COUNTING PATHS IN YOUNG'S LATTICE

Ira M. Gessel*
Department of Mathematics
Brandeis University
Waltham, MA 02254-9110

Abstract. Young's lattice is the lattice of partitions of integers, ordered by inclusion of diagrams. Standard Young tableaux can be represented as paths in Young's lattice that go up by one square at each step, and more general paths in Young's lattice correspond to more general kinds of tableaux. Using the theory of symmetric functions, in particular Pieri's rule for multiplying a Schur function by a complete symmetric function, we derive formulas for counting paths in Young's lattice that go up or down by horizontal or vertical strips. Our results are related to Richard Stanley's theory of differential posets in the special case of Young's lattice.

1. Introduction. Richard Stanley (1988) has studied paths in certain posets, called "differential posets," using properties of operators which move up and down by one rank in the poset. In the case of Young's lattice, the lattice of partitions ordered by inclusion of diagrams, these paths are sometimes called *oscillating* or *up-down tableaux*, and they generalize standard Young tableaux (i.e., with distinct entries). We describe here a related method, using symmetric functions, which can be used to extend some of these results to tableaux which may have repeated elements. We count paths in Young's lattice (or more precisely, walks in the Hasse graph of Young's lattice) that move up or down not only by one square at a time, but more generally by horizontal or vertical strips. Such paths have arisen in the work of Sundaram (1986, 1990) on the combinatorics of representations of symplectic groups.

A similar approach, in a more general context, has been developed independently by Fomin (1992).

2. Symmetric functions. We follow the notation of Macdonald (1979) for symmetric functions. In particular, s_λ, h_n, and e_n denote the Schur function, the complete symmetric function, and the elementary symmetric function.

---

*partially supported by NSF grant DMS-8902666

1991 *Mathematics Subject Classification.* Primary: 05E05. Secondary: 05E10, 05A15.

Key words and phrases. Young's lattice, tableaux, oscillating tableaux, symmetric functions, Schur functions, Pieri's rule, differential posets.

Pieri's rule (see Macdonald (1979, p. 42)) is the formula

$$s_\lambda h_n = \sum_\mu s_\mu, \tag{1}$$

where the sum is over all partitions μ such that $\mu - \lambda$ is a horizontal n-strip, i.e., a set of n squares no two of which are in the same column. We shall also need the analogous formula (Macdonald (1978), p. 42)

$$s_\lambda e_n = \sum_\mu s_\mu, \tag{2}$$

where the sum is over all partitions μ such that $\mu - \lambda$ is a vertical n-strip.

It follows from (1) by induction on k that the coefficient of s_μ in $s_\lambda h_{n_1} \cdots h_{n_k}$ is the number of paths in Young's lattice from λ to μ consisting of k steps, in which the ith step goes up by a horizontal n_i-strip. Such a path can be represented by a tableau of shape $\mu - \lambda$ with n_1 1's, n_2 2's, ..., n_k k's. More generally, the coefficient of s_μ in

$$s_\lambda h_{m_1} e_{n_1} h_{m_2} e_{n_2} \cdots h_{m_k} e_{n_k}$$

is the number of paths in Young's lattice from λ to μ consisting of $2k$ steps, which go up alternately by horizontal and vertical strips of sizes $m_1, n_1, \ldots, m_k, n_k$. (Since steps of size 0 are allowed, all possible paths from λ to μ that go up by horizontal and vertical strips are covered.) For further information on the corresponding generalized tableaux, see, for example, Remmel (1984).

We would now like to count paths that can go either up or down by horizontal or vertical strips. To do this we will work with linear operators on symmetric functions which are products of the operators of multiplication by h_n or e_n and the adjoints of these multiplication operators. It is convenient to write operators *after* their operands. Our notation does not distinguish between the symmetric function f and the operator of multiplication by f, but this should cause no problems.

If f is any symmetric function, then $D(f)$ is the adjoint of multiplication by f with respect to the usual scalar product on symmetric functions; that is, for any symmetric functions a and b, $\langle aD(f), b\rangle = \langle a, bf\rangle$. (See Macdonald (1979, pp. 43–45).) For convenience we write f^* for $D(f)$.

