits a weak lowering flip. Some partial results in this direction are obtained in Section 5.

Note that the definition of a weak raising flip was motivated by the Chamber Ansatz in [**BFZ**], more specifically, by the observation that the 3-term relation $M_{Lik}M_{Lj} = M_{Lij}M_{Lk} + M_{Ljk}M_{Li}$ in [**BFZ**], (2.5.4) does not involve subsets L and $Lijk$. Using this observation, we will show in Section 5 that every collection obtained from $\mathcal{C}_{\min}$ by a sequence of weak raising flips gives rise to a total positivity criterion for upper-triangular matrices (in the sense of [**BFZ**], Section 3.2). Thus, Conjecture 1.8 would imply that every collection $\mathcal{C} \in \mathbf{W}$ gives rise to a total positivity criterion.

The material of this paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.1. For the convenience of the reader, we recall here necessary definitions and results from [**TT**], [**FRT**] and [**BZ**].

Theorems 1.2, 1.3 and 1.4 are proved in Section 3, and Theorem 1.6 is proved in Section 4. As a by-product of these proofs, we obtain an inductive procedure for generating the set $\mathbf{W}_r$ and $\mathbf{S}_r$. This procedure is based on the following result. For a collection $\mathcal{C} \in \mathbf{W}_r$, let $\mathcal{C}^-$ denote the collection of all subsets $J \subset [1, r]$ such that $J = I \cap [1, r]$ for some $I \in \mathcal{C}$.

Theorem 1.9. *The correspondence $\mathcal{C} \mapsto \mathcal{C}^-$ is a surjection $\mathbf{W}_r \to \mathbf{W}_{r-1}$, and its restriction to $\mathbf{S}_r$ is a surjection $\mathbf{S}_r \to \mathbf{S}_{r-1}$.*

Note that Knuth [**Kn**] found and implemented an algorithm for enumerating commutation classes which is essentially equivalent to our inductive procedure for $\mathbf{S}_r$.

Finally, in Section 5 we prove Theorem 1.7 and some other results about the extended second Bruhat order.

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2. A criterion for the quasi-commutativity of quantum flag minors

In this section we prove Theorem 1.1 stated in the introduction. First, let us recall the defining relations of the algebra $\mathbf{Q}_q[\mathcal{F}]$ given in [**TT**]. Given two subsets I and J of $[1, r+1]$, we shall denote by $\mathrm{inv}(I, J)$ the number of pairs $(i, j) \in I \times J$ such that $i > j$. The relations from [**TT**] can be written in the following form:

$R_1(I,J):$ for all subsets I and J such that $|I| \le |J|$, we have

$$[I][J] = \sum_M (-q)^{\mathrm{inv}(J-M,M)-\mathrm{inv}(I,M)}[I \cup M][J-M]\,,$$

where the sum is over all $M \subset J - I$ with $|M| = |J| - |I|$;

$R_2(I,J):$ for all subsets I and J such that $|I| - 1 \ge |J| + 1$, we have

$$\sum_{i \in I-J} (-q)^{\mathrm{inv}(i,I-i)-\mathrm{inv}(i,J)}[I - \{i\}][J \cup \{i\}] = 0\,.$$

We will use the following notation. If the quantum minors $[I]$ and $[J]$ quasicommute, then we define the integer $c(I,J)$ from

$$[J][I] = q^{c(I,J)}[I][J]. \tag{2.1}$$

Clearly, we have

$$c(J,I) = -c(I,J). \tag{2.2}$$

We can now prove the "if" part of Theorem 1.1, which takes the following form.

Lemma 2.1. *Let I and J be weakly separated sets satisfying condition (1) in the Introduction. Then $[I]$ and $[J]$ quasicommute, and $c(I,J) = |J''| - |J'|$. In particular, if $J \subset I$ then $c(I,J) = 0$.*

Proof. In the case when $I \cap J = \emptyset$, Lemma 2.1 is an immediate consequence of [**KL**], Lemmas 3.6, 3.7. The general case can be reduced to this one with the help of the following observation.

Lemma 2.2. *The quantum flag minors $[I]$ and $[J]$ quasicommute if and only if $[I - J]$ and $[J - I]$ quasicommute. Moreover, we have $c(I,J) = c(I - J, J - I)$.*

Lemma 2.2 is an immediate consequence of the following.

Lemma 2.3. *For a subset L of $[1, r+1]$, let $\mathcal{A}_L$ (resp. $\mathcal{B}_L$) denote the subalgebra of $\mathbf{Q}_q[\mathcal{F}]$ generated by the quantum flag minors $[I]$ such that $I \cap L = \emptyset$ (resp. $L \subset I$). Then there is an algebra isomorphism $\mathcal{A}_L \to \mathcal{B}_L$ which sends each $[I]$ to $[L \cup I]$.*

Proof of Lemma 2.3. The defining relations of $\mathcal{A}_L$ (*resp.* $\mathcal{B}_L$) are just relations $R_1(I,J)$ and $R_2(I,J)$ with $L \cap (I \cup J) = \emptyset$ (*resp.* $L \subset I \cap J$). Suppose that $L \cap (I \cup J) = \emptyset$. Then, formally replacing every minor $[P]$ occuring in $R_1(I,J)$ (*resp.* $R_2(I,J)$) by $[L \cup P]$, we get exactly relation $R_1(L \cup I, L \cup J)$ (*resp.* $R_2(L \cup I, L \cup J)$). Conversely, if $L \subset I \cap J$, the substitution of $[P - L]$ in place of $[P]$ in $R_1(I,J)$ (*resp.* $R_2(I,J)$) gives relation $R_1(I - L, J - L)$ (*resp.* $R_2(I - L, J - L)$). This completes the proof of Lemma 2.3 and of the "if" part of Theorem 1.1.

To prove the "only if" part of Theorem 1.1, we will use another interpretation of $\mathbf{Q}_q[\mathcal{F}]$. Recall that the entries of a *generic q-matrix* $X = (x_{ij})$ $i, j \in [1, r+1]$ are indeterminates over $\mathbf{Q}(q)$ subject to the following relations: every 2×2 submatrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of X satisfies

$$ba = qab, \quad ca = qac, \quad bc = cb, \quad db = qbd, \quad dc = qcd, \quad da - ad = (q - q^{-1})bc. \tag{2.3}$$

The $\mathbf{Q}(q)$-algebra $\mathbf{Q}_q[\mathrm{Mat}]$ generated by the x_{ij} is the q-deformation of the ring of polynomials on the matrix algebra considered in [**FRT**]. For $J = \{j_1 < \cdots < j_l\} \subset [1, r+1]$, the generator $[J]$ of $\mathbf{Q}_q[\mathcal{F}]$ can be identified with the *quantum flag minor* of X on rows $1, 2, \ldots, l$ and columns $j_1, j_2, \ldots, j_l$ given by

$$[J] = \sum_{\sigma \in S_l} (-q)^{-l(\sigma)} x_{1,j_{\sigma(1)}} x_{2,j_{\sigma(2)}} \cdots x_{l,j_{\sigma(l)}} \tag{2.4}$$

(see [**LR**], [**TT**]). Thus, $\mathbf{Q}_q[\mathcal{F}]$ can be identified with the subalgebra of $\mathbf{Q}_q[\mathrm{Mat}]$ generated by quantum flag minors.

Let U_+ denote the algebra with unit over $\mathbf{Q}(q)$ generated by the elements $E_1, \ldots, E_r$ subject to the *quantum Serre relations*

$$\begin{gathered} E_i E_j = E_j E_i \text{ for } |i - j| > 1 \ , \\ E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \text{ for } |i - j| = 1 \ . \end{gathered} \tag{2.5}$$

This is the quantized universal enveloping algebra of the Lie algebra Lie$\,(N)$, where N is the maximal unipotent subgroup of upper unitriangular matrices in SL_{r+1}. Both algebras $\mathbf{Q}_q[\mathrm{Mat}]$ and U_+ are graded via

$$\deg x_{ij} = \varepsilon_j, \ \deg E_j = \alpha_j = \varepsilon_j - \varepsilon_{j+1} \ , \tag{2.6}$$

where $\varepsilon_1, \ldots, \varepsilon_{r+1}$ is the standard basis in the lattice $\mathbf{Z}^{r+1}$. We will use a q-deformation of the action of Lie$\,(N)$ on the matrix space by infinitesimal right translations. This quantized action is given by the following proposition, which can be proved by a straightforward calculation.

Proposition 2.4. *There is a unique action of U_+ on $\mathbf{Q}_q[\mathrm{Mat}]$ satisfying*

$$E_j(x_{kl}) = \delta_{j,l-1}\, x_{k,l-1}, \tag{2.7}$$

$$E_j(fg) = E_j(f)\, g + q^{-(\gamma,\alpha_j)} f\, E_j(g), \tag{2.8}$$

where in (2.8), γ denotes the degree of a homogeneous element f, and (γ, α) is the standard scalar product defined by $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$.

The formulas (2.7), (2.8) and (2.4) readily imply the following.

Lemma 2.5. *For any subset $I \subset [1, r+1]$, we have*

$$E_j([I]) = [I \cup \{j\} - \{j+1\}] \tag{2.9}$$

if $j \notin I$ and $j+1 \in I$, otherwise $E_j([I]) = 0$.

We will denote by δ_I the indicator function of a set I given by $\delta_I(i) = 1$ if $i \in I$, and $\delta_I(i) = 0$ otherwise.

Lemma 2.6. *The property that $[I]$ and $[J]$ quasicommute, is preserved by the following transformations of a pair (I, J):*

(1) replacing (I, J) by $(I \cup \{i\} - \{i+1\}, J)$ for some index i such that $i+1 \in I - J$ and $i \notin I$; in this case, $c(I \cup \{i\} - \{i+1\}, J) = c(I, J) + \delta_J(i)$.

(2) replacing (I, J) by $(I, J \cup \{j\} - \{j+1\})$ for some index j such that $j+1 \in J - I$ and $j \notin J$; in this case, $c(I, J \cup \{j\} - \{j+1\}) = c(I, J) - \delta_I(j)$.

Proof. Part (2) follows from (1) and (2.2), so it is enough to prove part (1). Note that $[J]$ is a homogeneous element of degree $\sum_{j \in J} \varepsilon_j$. Writing for short $I' = I \cup \{i\} - \{i+1\}$, and applying (2.8) and (2.9), we obtain

$$E_i([J][I]) = q^{-\delta_J(i)} [J][I'], \quad E_i([I][J]) = [I'][J],$$

which implies the desired statement.

Lemma 2.7. *Suppose I and J are two disjoint sets that are not weakly separated, but become weakly separated after any transformation as in Lemma 2.6. Interchanging I and J if necessary, the pair (I, J) assumes one of the following forms:*

$$I = [1, j-1] \cup [j+1, s], \ J = \{j\} \quad (1 < j < s); \tag{2.10}$$

$$I = [1, j-2] \cup \{j\}, \ J = \{j-1\} \cup [j+1, s] \quad (2 < j < s). \tag{2.11}$$

Proof. The conditions readily imply that $I \cup J = [1, s]$ for some s. Interchanging I and J if necessary, we can assume that $1 \in I$. The sets I and J subdivide $[1, s]$ into the union of disjoint intervals $I_1 \cup J_1 \cup I_2 \cup J_2 \cdots$ (reading in the increasing order) so that we have $I = I_1 \cup \cdots \cup I_k$, $J = J_1 \cup \cdots \cup J_{k-1}$ if $s \in I$, and $I = I_1 \cup \cdots \cup I_k$, $J = J_1 \cup \cdots \cup J_k$ if $s \in J$. Let A and B are two consequent intervals in this subdivision, i.e., $(A, B) = (I_p, J_p)$ or $(A, B) = (J_p, I_{p+1})$. We will call the (A, B)-*reduction* the following operation on a pair (I, J): removing the last point of A from one of these sets and the first point of B from the other one. The conditions of the lemma mean that any (A, B)-reduction transforms (I, J) into a weakly separated pair, in particular, leaves at most three disjoint intervals. Hence, the total number of intervals in I and J is at most five. Let us consider three cases.

Case 1: $I = I_1 \cup I_2 \cup I_3$, $J = J_1 \cup J_2$. Applying the (I_1, J_1)-reduction and the (J_2, I_3)-reduction, we conclude that $|J_1| = |J_2| = 1$. But then the (J_1, I_2)-reduction will not lead to a weakly separated pair, so this case is impossible.

Case 2: $I = I_1 \cup I_2$, $J = J_1 \cup J_2$. Suppose that $|J_1| > 1$. Applying the (I_1, J_1)-reduction, we see that $|I_1| = 1$, and $|I_2| \geq |J_1| + |J_2| - 1 \geq 2$. On the other hand, applying the (J_1, I_2)-reduction yields $|I_2| = 1$. This contradiction shows that $|J_1| = 1$. The same argument proves that $|I_2| = 1$, i.e., we are in a situation (2.11).

Case 3: $I = I_1 \cup I_2$, $J = J_1$. Suppose $|J_1| > 1$. Since I and J are not weakly separated, at least one of I_1 and I_2 must have more than one point. But then the reduction involving this interval together with J_1 will also give a pair that is not weakly separated. This contradiction shows that $|J_1| = 1$, i.e., we are in a situation (2.10). This completes the proof of Lemma 2.7.

Now everything is ready for the proof of the "only if" part of Theorem 1.1. Assume, on the contrary, that there exist two non weakly separated sets I and J such that $[I]$ and $[J]$ quasicommute. Consider such a pair with minimal possible value of $\sum_{i \in I} i + \sum_{j \in J} j$. Using Lemmas 2.2, 2.6 and 2.7, we can assume that (I, J) is one of the pairs given in (2.10) and (2.11).

First, suppose I and J are given by (2.10). Using Lemmas 2.1 and 2.6 (1), we obtain

$$c(I, J) = c([1, j] \cup [j + 2, s], \{j\}) + 1 = 1. \tag{2.12}$$

On the other hand, using Lemmas 2.1 and 2.6 (2), we obtain

$$c(I, J) = c([1, j - 1] \cup [j + 1, s], \{j - 1\}) - 1 = -1. \tag{2.13}$$

Comparing (2.12) and (2.13) yields a contradiction.

The case when I and J are given by (2.11), is treated similarly. On one hand, Lemmas 2.1 and 2.6 (1) imply that

$$\begin{aligned} c(I, J) = c([1, j - 1], \{j - 1\} \cup [j + 1, s]) + 1 &= c([1, j - 2], [j + 1, s]) + 1 \\ &= \min(j - 2, s - j) + 1. \end{aligned} \tag{2.14}$$

On the other hand, Lemmas 2.1 and 2.6 (2) imply that

$$\begin{aligned} c(I, J) = c([1, j - 2] \cup \{j\}, \{j - 2\} \cup [j + 1, s]) - 1 &= c([1, j - 3] \cup \{j\}, [j + 1, s]) - 1 \\ &= \min(j - 2, s - j) - 1. \end{aligned} \tag{2.15}$$

Comparing (2.14) and (2.15) again yields a contradiction, thus completing the proof of Theorem 1.1.

We conclude this section by some comments and complementary results. First, let us discuss the relationship between the present setup and that of [**BZ**]. The main object of study in [**BZ**] is the algebra $\mathcal{A}_r$ of non-commutative polynomials in the matrix entries t_{ij} $(i < j)$ of the matrix T described in [**BZ**], Section 1. Note that T is not a q-matrix, i.e., its entries do not satisfy the commutation relations (2.3). To make our notation more unified, we shall rename $\mathcal{A}_r$ to $\mathbf{Q}_q[N]$, where N is the group of unipotent upper triangular matrices in $G = SL_{r+1}$; the algebra $\mathbf{Q}_q[N]$ is indeed a natural q-deformation of the ring of polynomial functions on N.

Both algebras $\mathbf{Q}_q[\mathcal{F}]$ and $\mathbf{Q}_q[N]$ can be naturally embedded as subalgebras into the algebra $\mathbf{Q}_q[B]$, the q-deformation of the ring of regular functions on the upper triangular group B. To construct $\mathbf{Q}_q[B]$, we first consider the quotient algebra of $\mathbf{Q}_q[\mathrm{Mat}]$ modulo the two-sided ideal generated by the x_{ij}, $i > j$. We denote this algebra by $\mathbf{Q}_q[\mathrm{Upp}]$. For $i \le j$ let b_{ij} be the image of x_{ij} in $\mathbf{Q}_q[\mathrm{Upp}]$. It is natural to call the matrix (b_{ij}) a *generic upper triangular q-matrix.* Note that the diagonal entries b_{ii} commute with each other in $\mathbf{Q}_q[\mathrm{Upp}]$. Now $\mathbf{Q}_q[B]$ can be defined as the algebra obtained from $\mathbf{Q}_q[\mathrm{Upp}]$ by adjoining the inverses b_{ii}^{-1} of diagonal entries.

Finally, let θ denote the $\mathbf{Q}$-algebra automorphism of $\mathbf{Q}_q[N]$ defined by $\theta(t_{i\,i+1}) = t_{i\,i+1}$ and $\theta(q) = q^{-1}$.

The relationship between $\mathbf{Q}_q[\mathcal{F}]$ and $\mathbf{Q}_q[N]$ can be now summarized as follows.

Proposition 2.8.

(a) *The restriction of the projection $\mathbf{Q}_q[\mathrm{Mat}] \to \mathbf{Q}_q[B]$ to the subalgebra $\mathbf{Q}_q[\mathcal{F}]$ is injective.*

(b) *There is a natural algebra embedding $\mathbf{Q}_q[N] \to \mathbf{Q}_q[B]$ which sends $\theta(t_{ij})$ for $i < j$ to $b_{ij}b_{ii}^{-1}$.*

(c) *For every subset $J \subset [1, r+1]$ of size l, the embedding in (b) sends $\theta(\Delta([1,l]; J))$ to $[J][[1,l]]^{-1}$, where $\Delta([1,l]; J)$ is the minor defined in* [**BZ**], (1.6).

This proposition is an easy consequence of the definitions. It implies that two generators of $\mathbf{Q}_q[\mathcal{F}]$ quasicommute if and only if so do the corresponding flag minors of T. Thus, Theorem 1.1 provides the following criterion.

Corollary 2.9. *Two flag minors $\Delta([1,|I|]; I)$ and $\Delta([1,|J|]; J)$ quasicommute in $\mathbf{Q}_q[N]$ if and only if the sets I and J are weakly separated.*

Finally, let us briefly discuss algebraic significance of the strong separation (see the introduction for the definition of strongly separated sets). There is a natural involution $x \mapsto \overline{x}$ defined on $\mathbf{Q}_q[\mathrm{Mat}]$, which sends q to q^{-1} and preserves the canonical basis (see, e.g., [**K**]). It is known that all quantum minors are fixed under this involution, and that it satisfies

$$\overline{uv} = q^{(\lambda_r,\mu_r)-(\lambda_l,\mu_l)}\,\overline{v}\,\overline{u}$$

for all weight vectors $u, v \in \mathbf{Q}_q[Mat]$ [**K**]. Here $\mathbf{Q}_q[\mathrm{Mat}]$ is regarded as a bimodule over $U_q(sl_{r+1})$ and λ_r, λ_l (resp. μ_r, μ_l) denote the respective weights of u (resp. v) under the right and left actions (the right and left weights of x_{ij} are respectively ε_j and ε_i). It then follows from an easy calculation that

$$\overline{[I][J]} = q^{d(I,J)}[J][I] \tag{2.16}$$

for $|I| \geq |J|$, where

$$d(I,J) = \min(|I-J|, |J-I|) = \min(|I|,|J|) - |I \cap J|. \tag{2.17}$$

The following characterization of strongly separated subsets is easily obtained by comparing (2.16) with (1.1).

Corollary 2.10. *Two subsets I and J are strongly separated if and only if one of the products $[I][J]$ and $[J][I]$ is invariant under the involution $x \mapsto \overline{x}$.*

3. Weakly separated collections of maximal size

Our first goal in this section is to prove Theorem 1.2 stated in the introduction. The following lemma is an immediate consequence of the definition of weak separation.