By the definition of h_n^*, $\langle s_\lambda h_n^*, s_\mu\rangle = \langle s_\lambda, s_\mu h_n\rangle$. Then Pieri's rule, together with orthogonality of the Schur functions, implies that

$$s_\lambda h_n^* = \sum_\mu s_\mu \tag{3}$$

where the sum is over all partitions μ such that $\lambda - \mu$ is a horizontal n-strip, and similarly

$$s_\lambda e_n^* = \sum_\mu s_\mu \tag{4}$$

where the sum is over all partitions μ such that $\lambda - \mu$ is a vertical n-strip.

It follows that the coefficient of s_μ in

$$s_\lambda h_{k_1} e_{l_1} h^*_{m_1} e^*_{n_1} h_{k_2} e_{l_2} h^*_{m_2} e^*_{n_2} \cdots h_{k_r} e_{l_r} h^*_{m_r} e^*_{n_r}$$

is the number of paths in Young's lattice from λ to μ consisting of $4r$ steps, each of which goes up or down by a horizontal or vertical strip of the appropriate size.

We now describe a very useful notation for symmetric functions. (For some other applications of this notation, see, for example, Lascoux and Pragacz (1988).) Let $\alpha, \beta, \ldots$ be variables. By a *monomial* we mean an expression of the form $\alpha^r \beta^s \cdots$. Note that the coefficient of a monomial must be 1; thus $2\alpha^2\beta^3$ is not a monomial, but it may be written as a sum of two (equal) monomials. Note also that $1 = \alpha^0 \beta^0 \cdots$ is a monomial.

Now let $w_1, w_2, \ldots$ be monomials and let $f = f(x_1, x_2, \ldots)$ be a symmetric function. We define $f(w_1 + w_2 + \cdots)$ to be $f(w_1, w_2, \cdots)$. It is clear that if $w = w_1 + w_2 + \cdots$ and f and g are symmetric functions then

$$(f+g)(w) = f(w) + g(w) \qquad \text{and} \qquad (f \cdot g)(w) = f(w) \cdot g(w). \tag{5}$$

It is useful when working with vertical strips to extend this definition of $f(w)$ to sums of monomials with arbitrary coefficients. If $w = m_1 w_1 + m_2 w_2 \cdots$, where the m_i are nonnegative integers, then $f(w)$ is already defined as

$$f(\underbrace{w_1 + \cdots + w_1}_{m_1} + \underbrace{w_2 + \cdots + w_2}_{m_2} + \cdots).$$

In particular, for the power sum symmetric functions p_r, we have

$$p_r(m_1 w_1 + m_2 w_2 + \cdots) = m_1 w_1^r + m_2 w_2^r + \cdots \tag{6}$$

whenever the w_i are monomials and the m_i are nonnegative integers.

We now take (6) as a definition for arbitrary coefficients $m_1, m_2, \ldots$. Together with (5), this defines $f(w)$ for all symmetric functions f.

It is sometimes useful to let an unsubscripted letter denote the sum of the corresponding subscripted letters, which we take as variables. Thus if $x = x_1 + x_2 + \cdots$ and f is a symmetric function then $f(x)$ is actually equal to $f(x_1, x_2, \ldots)$. We adopt this convention for the letters x, y, u, d, U, and D.

We also use the notation $h = h(x)$ for $\sum_{n=0}^\infty h_n = \prod_i (1 - x_i)^{-1}$. One of the basic properties of h is that $h(x+y) = h(x)h(y)$. In particular, $1 = h(0) = h(x-x) = h(x)h(-x)$, so $h_n(-x) = (-1)^n e_n(x)$. This property will allow us to write formulas involving both complete and elementary symmetric functions compactly.