Lemma 3.1. *Let I and J be two subsets of $[1,r]$. Then the following conditions are equivalent:*
(1) I is weakly separated from $J \cup \{r+1\}$;
(2) either $|I| > |J|$ and I is weakly separated from J, or $|I| \leq |J|$ and $I - J \prec J - I$.

For a subset $I \subset [1, r+1]$, we will use the notation $I^- = I \cap [1,r]$. Lemma 3.1 implies the following.

Lemma 3.2. *If $I, J \subset [1, r+1]$ are weakly separated then I^- and J^- are also weakly separated.*

Now let $\mathcal{C}$ be a weakly separated collection of subsets in $[1, r+1]$. Let $\mathcal{C}^-$ denote the collection of all subsets $J \subset [1,r]$ such that $J = I^-$ for some $I \in \mathcal{C}$. Thus, the correspondence $I \mapsto I^-$ is a surjective map $\mathcal{C} \to \mathcal{C}^-$. Obviously, the fiber of this map over each subset $J \in \mathcal{C}^-$ has at most two elements, namely J and $J \cup \{r+1\}$. We will use the notation

$$\mathcal{C}_p^- = \{J \in \mathcal{C}^- : |J| = p\}.$$

Lemma 3.3. *For any $p = 0, 1, \ldots, r$, there is at most one $J \in \mathcal{C}_p^-$ such that both J and $J \cup \{r+1\}$ belong to $\mathcal{C}$.*

Proof. Suppose that $I, J \in \mathcal{C}_p^-$ are such that $I, I \cup \{r+1\}, J$ and $J \cup \{r+1\}$ all belong to $\mathcal{C}$. Since $\mathcal{C}$ is weakly separated, Lemma 3.1 implies that $I - J \prec J - I$ and $J - I \prec I - J$. Hence $I = J$, as desired.

Now everything is ready for the proof of Theorem 1.2. The collection $\mathcal{C}_{\min}$ of all intervals provides an example of a weakly separated collection of size $\binom{r+2}{2}+1$. It remains to show that every weakly separated collection $\mathcal{C}$ of subsets in $[1, r+1]$ has size $\leq \binom{r+2}{2}+1$. The proof is by induction on r. For $r = 0$ or $r = 1$, we have $\binom{r+2}{2}+1 = 2^{r+1}$, so our statement is obvious. So we will assume that $r \geq 2$, and that every weakly separated collection of subsets in $[1, r]$ is known to have size $\leq \binom{r+1}{2}+1$. In view of Lemma 3.3 and the inductive assumption, we have

$$|\mathcal{C}| \leq r+1+|\mathcal{C}^-| \leq r+1+\binom{r+1}{2}+1 = \binom{r+2}{2}+1. \tag{3.1}$$

Theorem 1.2 is proved.

Let $\mathbf{W}_r$ denote the set of all weakly separated collections $\mathcal{C}$ of subsets of $[1, r+1]$, that have maximal possible size $|\mathcal{C}| = \binom{r+2}{2}+1$. The estimate in (3.1) implies the following.

Corollary 3.4. *For every $\mathcal{C} \in \mathbf{W}_r$, the collection $\mathcal{C}^- = \{I^- : I \in \mathcal{C}\}$ belongs to $\mathbf{W}_{r-1}$. Furthermore, for each $p = 0, 1, \ldots, r$, there is exactly one subset $J \in \mathcal{C}_p^-$ such that both J and $J \cup \{r+1\}$ belong to $\mathcal{C}$.*

Thus, the correspondence $\mathcal{C} \mapsto \mathcal{C}^-$ is a map $\mathbf{W}_r \to \mathbf{W}_{r-1}$. Let us describe the fibers of this map.

For $\mathcal{C} \in \mathbf{W}_r$, let $\Sigma(\mathcal{C})$ denote the family of subsets $J \in \mathcal{C}^-$ such that both J and $J \cup \{r+1\}$ belong to $\mathcal{C}$. By Corollary 3.4, for each $p = 0, 1, \ldots, r$, the family $\Sigma(\mathcal{C})$ contains a unique subset of size p.

Now let $\mathcal{C}' \in \mathbf{W}_{r-1}$. By a *section* of $\mathcal{C}'$ we will mean a family of subsets $\Sigma = (J_0, J_1, \ldots, J_r)$ satisfying the following two conditions:

(1) each subset J_p belongs to $\mathcal{C}'_p$ and is strongly separated from all subsets $J \in \mathcal{C}'_p$;

(2) $J_p - J_{p+1} \prec J_{p+1} - J_p$ for $p < r$.

The following result is a refinement of the first statement in Theorem 1.9 from the introduction.

Theorem 3.5. *For any $\mathcal{C} \in \mathbf{W}_r$, the family $\Sigma(\mathcal{C})$ is a section of $\mathcal{C}^-$. Furthermore, for any $\mathcal{C}' \in \mathbf{W}_{r-1}$, the correspondence $\mathcal{C} \mapsto \Sigma(\mathcal{C})$ is a bijection between the set of collections $\mathcal{C} \in \mathbf{W}_r$ such that $\mathcal{C}^- = \mathcal{C}'$ and the set of sections Σ of $\mathcal{C}'$. The inverse bijection sends a section $\Sigma = (J_0, J_1, \ldots, J_r)$ of $\mathcal{C}'$ to the collection $\mathcal{C}$ given by*

$$\mathcal{C}_p = \{J \in \mathcal{C}'_p : J - J_p \prec J_p - J\} \cup \{J \cup \{r+1\} : J \in \mathcal{C}'_{p-1},\ J_{p-1} - J \prec J - J_{p-1}\}. \tag{3.2}$$

Proof. Let $\mathcal{C} \in \mathbf{W}_r$, and $\Sigma(\mathcal{C}) = (J_0, J_1, \ldots, J_r)$. Let $J \in \mathcal{C}_p^-$ be a subset different from J_p. Then exactly one of the subsets J and $J \cup \{r+1\}$ belongs to $\mathcal{C}$. Using Lemma 3.1, we see that if $J \in \mathcal{C}$ then $J - J_p \prec J_p - J$; similarly, if $J \cup \{r+1\} \in \mathcal{C}$ then $J_p - J \prec J - J_p$. In both cases, J_p is strongly separated from J, so $\Sigma(\mathcal{C})$ satisfies the first condition in the definition of a section. The second condition follows from Lemma 3.1 applied to $I = J_p$ and $J = J_{p+1}$. Thus, $\Sigma(\mathcal{C})$ is a section of $\mathcal{C}^-$. Furthermore, the above argument shows that the collection $\mathcal{C}$ is obtained from $\mathcal{C}' = \mathcal{C}^-$ and $\Sigma = \Sigma(\mathcal{C}^-)$ by means of (3.2).

It remains to show that, conversely, for any $\mathcal{C}' \in \mathbf{W}_{r-1}$ and any section Σ of $\mathcal{C}'$, the collection $\mathcal{C}$ defined by (3.2) is weakly separated. In view of Lemma 3.1, we only need to show the following: if $I \in \mathcal{C}'_p$ and $J \in \mathcal{C}'_q$ are such that $p \le q$, $I - J_p \prec J_p - I$ and $J_q - J \prec J - J_q$ then $I - J \prec J - I$. This is a consequence of the following lemma.

Lemma 3.6. *Suppose that $I, J, K \subset [1, r]$ are such that $|I| \le |J| \le |K|$, $I - J \prec J - I$, $J - K \prec K - J$, and I is weakly separated from K. Then $I - K \prec K - I$.*

Before proving Lemma 3.6, we state the following obvious reformulations of strong and weak separation.

Lemma 3.7. *Two sets I and J are strongly separated if and only if there do not exist three indices $a < b < c$ such that either $I \cap \{a, b, c\} = \{b\}$, $J \cap \{a, b, c\} = \{a, c\}$ or $I \cap \{a, b, c\} = \{a, c\}$, $J \cap \{a, b, c\} = \{b\}$.*

Lemma 3.8.

(a) *Suppose $|I| < |J|$. Then I and J are weakly separated if and only if there do not exist three indices $a < b < c$ such that $I \cap \{a, b, c\} = \{b\}$, $J \cap \{a, b, c\} = \{a, c\}$.*

(b) *Suppose $|I| = |J|$. Then I and J are weakly separated if and only if there do not exist four indices $a < b < c < d$ such that either $I \cap \{a, b, c, d\} = \{b, d\}$, $J \cap \{a, b, c, d\} = \{a, c\}$ or $I \cap \{a, b, c, d\} = \{a, c\}$, $J \cap \{a, b, c, d\} = \{b, d\}$.*

Proof of Lemma 3.6. Suppose there exist $a \in K - I$ and $b \in I - K$ such that $a < b$. We will show that this leads to a contradiction. First, the following two chains of implications show that either $\{a, b\} \subset J$ or $J \cap \{a, b\} = \emptyset$:

$$a \in J \Rightarrow a \in J - I \Rightarrow b \notin I - J \Rightarrow b \in J \ ;$$

$$b \in J \Rightarrow b \in J - K \Rightarrow a \notin K - J \Rightarrow a \in J \ .$$

Suppose $\{a, b\} \subset J$. Since $b \in J - K$ and $|J| \le |K|$, it follows that $K - J \ne \emptyset$. Choose $c \in K - J$; since $J - K \prec K - J$, it follows that $b < c$. Since $a \in J - I$ and $I - J \prec J - I$, it follows that $c \notin I$. Thus, $I \cap \{a, b, c\} = \{b\}$ and $K \cap \{a, b, c\} = \{a, c\}$. By Lemma 3.8 (a), we must have $|I| \ge |K|$, hence $|I| = |J| = |K|$. It follows that $I - J \ne \emptyset$. Choose $d \in I - J$; since $I - J \prec J - I$, it follows that $d < a$. But then both possibilities $d \in K$ and $d \notin K$ lead to a contradiction: if $d \in K$ then $d < b$ contradicts $J - K \prec K - J$; if $d \notin K$ then $I \cap \{d, a, b, c\} = \{d, b\}$ and $K \cap \{d, a, b, c\} = \{a, c\}$, which contradicts Lemma 3.8 (b).

The case $J \cap \{a, b\} = \emptyset$ is treated in the same way; we leave details to the reader. Lemma 3.6 and Theorem 3.5 are proved.

Theorem 3.5 provides an inductive procedure for generating all the collections $\mathcal{C} \in \mathbf{W}_r$. Let us illustrate this procedure by some examples. In listing the subsets in a collection $\mathcal{C} \in \mathbf{W}_r$, we will omit the subsets $[1, a]$ and $[a+1, r+1]$, that always belong to $\mathcal{C}$. The remaining subsets will be called *bounded.* (This terminology comes from the geometric interpretation of strongly separated subsets as chamber sets of an arrangement of pseudo-lines, that will be discussed later.) Every $\mathcal{C} \in \mathbf{W}_r$ contains $\binom{r}{2}$ bounded subsets. We will also write a section Σ of $\mathcal{C}' \in \mathbf{W}_{r-1}$ as $(J_1, \dots, J_{r-1})$, omitting the subsets $J_0 = \emptyset$ and $J_r = [1, r]$.

There is only one collection in $\mathbf{W}_1$, consisting of all four subsets in $[1, 2]$. It has two sections: $\Sigma = (1)$ and $\Sigma' = (2)$. The corresponding two collections in $\mathbf{W}_2$ are (listing only bounded sets) $\mathcal{C} = \{13\}$ and $\mathcal{C}' = \{2\}$. The collection $\mathcal{C} = \{13\}$ has five sections:

$$(1, 12),\ (1, 13),\ (1, 23),\ (3, 13),\ (3, 23);$$

the corresponding five collections in $\mathbf{W}_3$ are

$$\{14, 124, 134\},\ \{14, 13, 134\},\ \{14, 13, 23\},\ \{3, 13, 134\},\ \{3, 13, 23\}.$$

The collection $\mathcal{C}' = \{2\}$ also has five sections:

$$(1, 12),\ (1, 23),\ (2, 12),\ (2, 23),\ (3, 23);$$

the corresponding five collections in $\mathbf{W}_3$ are

$$\{14, 24, 124\},\ \{14, 24, 23\},\ \{2, 24, 124\},\ \{2, 24, 23\},\ \{2, 3, 23\}.$$

Thus, $|\mathbf{W}_3| = 10$.

Let us turn to the proof of Theorem 1.3. We will need the following lemma which is an immediate consequence of Lemma 3.8.

Lemma 3.9. *Any interval $I = [a, b] \subset [1, r+1]$ is weakly separated from all subsets $J \subset [1, r+1]$ with $|J| \leq |I|$. Any subset $I = [1, r+1] - [a, b]$ which is the complement of an interval, is weakly separated from all subsets $J \subset [1, r+1]$ with $|J| \geq |I|$.*

Proof of Theorem 1.3. Let $\mathcal{C}$ be a weakly separated collection of subsets in $[1, r+1]$ such that $k \leq |I| \leq l$ for all $I \in \mathcal{C}$. Let $\tilde{\mathcal{C}}$ be the collection obtained from $\mathcal{C}$ by adjoining all intervals $[a, b]$ in $[1, r+1]$ with $|[a, b]| > l$, and all complements of intervals $[1, r+1] - [a, b]$ with $|[1, r+1] - [a, b]| < k$. By Lemma 3.9, $\tilde{\mathcal{C}}$ is a weakly separated collection. By Theorem 1.2, $|\tilde{\mathcal{C}}| \leq \binom{r+2}{2} + 1$. Now the desired upper bound

$$|\mathcal{C}| \leq \binom{r+2}{2} - \binom{r+2-l}{2} - \binom{k+1}{2} + 1$$

becomes a consequence of the following obvious statement.

Lemma 3.10. *The number of intervals $[a,b]$ in $[1,r+1]$ with $|[a,b]|>l$ is equal to $\binom{r+2-l}{2}$.*

To complete the proof of Theorem 1.3, it remains to construct a weakly separated collection $\mathcal{C}$ as above, of size $\binom{r+2}{2}-\binom{r+2-l}{2}-\binom{k+1}{2}+1$. We define

$$\mathcal{C}_k=\{[1,k]\}\cup\{[1,a]\cup[b+1,b+k-a]:0\le a\le k-1,\ a+1\le b\le a+r+1-k\}\ ,\tag{3.3}$$

and, for $k<j\le l$, we define $\mathcal{C}_j$ to be the collection of all intervals of size j in $[1,r+1]$. Using Lemmas 3.8, 3.9 and 3.10, it is easy to check that $\mathcal{C}=\bigcup_{j=k}^{l}\mathcal{C}_j$ is a weakly separated collection of required size. Theorem 1.3 is proved.

We now turn to the proof of Theorem 1.4. The above proof of Theorem 1.2 extends to this more general case with the following modifications. Fix $w\in S_{r+1}$ and let $s=w(r+1)$. We associate to w a permutation $w^-\in S_r$ as follows: for $i\in[1,r]$ set $w^-(i)=w(i)$ if $w(i)<s$, and $w^-(i)=w(i)-1$ if $w(i)>s$. Counting inversions of w and w^-, we see that

$$l(w)=l(w^-)+r+1-s.\tag{3.4}$$

The next lemma follows at once from definitions.

Lemma 3.11. *Let $J\subset[1,r]$. Then J is a w-chamber set if and only if J is a w^--chamber set. The subset $J\cup\{r+1\}$ is a w-chamber set if and only if J is a w^--chamber set and $w^{-1}([1,s-1])\subset J$.*

Now let $\mathcal{C}$ be a weakly separated collection of w-chamber sets. By Lemma 3.11, the collection $\mathcal{C}^-$ consists of w^--chamber sets. Furthermore, if $J\in\mathcal{C}_p^-$ and $p<s-1$ then $(J\cup\{r+1\})\notin\mathcal{C}$. Combining this with Lemma 3.3 and using induction on r, we obtain the following generalization of (3.1):

$$|\mathcal{C}|\le|\mathcal{C}^-|+r+2-s\le l(w^-)+r+1+r+2-s=l(w)+r+2\tag{3.5}$$

(the last equality in (3.5) follows from (3.4)).

To complete the proof of Theorem 1.4, it remains to give an example of a collection that turns (3.5) into the equality. This can be easily done "by hand" but we prefer to give a procedure generating all such collections; this procedure generalizes Theorem 3.5.

Let $\mathbf{W}(w)$ denote the set of all weakly separated collections $\mathcal{C}$ of w-chamber sets that have maximal possible size $|\mathcal{C}|=l(w)+r+2$. The following statement generalizes Corollary 3.4 and is proved in the same way.

Corollary 3.12. *For every $\mathcal{C}\in\mathbf{W}(w)$, the collection $\mathcal{C}^-$ belongs to $\mathbf{W}(w^-)$. Furthermore, for each $p=s-1,s,\dots,r$, there is exactly one subset $J\in\mathcal{C}_p^-$ such that both J and $J\cup\{r+1\}$ belong to $\mathcal{C}$.*

Thus, the correspondence $\mathcal{C}\mapsto\mathcal{C}^-$ is a map $\mathbf{W}(w)\to\mathbf{W}(w^-)$. Generalizing Theorem 3.5, we will describe fibers of this map. By a *section* of a collection $\mathcal{C}'\in\mathbf{W}(w^-)$ we will mean a family of subsets $\Sigma=(J_{s-1},J_s,\dots,J_r)$ satisfying conditions (1) and (2) before Theorem 3.5 and such that $J_{s-1}=w^{-1}([1,s-1])$.

Theorem 3.13. *For any $\mathcal{C}' \in \mathbf{W}(w^-)$, there is a bijection between sections Σ of $\mathcal{C}'$ and collections $\mathcal{C} \in \mathbf{W}(w)$ such that $\mathcal{C}^- = \mathcal{C}'$. Under this bijection, the collection $\mathcal{C}$ corresponding to a section $\Sigma = (J_{s-1}, J_s, \ldots, J_r)$ of $\mathcal{C}'$, has $\mathcal{C}_p$ given by (3.2) for $p \geq s$, and $\mathcal{C}_p = \mathcal{C}'_p$ for $0 \leq p \leq s-1$.*

This is proved by the same arguments as Theorem 3.5. The only additional statement to check is the following: if $J \in \mathcal{C}'_p$ for some $p \geq s$ is such that $J_p - J \prec J - J_p$ then $J_{s-1} \subset J$ (by Lemma 3.11, this is needed to guarantee that $J \cup \{r+1\}$ is a w-chamber set). To prove this, we set $I = J_{s-1} = w^{-1}([1, s-1])$ and $K = (w^-)^{-1}([1,p]) = w^{-1}([1,p] - \{s\})$. Then I, J and K satisfy the conditions of the next lemma, which implies the desired statement.

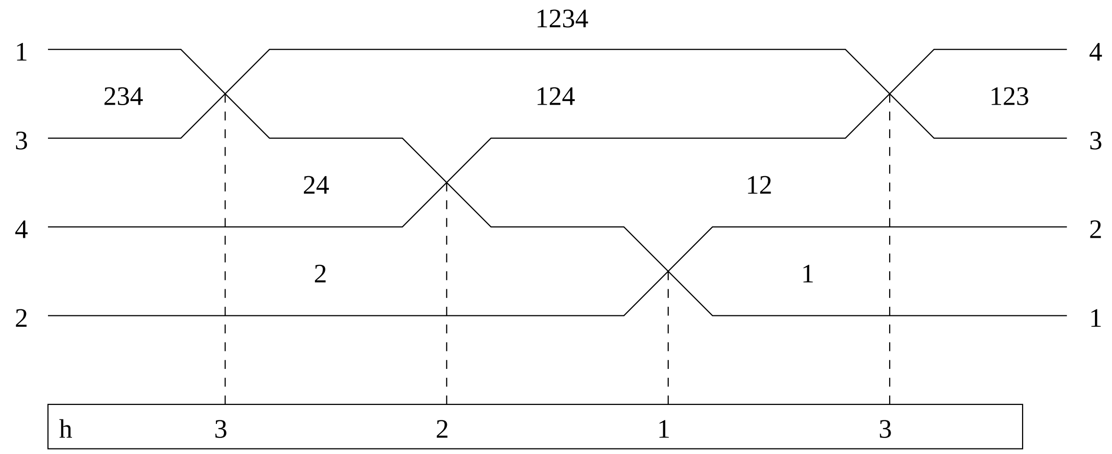
Lemma 3.14. *Suppose that $I, J, K \subset [1, r]$ are such that $|J| = |K|$, $I - J \prec J - I$, $J - K \prec K - J$, and $I \subset K$. Then $I \subset J$.*

Proof. Suppose our statement is not true, i.e., $I - J \neq \emptyset$. Let $b \in I - J$. Since $I \subset K$, we also have $b \in K$, so $b \in K - J$. Since $|J| = |K|$, it follows that $J - K \neq \emptyset$. Let $a \in J - K$. Since $J - K \prec K - J$, it follows that $a < b$. However, since $I \subset K$, we also have $a \in J - I$, which contradicts the condition that $I - J \prec J - I$.