Rather than working with h_n and h_n^* directly, it will be more convenient to work with their generating functions. If s is a variable, $\sum_{n=0}^\infty s^n h_n(x) = h(sx)$. There is also a simple description of the action of $\sum_{n=0}^\infty t^n h_n^*(x) = h^*(tx)$, where t is a variable. It is not hard to show that for any symmetric functions $f = f(x)$ and $g = g(x)$, $gf^* = \langle g(x+y), f(y) \rangle_y$, where $\langle\ ,\ \rangle_y$ denotes the scalar product in the y variables only.[1] We also recall that for any symmetric function $f(x)$, $\langle f(y), h(zy)\rangle_y = f(z)$. Thus

$$g(x)h^*(tx) = \langle g(x+y), h(ty)\rangle_y = g(x+t), \tag{7}$$

so $h^*(tx)$ is a homomorphism. (In fact (7) is true for arbitrary t.) Although $h(sx)$ and $h^*(tx)$ don't commute, (7) makes it easy to evaluate the products in which they appear.

3. Paths. We first consider the case of paths that go up or down by horizontal strips only. Note that

$$s_\lambda h_{m_1} h^*_{n_1} h_{m_2} h^*_{n_2} \cdots$$

is the coefficient of $u_1^{m_1} d_1^{n_1} u_2^{m_2} d_2^{n_2} \cdots$ in

$$s_\lambda h(u_1 x) h^*(d_1 x) h(u_2 x) h^*(d_2 x) \cdots . \tag{8}$$

We recall that

$$h(xy) = \sum_\lambda s_\lambda(x) s_\lambda(y). \tag{9}$$

Thus (8) is the coefficient of $s_\lambda(y)$ in

$$h(xy)h(u_1x)h^*(d_1x)h(u_2x)h^*(d_2x)\cdots .$$

Now $h(xy)h(u_1x)h^*(d_1x) = h(xy + u_1x + d_1y + u_1d_1)$, and an easy induction shows that in general

$$h(xy)h^*(d_1x)h(u_1x)h^*(d_2x)h(u_2x)\cdots = h\Bigl(xy + ux + dy + \sum_{i\le j} u_i d_j\Bigr). \tag{10}$$

Thus the coefficient of $s_\lambda(x)s_\mu(y)$ in (10) is the generating function for paths from μ to λ; that is, the coefficient of $s_\lambda(x)s_\mu(y)u_1^{m_1}d_1^{n_1}u_2^{m_2}d_2^{n_2}\cdots$ in (10) is the number of paths in Young's lattice from λ to μ that first go up by a horizontal m_1-strip, then down by a horizontal n_1-strip, then up by a horizontal m_2-strip, and so on. A straightforward calculation, which we omit, shows that the coefficient of $s_\lambda(x)s_\mu(y)$ in (10) may be expressed as

$$\sum_\nu s_{\lambda/\nu}(u) s_{\mu/\nu}(d) \Big/ \prod_{i\le j}(1 - u_i d_j) \tag{11}$$

In particular, the generating function for paths from $\hat{0}$ to λ is

$$s_\lambda(u) \Big/ \prod_{i\le j}(1 - u_i d_j). \tag{12}$$

[1]This follows easily from the fact that the comultiplication $f(x) \mapsto f(x+y)$ is the dual of multiplication; i.e., $\langle a(x+y), b(x)c(y)\rangle_{xy} = \langle a(x), b(x)c(x)\rangle$.

and the generating function for paths from $\hat{0}$ to $\hat{0}$ is

$$\prod_{i\le j} \frac{1}{1-u_i d_j}. \tag{13}$$

To count paths in which all the up steps come before all the down steps, we set some of the u_i and d_j equal to zero so that $i \le j$ for every nonzero u_i and v_j, and (13) becomes $h(ud)$. In this case a path can be represented by a pair of tableaux of the same shape, and our result reduces to (9), with u and d replacing x and y.

For a generating function of a slightly different type, let us count closed paths, where instead of keeping track of the starting and ending point, we weight a path from λ to λ by $q^{|\lambda|}$. Then the generating function for these paths is

$$\sum_{\lambda,\nu} s_{\lambda/\nu}(u) s_{\lambda/\nu}(d) q^{|\lambda|} \Big/ \prod_{i\le j}(1-u_i d_j),$$

which by Corollary 6.7 of Sagan and Stanley (1990) is equal to

$$h\left(q\frac{1+ud}{1-q}\right)\Big/\prod_{i\le j}(1-u_i d_j) = \prod_{m=1}^{\infty}(1-q^m)^{-1} \prod_{k,l,m=1}^{\infty}(1-u_k d_l q^m)^{-1} \prod_{i\le j}(1-u_i d_j)^{-1}. \tag{14}$$

When we allow vertical strips as steps, it is convenient to work with $(-1)^n e_n = h_n(-x)$ and $(-1)^n e_n^* = h_n^*(-x)$ rather than e_n and e_n^*, since this allows us to use a more compact notation. Then in the generating functions obtained using this convention, the term in the corresponding to a path will be multiplied by $(-1)^N$, where N is the sum of the lengths of all the vertical strips, up and down, in the path. When our final result is obtained and written out explicitly, we can make all signs positive by replacing each U_i with $-U_i$ and each D_j with $-D_j$.

We have

$$\sum_{n=0}^{\infty}(-1)^n e_n(x) U_i^n = h(-U_i x),$$
$$\sum_{n=0}^{\infty}(-1)^n e_n^*(x) D_i^n = h^*(-D_i x),$$

and with the formula $g(x)h^*(-tx) = g(x-t)$ we find as before that

$$\begin{aligned} &h(xy)h(u_1 x)h(-U_1 x)h^*(d_1 x)h^*(-D_1 x)\cdots \\ &\qquad = h\Bigl(xy + (u-U)x + (d-D)y + \sum_{i\le j}(u_i - U_i)(d_j - D_j)\Bigr). \end{aligned} \tag{15}$$

Thus, for example, setting $x = y = 0$ and fixing the signs, we find that the generating function for paths from $\hat{0}$ to $\hat{0}$ is

$$\prod_{i\le j}\frac{(1+u_i D_j)(1+U_i d_j)}{(1-u_i d_j)(1-U_i D_j)}.$$

Similar formulas can be obtained if we don't care where the paths start, by using

$$\sum_\lambda s_\lambda(x) = h\big(x + e_2(x)\big) = \prod_i \frac{1}{1-x_i} \prod_{i\le j} \frac{1}{1-x_i x_j} \tag{16}$$

in place of (9), and more generally, we may use any of Littlewood's formulas described in Macdonald (1979, pp. 45–47). Thus, for example, we can count paths that begin at a partition with an even number of parts or paths that begin at a partition of the form $(\alpha_1 - 1, \ldots, \alpha_p - 1 \mid \alpha_1, \ldots, \alpha_p)$ in Frobenius notation.

4. An R-S-K bijection. As noted above, the generating function (13) for paths from $\hat{0}$ to $\hat{0}$ is a generalization of the generating function for pairs of tableaux of the same shape. It is natural to ask whether a bijective proof can be given by a generalization of the Robinson-Schensted-Knuth correspondence (Knuth, 1970). We describe a bijection that proves the "dual" of (12), that the generating function for paths from μ to $\hat{0}$ is $s_\mu(d)/\prod_{i\le j}(1-u_i d_j)$. We omit the proof, which follows easily from properties of Knuth's insertion algorithm.

The bijection is given by an algorithm which constructs a sequence of column-strict decreasing tableaux P_i from a two-line array of positive integers

$$\begin{pmatrix} q_1 \cdots q_N \\ p_1 \cdots p_N \end{pmatrix}. \tag{17}$$

together with a column-strict decreasing tableau T of shape μ. The two-line array must satisfy $q_i \le p_i$ and the lexicographic conditions $q_i \le q_{i+1}$ and if $q_i = q_{i+1}$ then $p_i \ge p_{i+1}$. The sequence of shapes of the P_i is the desired path from μ to $\hat{0}$ in Young's lattice. Given a two line array (17) and a tableau T, the tableaux P_i are constructed as follows:

> Let M be the largest integer occurring in T or among the p_i. Let σ_k for $1 \le k \le N$ be the word in the p's lying underneath the occurrences of k among the q's (so σ_k is weakly decreasing). By "insertion" of σ_k into a column-strict decreasing tableau we mean the usual Knuth row insertion.