Lemma 3.14 and Theorem 3.13 are proved. In view of Theorem 3.13, an inductive argument shows that $\mathbf{W}(w) \neq \emptyset$ for $w \in S_{r+1}$. This completes the proof of Theorem 1.4.

4. Maximal strongly separated collections

In this section we prove Theorem 1.6 from the introduction. Our proof will be based on the geometric description of chamber sets given in [**BFZ**]. Let us recall this description. We fix a permutation $w \in S_{r+1}$ of length $l(w) = l$. Let $\mathbf{h} = (h_1, \ldots, h_l) \in R(w)$ be a reduced word for w. We associate with $\mathbf{h}$ a *pseudo-line arrangement* $\mathrm{Arr}(\mathbf{h})$. This is a configuration of $r+1$ pseudo-lines in the plane (modulo natural isotopy) which may be defined recursively as follows. If $\mathbf{h}$ is the empty word, i.e., the unique element of $R(\mathrm{id})$, then $\mathrm{Arr}(\mathbf{h})$ consists of $r+1$ parallel horizontal lines labelled $1, 2, \ldots, r+1$ from bottom to top. Otherwise, $\mathrm{Arr}(\mathbf{h})$ is obtained from $\mathrm{Arr}(\mathbf{h}^-)$, where $\mathbf{h}^- = (i_2, \cdots, i_l)$, by adding on the left a crossing between the i_1'th and (i_1+1)'th pseudo-lines from the bottom. For example, if $r = 3$ and $\mathbf{h} = (3, 2, 1, 3)$ then $\mathrm{Arr}(\mathbf{h})$ is the configuration:



The *chambers* of Arr($\mathbf{h}$) are the connected components of the complement of the union of all pseudo-lines. It is easy to see that there are $l + r + 2$ of them. We label every chamber A by the set $L(A)$ of indices of all pseudo-lines that pass below A. The sets $L(A)$ are the *chamber sets* of Arr($\mathbf{h}$) (or simply of $\mathbf{h}$). We denote by $\mathcal{C}(\mathbf{h})$ the collection of all chamber sets of $\mathbf{h}$. For instance, we see from the figure above that

$$\mathcal{C}(3, 2, 1, 3) = \{\emptyset, 1, 2, 12, 24, 123, 124, 234, 1234\} \ .$$

It is easy to see that this description is equivalent to that in the introduction.

As a first step towards Theorem 1.6, we prove the following lemma.

Lemma 4.1. *For any reduced word $\mathbf{h} \in R(w)$, the collection $\mathcal{C}(\mathbf{h})$ is a strongly separated collection of w-chamber sets.*

Proof. Recall that a subset $I \subset [1, r+1]$ is a *w-chamber set* if for each $j \in I$ the set I also contains all indices i such that $i < j$ and $w(i) < w(j)$. The definition of Arr($\mathbf{h}$) implies easily that the pseudo-lines Line_i and Line_j with $i < j$ cross each other if and only if $w(i) > w(j)$. It follows that every chamber set of $\mathbf{h}$ is a w-chamber set.

To prove that $\mathcal{C}(\mathbf{h})$ is strongly separated, consider two chamber sets $I = L(A)$ and $J = L(B)$. Suppose $I - J \not\prec J - I$. This means that there exist indices $i < j$ such that the pseudo-line Line_i passes above the chamber A but below B, while Line_j passes below A but above B. In other words, the chambers A and B lie inside two opposite angles formed by Line_i and Line_j at their crossing point: the chamber A is to the left and B is to the right of this crossing point. In particular, this implies that every point of A is to the left of every point of B. Similarly, the assumption $J - I \not\prec I - J$ implies that every point of B is to the left of every point of A. It follows that at least one of the conditions $I - J \prec J - I$ and $J - I \prec I - J$ must hold, i.e., I and J are strongly separated. Lemma 4.1 is proved.

Recall from Section 3 that $\mathbf{W}(w)$ denotes the set of all weakly separated collections of w-chamber sets in $[1, r+1]$ that have maximal possible size $l + r + 2$. Let $\mathbf{S}(w)$ denote the subset of $\mathbf{W}(w)$ formed by strongly separated collections. By Lemma 4.1, for every $\mathbf{h} \in R(w)$, the chamber set collection $\mathcal{C}(\mathbf{h})$ belongs to $\mathbf{S}(w)$. As a next step in the proof of Theorem 1.6, we now establish the reverse inclusion.

Lemma 4.2. *Every collection $\mathcal{C} \in \mathbf{S}(w)$ is of the form $\mathcal{C}(\mathbf{h})$ for some $\mathbf{h} \in R(w)$.*

Proof. We will use the surjection $\mathcal{C} \mapsto \mathcal{C}^-$ from $\mathbf{W}(w)$ to $\mathbf{W}(w^-)$ and the description of its fibers given by Theorem 3.13.

Lemma 4.3. *A collection $\mathcal{C} \in \mathbf{W}(w)$ is strongly separated if and only if $\mathcal{C}^-$ is strongly separated and the section $\Sigma = \Sigma(\mathcal{C}) = (J_{s-1}, J_s, \ldots, J_r)$ forms a flag, i.e., $J_{p-1} \subset J_p$ for $s \leq p \leq r$.*

Proof of Lemma 4.3. Clearly, if $\mathcal{C}$ is strongly separated then so is $\mathcal{C}^-$. It is also clear that if $J_{p-1} \not\subset J_p$ then J_p is not strongly separated from $J_{p-1} \cup \{r+1\}$. This proves the "only if" part of our lemma. To prove the "if" part assume that $\mathcal{C}^-$ is strongly separated and the section $\Sigma = \Sigma(\mathcal{C}) = (J_{s-1}, J_s, \ldots, J_r)$ forms a flag. It is enough to check the following: if $s-1 \leq p < p' \leq r$, and $J \in \mathcal{C}_p^-$, $J' \in \mathcal{C}_{p'}^-$ are such that $J_p - J \prec J - J_p$, $J' - J_{p'} \prec J_{p'} - J'$ then $J' - J \prec J - J'$ (this guarantees that J' is strongly separated from $J \cup \{r+1\}$). Since $\mathcal{C}^-$ is strongly separated, we can assume that $J - J' \prec J' - J$; our goal then is to prove that $J \subset J'$. By Lemma 3.6, the conditions $J_p - J \prec J - J_p$ and $J - J' \prec J' - J$ imply that $J_p - J' \prec J' - J_p$. Applying Lemma 3.14 to the triple $(I, J, K) = (J_p, J', J_{p'})$, we obtain that $J_p \subset J'$. But then the conditions $J_p - J \prec J - J_p$ and $J - J' \prec J' - J$ imply that $J \subset J'$, as desired (this is proved by exactly the same argument as Lemma 3.14). Lemma 4.3 is proved.

Note that Lemma 4.3 establishes the following generalization of the second statement in Theorem 1.9.

Corollary 4.4. *The correspondence $\mathcal{C} \mapsto \mathcal{C}^-$ is a surjection $\mathbf{S}(w) \to \mathbf{S}(w^-)$.*

Continuing the proof of Lemma 4.2, let us assume that $\mathcal{C} \in \mathbf{S}(w)$. By Lemma 4.3, $\mathcal{C}^- \in \mathbf{S}(w^-)$. Using induction on r, we can assume that $\mathcal{C}^- = \mathcal{C}(\mathbf{h}')$ for some reduced word $\mathbf{h}'$ of w^-. The condition $J_{p-1} \subset J_p$ means that the corresponding chambers in the arrangement $\mathrm{Arr}(\mathbf{h}')$ are adjacent to each other (they are separated by the pseudo-line Line_j where $\{j\} = J_p - J_{p-1}$). Therefore, we can extend the arrangement $\mathrm{Arr}(\mathbf{h}')$ to an arrangement of $r+1$ pseudo-lines, by adding one more pseudo-line Line_{r+1} that passes through all chambers in $\mathrm{Arr}(\mathbf{h}')$ with chamber sets $J_{s-1}, J_s, \ldots, J_r$. Clearly, this new arrangement is isotopic to $\mathrm{Arr}(\mathbf{h})$ for some reduced word $\mathbf{h}$ of w, and $\mathcal{C}$ is exactly the collection of its chamber sets. Lemma 4.2 is proved.

To complete the proof of Theorem 1.6, it remains to show the following.

Lemma 4.5. *Any strongly separated collection $\mathcal{C}$ of w-chamber sets in $[1, r+1]$ is contained in $\mathcal{C}(\mathbf{h})$ for some $\mathbf{h} \in R(w)$.*

Proof. Let $s = w(r+1)$ and let $u = w^- \in S_r$. We will write the collection $\mathcal{C}^-$ of subsets in $[1, r]$ as $\mathcal{C}^- = \mathcal{C}' \cup \mathcal{C}''$, where

$$\mathcal{C}' = \{I \subset [1, r] : I \in \mathcal{C}\} \ , \ \mathcal{C}'' = \{I \subset [1, r] : I \cup \{r+1\} \in \mathcal{C}\} \ .$$

Clearly, $\mathcal{C}^-$ is a strongly separated collection of u-chamber sets. Using induction on r, we can assume that $\mathcal{C}^- \subset \mathcal{C}(\mathbf{h}')$ for some $\mathbf{h}' \in R(u)$. Furthermore, the subcollections $\mathcal{C}'$ and $\mathcal{C}''$ are easily seen to have the following property:

(1) for any $I \in \mathcal{C}'$ and $J \in \mathcal{C}''$, we have $I - J \prec J - I$, and $u^{-1}[1, s-1] \subset J$.