(1) Start with $P_0 = T$.
(2) For $k = 1$ to M do
  Step $2k-1$: Insert σ_k into P_{2k-2} to obtain P_{2k-1}.
  Step $2k$: Delete all k's from P_{2k-1} to obtain P_{2k}.

As an example we take the tableau

$$T = \begin{array}{|c|c|c|} \hline 3 & 3 & 1 \\ \hline 2 \\ \cline{1-1} \end{array}$$

and the two-line array

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 3 & 2 \end{pmatrix}$$

Then we have $P_0 = T$,

$$P_1 = \begin{array}{|c|c|c|c|c|}\hline 3 & 3 & 2 & 2 & 1 \\ \hline 2 & 1 \\ \cline{1-2}\end{array} \qquad P_2 = \begin{array}{|c|c|c|c|}\hline 3 & 3 & 2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$P_3 = \begin{array}{|c|c|c|c|c|}\hline 3 & 3 & 3 & 2 & 2 \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \qquad P_4 = \begin{array}{|c|c|c|}\hline 3 & 3 & 3 \\ \hline\end{array}$$

$$P_5 = \begin{array}{|c|c|c|}\hline 3 & 3 & 3 \\ \hline\end{array} \qquad P_6 = \hat{0}$$

Similar algorithms in the special case of oscillating tableaux were given by Delest, Dulucq, and Favreau (1988) Sundaram (1986, Lemma 8.7), R. P. Stanley (unpublished), and G. X. Viennot (unpublished). The general case was found independently by Roby (1991).

5. Exponential generating functions. As special cases of his results on differential posets, Stanley obtained numerous exponential generating functions for paths in Young's lattice that go up or down by only one square at a time. We indicate how some of these results can be derived from the results of Section 3. The key to obtaining exponential generating functions is a well-known homomorphism from symmetric functions to exponential generating functions. (See, e.g., Macdonald (1979, p. 18, Ex. 2) or Gessel (1990 Theorem 1).) We may define a linear map θ from power series in $u_1, u_2, \ldots$ to power series in z by

$$\theta(u_1 u_2 \cdots u_n) = \frac{z^n}{n!}, \quad \text{including } \theta(1) = 1;$$
$$\theta\left(\prod u_i^{m_i}\right) = 0 \quad \text{for all other monomials.}$$

Then the restriction of θ to symmetric power series is a homomorphism that also satisfies

$$\theta\big(p_n(u)\big) = \begin{cases} z, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases}$$

and it is clear that

$$\theta\big(s_\lambda(u)\big) = f^\lambda \frac{z^{|\lambda|}}{|\lambda|!},$$

where f^λ is the number of standard Young tableaux of shape λ.

To count paths that go up or down by 1 square at each step, we set $d_i = u_i$ in the relevant generating function and extract the coefficient of $u_1 u_2 \cdots u_n$. Since the substitution yields a symmetric function of u, an exponential generating function can be obtained by applying θ. It is easily checked that $\theta\left(\prod_{i \le j}(1 - u_i u_j)^{-1}\right) = e^{z^2/2}$.

Thus from (12) we obtain the generating function $f_\lambda e^{z^2/2} z^{|\lambda|}/|\lambda|!$ for oscillating tableaux of shape λ, as found by Sundaram (1986) and Delest, Dulucq, and Favreau (1988). Similarly, from (14) we obtain the exponential generating function

$$\exp\left(\frac{z^2}{2} + \frac{qz^2}{1-q}\right) \Big/ \prod_{m=1}^{\infty} (1-q^m)$$

in which the coefficient of $q^k z^n/n!$ is the number of closed paths in Young's lattice with n steps (each up or down by one square), beginning and ending at the same partition of k. This is the case $r = 1$ of Stanley (1988, Corollary 3.14).