We will say that a subset $K \subset [1, r]$ *separates* $\mathcal{C}'$ and $\mathcal{C}''$ if $I - K \prec K - I$ and $K - J \prec J - K$ for all $I \in \mathcal{C}'$, $J \in \mathcal{C}''$. Taking into account Lemmas 4.2 and 4.3, we see that Lemma 4.5 becomes a consequence of the following.

Lemma 4.6. *Let $\mathbf{h}' \in R(u)$ and suppose that $\mathcal{C}'$ and $\mathcal{C}''$ are two subcollections of the chamber set collection $\mathcal{C}(\mathbf{h}')$ that satisfy (1). Then there exists a flag $(u^{-1}[1, s-1] = J_{s-1} \subset J_s \subset \cdots \subset J_r = [1, r])$ in $\mathcal{C}(\mathbf{h}')$ such that each J_p separates $\mathcal{C}'$ and $\mathcal{C}''$.*

Proof of Lemma 4.6. First of all, it is clear that $u^{-1}[1, s-1] = J_{s-1}$ separates $\mathcal{C}'$ and $\mathcal{C}''$ (we have $I - J_{s-1} \prec J_{s-1} - I$ for all u-chamber sets I).

Now suppose we have already constructed the chamber sets $J_{s-1} \subset J_s \subset \cdots \subset J_p$ separating $\mathcal{C}'$ and $\mathcal{C}''$, for some p with $s - 1 \leq p < r$. To conclude the proof, it remains to construct the chamber set J_{p+1} of size $p+1$ such that $J_p \subset J_{p+1}$, and J_{p+1} separates $\mathcal{C}'$ and $\mathcal{C}''$. In view of Lemma 3.6, we can and will assume that all subsets from $\mathcal{C}'$ and $\mathcal{C}''$ have size $> p$.

Let B be a chamber in the arrangement $\mathrm{Arr}(\mathbf{h}')$. By the *shadow* of B we will mean the union of all chambers below B (including B itself) that can be joined with B by a polygonal line that crosses only horizontal segments of $\mathrm{Arr}(\mathbf{h}')$. It is easy to see that a chamber A belongs to the shadow of B if and only if $L(A) \subset L(B)$. We will use the following geometrically obvious lemma.

Lemma 4.7. *Suppose $I = L(A)$ and $J = L(B)$ are two chamber sets of $\mathrm{Arr}(\mathbf{h}')$ such that $|I| \leq |J|$ (i.e., the chamber A lies below B or on the same horizontal level with B). Then $I - J \prec J - I$ if and only if A either belongs to the shadow of B, or lies to the right of the shadow.*

Now let A be the chamber in the arrangement $\mathrm{Arr}(\mathbf{h}')$ with the chamber set $L(A) = J_p$. Lemma 4.7 implies easily that for every two chambers B' and B'' such that $L(B') \in \mathcal{C}'$, $L(B'') \in \mathcal{C}''$, there exists a chamber B adjacent from above to A and lying inside or to the left of the shadow of B' and inside or to the right of the shadow of B''. Let us choose B' so that the right end of its shadow on the level immediately above A is as far to the left as possible; similarly, choose B'' so that the left end of its shadow on the level immediately above A is as far to the right as possible. Taking the corresponding chamber B as above, we conclude that the chamber set $J_{p+1} = L(B)$ contains J_p and separates $\mathcal{C}'$ and $\mathcal{C}''$. This completes the proof of Lemmas 4.5 and 4.6 and of Theorem 1.6.

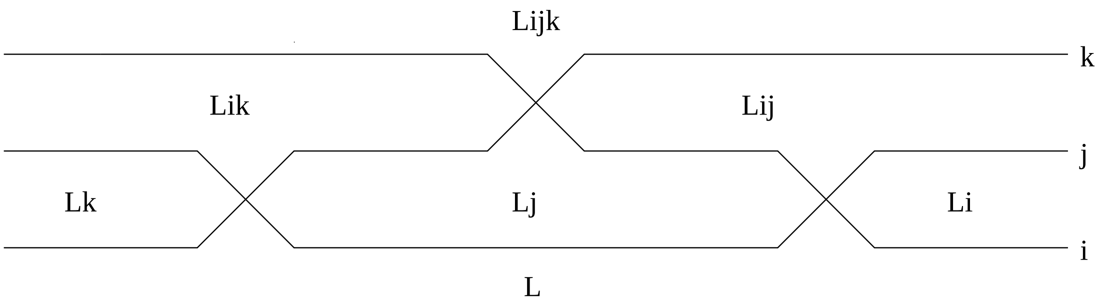
5. Extension of the second Bruhat order

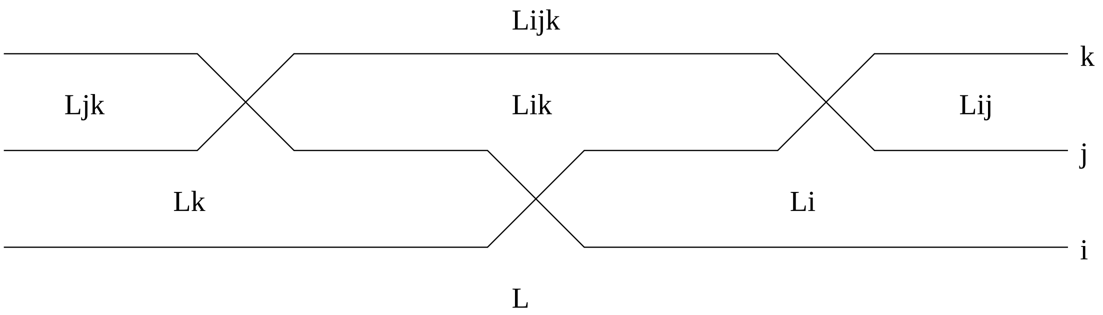
In this Section we prove Theorem 1.7 from the introduction. Recall from [**MS**] the definition of the second Bruhat order on the set $R^c(w_0)$ of commutation classes of reduced words for w_0. This partial order is the transitive closure of the following relation on reduced words: $\mathbf{h}' \geq \mathbf{h}$ if $\mathbf{h}'$ is obtained from $\mathbf{h}$ by a *raising 3-move* replacing three consecutive indices $i, i+1, i$ by $i+1, i, i+1$. By Theorem 1.6, the correspondence $\mathbf{h} \to \mathcal{C}(\mathbf{h})$ that associates to each reduced word $\mathbf{h}$ the chamber set collection of the pseudo-line arrangement $\mathrm{Arr}(\mathbf{h})$, allows us to identify $R^c(w_0)$ with the set $\mathbf{S}_r$ of all maximal strongly separated collections of subsets in $[1, r+1]$. Let us describe how this identification transforms the second Bruhat order.

In terms of the pseudo-line arrangements $\mathrm{Arr}(\mathbf{h})$ and $\mathrm{Arr}(\mathbf{h}')$, a raising 3-move translates into the following *Yang-Baxter raising flip*. If $\mathrm{Arr}(\mathbf{h})$ has a "triangular" chamber bounded by three pseudo-lines Line_i, Line_j and Line_k such that Line_j passes below the crossing of Line_i and Line_k, then $\mathrm{Arr}(\mathbf{h}')$ is obtained from $\mathrm{Arr}(\mathbf{h})$ by deforming Line_j so that it will now pass above the crossing of Line_i and Line_k (and keep the same position with respect to all other crossings).

It is geometrically obvious that $\mathrm{Arr}(\mathbf{h})$ has a "triangular" chamber A as above if and only if its chamber set collection $\mathcal{C}(\mathbf{h})$ contains seven subsets L, Li, Lj, Lk, Lij, Ljk and $Lijk$, where the chamber set $L(A)$ is equal to $Lj := L \cup \{j\}$. Furthermore, $\mathcal{C}(\mathbf{h}')$ is obtained from $\mathcal{C}(\mathbf{h})$ by replacing Lj with Lik. We see that a raising 3-move translates into the following operation on a maximal strongly separated collection $\mathcal{C}$: replacing $Lj \in \mathcal{C}$ with Lik for some indices $i < j < k$ and a subset $L \subset [1, r+1]$ disjoint from $\{i, j, k\}$, provided that $\mathcal{C}$ contains six "witnesses" L, Li, Lk, Lij, Ljk and $Lijk$. We call this operation a *strong raising flip*. Thus, $\mathcal{C}' \geq \mathcal{C}$ in $\mathbf{S}_r$ if $\mathcal{C}'$ is obtained from $\mathcal{C}$ by a sequence of strong raising flips.

These definitions are illustrated by the following picture.





We now define a *weak raising flip* on a collection of subsets $\mathcal{C}$ as the operation of replacing $Lj \in \mathcal{C}$ with Lik for some indices $i < j < k$ and a subset $L \subset [1, r+1]$ disjoint from $\{i, j, k\}$, provided that $\mathcal{C}$ contains four "witnesses" Li, Lk, Lij and Ljk. The inverse operation will be called a *weak lowering flip*.

Theorem 1.7 from the introduction is a consequence of the following statement.

Theorem 5.1. *Weak raising and lowering flips preserve the set of weakly separated collections.*

Proof of Theorem 5.1. It is enough to prove the following.

Lemma 5.2. *Suppose $i < j < k$ and $L \cap \{i, j, k\} = \emptyset$. If a subset $I \subset [1, r+1]$ is weakly separated from each of the four subsets Li, Lk, Lij, Ljk and is different from Lj and Lik, then I is weakly separated from Lj and from Lik.*

Proof of Lemma 5.2. Suppose that I is not weakly separated from Lj. Consider two cases.

Case 1. Suppose $|I| \le |Lj|$. By Lemma 3.8, there exist three indices $a < b < c$ such that $I \cap \{a, b, c\} = \{b\}$, $Lj \cap \{a, b, c\} = \{a, c\}$. Since at least one of the indices i and k is different from b, at least one of the intersections $Lij \cap \{a, b, c\}$ and $Ljk \cap \{a, b, c\}$ is equal to $\{a, c\}$. Hence at least one of Lij and Ljk is not weakly separated from I, which contradicts our assumptions.

Case 2. Suppose $|I| > |Lj|$. By Lemma 3.8, there exist three indices $a < b < c$ such that $I \cap \{a, b, c\} = \{a, c\}$, $Lj \cap \{a, b, c\} = \{b\}$.

Subcase 2.1. Suppose $b = j$. Then $j \notin I$. We claim that in this case I contains i and k. Suppose that $i \notin I$. Then $i \neq a$, so we have either $i < a$, or $a < i < j = b$. If $i < a$, then $I \cap \{i, a, j, c\} = \{a, c\}$, $Lij \cap \{i, a, j, c\} = \{i, j\}$, which contradicts the assumption that I and Lij are weakly separated. If $a < i < j$, then $I \cap \{a, i, c\} = \{a, c\}$, $Li \cap \{a, i, c\} = \{i\}$, which contradicts the assumption that I and Li are weakly separated. We see that $i \in I$. The fact that $k \in I$ is proved in a similar way. Thus, $I = I'ik$ with $I' \cap \{i, j, k\} = \emptyset$. We have $|I'| \geq |L|$ and $I' \neq L$, because I is assumed to be different from Lik. Thus, $I' - L \neq \emptyset$.

Subsubcase 2.1.1. Suppose $L \subset I'$. Then $Lij - I = Ljk - I = \{j\}$. Since $I - Lij = (I' - L)k$, $I - Ljk = (I' - L)i$, the conditions that I is weakly separated from Lij and Ljk imply that $j \prec I' - L$ and $I' - L \prec j$, a contradiction.

Subsubcase 2.1.2. Suppose $L \not\subset I'$, i.e., $L - I' \neq \emptyset$. Since $Li - I = Lk - I = L - I'$ and $I - Li = (I' - L)k$, $I - Lk = (I' - L)i$, the conditions that I is weakly separated from Li and Lk imply that the set $(I' - L)ik$ lies between two consecutive elements of $L - I'$. Therefore, $L - I'$ either contains some l such that $(I' - L)k \prec l$, or contains some h such that $h \prec (I' - L)i$. Suppose, $L - I'$ contains some l such that $(I' - L)k \prec l$. Then $Lij \cap \{j, k, l\} = \{j, l\}$ and $I \cap \{j, k, l\} = \{k\}$, so the fact that I and Lij are weakly separated, implies that $j \prec I - Lij = (I' - L)k \prec l$. Hence $I' - L$ contains some m such that $j < m < l$. But then we have $Ljk \cap \{i, j, m, l\} = \{j, l\}$ and $I \cap \{i, j, m, l\} = \{i, m\}$; in view of Lemma 3.8, this contradicts the assumption that I and Ljk are weakly separated. The case when $L - I'$ contains some h such that $h \prec (I' - L)i$, is treated in the same way.

Subcase 2.2. Suppose $b \neq j$, i.e., $b \in L - I$. Then the condition that I is weakly separated from Li and Lk, can be satisfied only if $a = i$ and $c = k$, i.e., we again have $I = I'ik$ for some subset I' such that $|I'| \geq |L|$ and $I' \neq L$. Now we have $b \in L - I' = Li - I$ and $k \in (I' - L)k = I - Li$, so the condition that I and Li are weakly separated, implies that $b \prec I' - L$. However, by a similar argument, the condition that I and Lk are weakly separated, implies that $I' - L \prec b$. This contradiction completes the proof of the fact that I is weakly separated from Lj. The remaining part of Lemma 5.2 (that I is weakly separated from Lik) can be proved in a similar way, or deduced from the first part by passing to complementary subsets. Lemma 5.2 and Theorem 5.1 are proved.

Recall that $\mathbf{W} = \mathbf{W}_r$ is the set of all weakly separated collections of subsets in $[1, r+1]$ that have maximal possible size $\binom{r+2}{2} + 1$. We introduce a partial order on $\mathbf{W}_r$ as follows: $\mathcal{C}' \geq \mathcal{C}$ if $\mathcal{C}'$ is obtained from $\mathcal{C}$ by a sequence of weak raising flips. This definition makes $\mathbf{W}_r$ into a ranked poset with respect to the rank function

$$r(\mathcal{C}) = \sum_{I \in \mathcal{C}} |I| \tag{5.1}$$

(this is obvious, since any weak raising flip increases rank by 1).

Clearly, the collection $\mathcal{C}_{\min}$ of all intervals in $[1, r+1]$ is a minimal element of the poset $\mathbf{W}_r$ because every weak lowering flip involves a subset of the form Lik which is not an interval. Similarly, the collection $\mathcal{C}_{\max}$ of all interval complements is a maximal element of $\mathbf{W}_r$. This is also a special case of the following statement which is an immediate consequence of the definitions.

Proposition 5.3. *For a collection $\mathcal{C}$ of subsets in $[1, r+1]$ let $\overline{\mathcal{C}}$ denote the collection of complements of subsets from $\mathcal{C}$. Then the correspondence $\mathcal{C} \to \overline{\mathcal{C}}$ is an order-reversing involution of $\mathbf{W}_r$.*

Conjecture 1.8 from the introduction claims that $\mathcal{C}_{\min}$ is the only minimal element of $\mathbf{W}_r$, while $\mathcal{C}_{\max}$ is the only maximal element. Let us present some partial results in this direction.

Consider an increasing filtration

$$\{\mathcal{C}_{\min}\} = \mathbf{W}^1 \subset \mathbf{W}^2 \subset \cdots \subset \mathbf{W}^r = \mathbf{W}$$

on $\mathbf{W} = \mathbf{W}_r$, where $\mathbf{W}^p$ consists of collections $\mathcal{C} \in \mathbf{W}$ such that all subsets $I \in \mathcal{C}$ with $|I| > p$ are intervals. Clearly, each $\mathbf{W}^p$ is invariant under weak lowering flips, i.e., is an order ideal in $\mathbf{W}$. The following easy lemma gives several equivalent descriptions of these order ideals.

Lemma 5.4. *Let $\mathcal{C} \in \mathbf{W}_r$, and $p = 1, \ldots, r$. The following conditions are equivalent:*
(1) $\mathcal{C} \in \mathbf{W}^p$;
(2) $\mathcal{C}$ contains all intervals of size $\geq p$;
(3) $\mathcal{C}$ contains all intervals of size p.

Proof. Since $\mathcal{C}$ is a maximal (by inclusion) weakly separated collection, the implication $(1) \Rightarrow (2)$ follows from Lemma 3.9. The implication $(2) \Rightarrow (3)$ is trivial, and $(3) \Rightarrow (1)$ follows form an easy observation that every non-interval set I is not weakly separated from some interval $[a, b]$ of smaller size.

The following conjecture sharpens Conjecture 1.8.

Conjecture 5.5. For $p \geq 2$, every collection $\mathcal{C} \in \mathbf{W}^p - \mathbf{W}^{p-1}$ admits a weak lowering flip $Lik \to Lj$ with $|Lik| = p$.

This conjecture was computer checked for $r \leq 4$ and all $p \geq 2$, and for $r = 5, p \leq 3$. We shall prove it for $p = 2$ and r arbitrary, by giving an explicit combinatorial description of the poset $\mathbf{W}^2$.

Our answer will be given in terms of parenthesizings of strings of letters. They are defined recursively as follows. For one letter a there is only one parenthesizing (a). Now a parenthesizing of a string $a_1 \cdots a_n$ for $n > 1$ is obtained by splitting it into two non-empty substrings $a_1 \cdots a_k$ and $a_{k+1} \cdots a_n$ and taking parenthesizings of each of them plus the pair of parentheses surrounding the whole string. Thus, each parenthesizing of $a_1 \cdots a_n$ has $2n-1$ pairs of matching opening and closing parentheses. It is well known that the number of such parenthesizings is the Catalan number $\frac{1}{n}\binom{2n-2}{n-1}$. For example, a string $a_1 a_2 a_3 a_4$ has five different parenthesizings:

$$((a_1)((a_2)((a_3)(a_4)))),\ ((a_1)(((a_2)(a_3))(a_4))),\ (((a_1)(a_2))((a_3)(a_4))),$$

$$(((a_1)((a_2)(a_3)))(a_4)),\ ((((a_1)(a_2))(a_3))(a_4))\ .$$

We will also need *partial parenthesizings*: a partial parenthesizing of a string $a_1 \cdots a_n$ is obtained by splitting it into any number of substrings and taking a parenthesizing of each of these substrings; in particular, any parenthesizing is considered as a partial parenthesizing. In depicting partial parenthesizings, we put substrings into which our string is split, into brackets. For example, a string $a_1 a_2 a_3$ has five partial parenthesizings:

$$[((a_1)((a_2)(a_3)))],\ [(((a_1)(a_2))(a_3))],\ [((a_1)(a_2))][(a_3)],\ [(a_1)][((a_2)(a_3))],\ [(a_1)][(a_2)][(a_3)]\ . \tag{5.2}$$

Theorem 5.6. *The subset $\mathbf{W}^2 \subset \mathbf{W}_r$ is in a bijection with all partial parenthesizings of the string $a_{12}a_{23} \cdots a_{r,r+1}$. The collection $\mathcal{C} \in \mathbf{W}^2$ corresponding to a partial parenthesizing π is described as follows. One-element members of $\mathcal{C}$ are $\{1\}, \{r+1\}$ and all $\{i\}$ such that $a_{i-1,i}$ and $a_{i,i+1}$ belong to two different brackets in π. Two-element members of $\mathcal{C}$ are $\{i<j\}$ such that $a_{i,i+1}$ and $a_{j-1,j}$ belong to the same bracket of π, and the parenthesizing of this bracket in π has a matching pair of parentheses with the opening one right before $a_{i,i+1}$ and the closing one right after $a_{j-1,j}$.*

For example, if $r=3$, there are five collections in $\mathbf{W}^2$ corresponding to five partial parenthesizings in (5.2) (where each a_i is replaced by $a_{i,i+1}$). Skipping the subsets $\{1\}, \{4\}, \{1,2\}, \{3,4\}$ that are always present, the lists of remaining one- and two-element subsets in these collections are the following:

$$\{14, 24, 23\},\ \{14, 13, 23\},\ \{3, 13, 23\},\ \{2, 24, 23\},\ \{2, 3, 23\}\ .$$

Proof. Let $\mathcal{C} \in \mathbf{W}^2$. By Lemma 5.4, $\mathcal{C}$ contains all subsets $\{i, i+1\}$ for $i = 1, \ldots, r$. Let $\{i_0\}, \{i_1\}, \ldots, \{i_{s+1}\}$ be all one-element subsets in $\mathcal{C}$, where $s \geq 1$ and $1 = i_0 < i_1 < \cdots < i_s < i_{s+1} = r+1$. Clearly, for every two-element subset $\{i<j\} \in \mathcal{C}$, the interval $[i,j]$ is contained in one of $[i_k, i_{k+1}]$, $k = 0, \ldots, s$. Since $\mathcal{C}$ is a maximal weakly separated collection, it follows that $\mathcal{C}$ contains all subsets $\{i_k, i_{k+1}\}$. Furthermore, we claim that if $i_{k+1} - i_k > 1$ for some k then there exists j such that $i_k < j < i_{k+1}$ and both $\{i_k, j\}$ and $\{j, i_{k+1}\}$ belong to $\mathcal{C}$. Indeed, take

$$j = \max\{i : i_k < i < i_{k+1},\ \{i_k, i\} \in \mathcal{C}\};$$

the fact that $\{j, i_{k+1}\} \in \mathcal{C}$ again follows from the maximality of $\mathcal{C}$. Repeating this argument, we conclude that the subsets $\{i, j\} \in \mathcal{C}$ that are contained in $[i_k, i_{k+1}]$ are exactly the ones corresponding to some parenthesizing of the string $a_{i_k, i_k+1} \cdots a_{i_{k+1}-1, i_{k+1}}$, as described in the theorem.

Corollary 5.7. *Conjecture 5.5 holds for $p = 2$. Lowering flips on a collection $\mathcal{C} \in \mathbf{W}^2 - \mathbf{W}^1$ correspond to the following operations on a partial parenthesizing:*

$$[((A)(B))] \to [(A)][(B)]$$

(splitting one bracket into two).

Let C_r be the cardinality of the subset $\mathbf{W}^2 \subset \mathbf{W}_r$ (we set $C_0 = 1$). Theorem 5.6 easily implies the following expression for the generating function of the sequence (C_r):

$$\sum_{r \geq 0} C_r x^r = \frac{2}{1 + (1 - 4x)^{1/2}}. \tag{5.3}$$

We leave the proof of (5.3) as an exercise for the reader.

Note also that a collection $\mathcal{C} \in \mathbf{W}^2$ is strongly separated if and only if the corresponding partial parenthesizing has at most two symbols $a_{i,i+1}$ inside each bracket. The number F_r of such collections is easily seen to be the Fibonacci number (i.e., we have $F_0 = F_1 = 1$ and $F_r = F_{r-1} + F_{r-2}$ for $r \geq 2$).

Now let us consider another interesting subposet of $\mathbf{W}_r$. Namely, let $\mathcal{C}' \in \mathbf{W}_{r-1}$ be the collection of all intervals in $[1, r]$, and let

$$\mathbf{M} = \mathbf{M}_r = \{\mathcal{C} \in \mathbf{W}_r \ : \ \mathcal{C}^- = \mathcal{C}'\} \tag{5.4}$$

be the fiber over $\mathcal{C}'$ of the natural projection $\mathbf{W}_r \to \mathbf{W}_{r-1}$. According to Theorem 3.5, the collections from $\mathbf{M}$ are in a bijection with sections $(\emptyset = J_0, J_1, \ldots, J_r = [1, r])$ of $\mathcal{C}'$; this bijection is given by (3.2).

The definitions in Section 3 readily imply that every section of $\mathcal{C}'$ has the form $(J_p = [b_p - p + 1, b_p])$, where $(b_0, b_1, \ldots, b_r)$ is a sequence of integers satisfying

$$0 = b_0 \leq b_1 \leq \cdots \leq b_r = r, \ \ b_p \geq p\ . \tag{5.5}$$

We will denote the corresponding collection $\mathcal{C} \in \mathbf{M}$ by $\mathcal{C}(b_1, \ldots, b_{r-1})$. Thus, the collection $\mathcal{C}(b_1, \ldots, b_{r-1})$ consists of intervals $[b - p + 1, b]$ with $0 \leq p \leq b \leq b_p$ and of subsets $[b - p + 1, b] \cup \{r + 1\}$ with $b_p \leq b \leq r$. In particular, this implies that the intersections of $\mathbf{M}$ with the terms of the filtration $(\mathbf{W}^p)$ in $\mathbf{W}_r$ are given by

$$\mathbf{M} \cap \mathbf{W}^p = \{\mathcal{C}(b_1, \ldots, b_{r-1}) \ : \ b_p = r\}\ . \tag{5.6}$$

The cardinality of $\mathbf{M}$ is easily seen to be the Catalan number $\frac{1}{r+1}\binom{2r}{r}$. One way to see this is to establish a bijection between sequences satisfying (5.5) and lattice paths from $(0,0)$ to (r,r) lying below the diagonal $x = y$; to construct such a bijection, we associate to $(b_0, b_1, \ldots, b_r)$ the lattice path whose horizontal segments are $[(b_p, p), (b_{p+1}, p)]$ for $p = 0, 1, \ldots, r-1$.

The proof of the following proposition is straightforward.

Proposition 5.8. *The subset $\mathbf{M}$ is an order ideal in $\mathbf{W}_r$. A collection $\mathcal{C}(b'_1, \ldots, b'_{r-1})$ is obtained from $\mathcal{C}(b_1, \ldots, b_{r-1})$ by a weak lowering flip if and only if*

$$(b'_1, \ldots, b'_{r-1}) = (b_1, \ldots, b_{p-1}, b_p + 1, b_{p+1}, \ldots, b_{r-1})$$

for some p such that $b_p < b_{p+1}$; this flip replaces Lik with Lj, where $i = b_p - p + 1, j = b_p + 1, k = r + 1$ and $L = [b_p - p + 2, b_p]$.

Comparing this description of weak lowering flips with (5.6), we obtaing the following corollary.

Corollary 5.9. *Conjecture 5.5 holds for any collection $\mathcal{C}(b_1, \ldots, b_{r-1}) \in \mathbf{M}$ different from $\mathcal{C}_{\min} = \mathcal{C}(r, r, \ldots, r)$.*

Note that, in view of Lemma 4.3, a collection $\mathcal{C}(b_1, \ldots, b_{r-1})$ is strongly separated if and only if $b_{p+1} \le b_p + 1$ for $1 \le p \le r-1$. Such sequences $(b_1, \ldots, b_{r-1})$ are in a bijection with binary strings of length $r-1$ via the correspondence

$$(b_1, \ldots, b_{r-1}) \mapsto (b_2 - b_1, b_3 - b_2, \ldots, b_r - b_{r-1}) \ .$$

In particular, the number of strongly separated collections in $\mathbf{M}$ is 2^{r-1}.

We conclude this paper by a "classical" application of weak flips to total positivity criteria. Consider the subgroup $B \subset GL_{r+1}$ of upper-triangular matrices. For a subset $J \subset [1, r+1]$ of size l let $\Delta^J(x)$ denote the minor of x with the row set $[1, l]$ and column set J; we will think of Δ^J as an element of the algebra $\mathbf{Q}[B]$ of polynomial functions on B. A collection $\mathcal{C}$ of non-empty subsets in $[1, r+1]$ will be called a *total positivity base* for B if $|\mathcal{C}| = \dim\,(B) = \binom{r+2}{2}$ and every flag minor $\Delta^I \in \mathbf{Q}[B]$ is a subtraction-free rational expression in the minors Δ^J $(J \in \mathcal{C})$. In view of [**BFZ**], Theorem 3.2.1, every such collection $\mathcal{C}$ gives rise to a total positivity criterion: a matrix $x \in B$ is totally positive (i.e., all minors not identically equal to 0 on B take positive real values at x) if and only if $\Delta^J(x) > 0$ for all $J \in \mathcal{C}$. The classical Fekete criterion (see [**BFZ**], Theorem 3.2.2) says that the collection $\mathcal{C}_{\min} - \{\emptyset\}$ of all non-empty intervals in $[1, r+1]$ is a total positivity base for B.

Proposition 5.10. *Weak raising and lowering flips preserve the set of all totally positivity bases for B.*

Proof. Suppose $i < j < k$ and $L \cap \{i, j, k\} = \emptyset$. Our statement follows at once from the fact that the flag minors satisfy the 3-term Plücker relation

$$\Delta^{Lj}\Delta^{Lik} = \Delta^{Li}\Delta^{Ljk} + \Delta^{Lk}\Delta^{Lij} \tag{5.7}$$

(this relation is obtained from the relation $R_2(Lijk, L)$ in Section 2 by the specialization $q = 1$).

In particular, combining this proposition with the Fekete criterion, we obtain that all maximal strongly separated collections are total positivity bases (cf. [**BFZ**], Proposition 3.2.3). Conjecture 1.8 would imply that the same is true for all maximal weakly separated collections. In any case, in view of Corollaries 5.7 and 5.9, all the collections in Theorem 5.6 and Proposition 5.8 are total positivity bases, hence give rise to new criteria for total positivity.

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B.L. Département de Mathématiques, Université de Caen, 14032 Caen cedex, FRANCE.

A.Z. Department of Mathematics, Northeastern University, Boston MA 02115, USA.