6. Commutation relations. The operators U and D of Stanley (1988) (not to be confused with U and D as used in this paper), when restricted to Young's lattice, are h_1 and h_1^*. Stanley's theory is based on the fact that their commutator is 1. Thus it is natural to try to generalize these commutation relations to h_m and h_n^*, and to ask whether analogous operators satisfying these relations exist for other posets. We note also that the h_i^*, like Stanley's D, are differential operators, but not of first order (Macdonald (1978, pp. 43–45)).

The commutation relations are easily derived by considering the action of their generating functions: If f is any symmetric function then

$$f(x)h^*(d_1x)h(u_1x) = f(x+d_1)h(u_1x).$$

and

$$\begin{aligned} f(x)h(u_1x)h^*(d_1x) &= f(x+d_1)h(u_1x+u_1d_1) \\ &= f(x+d_1)h(u_1x)h(u_1d_1) \\ &= f(x)h^*(d_1x)h(u_1x)h(u_1d_1). \end{aligned} \tag{18}$$

Equating coefficients of $u_1^m d_1^n$ in the extremes of (18) we obtain

$$h_m h_n^* = \sum_{i\ge 0} h_{n-i}^* h_{m-i}. \tag{19}$$

Equivalently, multiplying both sides of (18) by $1 - u_1v_1 = h(u_1d_1)^{-1}$, we obtain

$$h_n^* h_m = h_m h_n^* - h_{m-1}h_{n-1}^*.$$

Similarly,

$$h_m e_n^* = e_n^* h_m + e_{n-1}^* h_{m-1}$$

and analogous formulas hold for the other operators.

It is not too difficult to give combinatorial proofs of these commutation relations. Thus to prove (19) combinatorially in our context, given partitions λ and μ, we must find a bijection from the set of partitions ν such that $\nu - \lambda$ is a horizontal m-strip and $\nu - \mu$ is a horizontal n-strip to the set of partitions $\bar{\nu}$ such that for some

$i \geq 0$, $\lambda - \bar{\nu}$ is a horizontal $(n-i)$-strip and $\mu - \bar{\nu}$ is a horizontal $(m-i)$-strip. It can be shown that the following correspondence is such a bijection: Given ν, let $\alpha_i = \nu_i - \lambda_i$ and $\beta_i = \nu_i - \mu_i$. Let $\bar{\alpha}_i = \min(\alpha_{i+1}, \beta_{i+1}) + \max(0, \beta_i - \alpha_i)$. Then define $\bar{\nu}$ by $\bar{\nu}_i = \lambda_i - \bar{\alpha}_i$.

An informal description of this bijection may be helpful: When we go from λ up to ν then down to μ, in row i there is an "overlap" of $\min(\alpha_i, \beta_i)$ squares which are added and then removed and a "net gain" of $\alpha_i - \beta_i$ squares or a "net loss" of $\beta_i - \alpha_i$ squares, whichever is nonnegative. A similar statement holds when we go from λ down to $\bar{\nu}$ then up to μ. Then $\bar{\nu}$ is chosen so that the overlap in row i of $\lambda \to \bar{\nu} \to \mu$ is equal to the overlap in row $i+1$ of $\lambda \to \nu \to \mu$. We note that the special case $\lambda = \mu$ (and consequently $m = n$) of this bijection was given by Macdonald (1979, Ex. 4, p. 45) who used it to prove (16).

Young's lattice is, in Stanley's terminology, a "1-differential poset"; but Stanley's theory applies more generally to "r-differential posets," of which the r-fold product of Young's lattice, Y^r is an example. We can study paths in Y^r by representing its elements as products of Schur functions in r sets of variables. Thus the element (λ, μ) of Y^2 may be represented by $s_\lambda(x)s_\mu(y)$ and we may count paths in Y^2 by using the operators $h_n(x+y) = \sum_{i+j=n} h_i(x)h_j(y)$ and $h_n^*(x+y) = \sum_{i+j=n} h_i^*(x)h_j^*(y)$. The commutation relations in the general case of Y^r, which correspond to the defining property of an r-differential poset, are easily shown to be

$$h_m h_n^* = \sum_{i \geq 0} \binom{r+i-1}{r-1} h_{n-i}^* h_{m-i};$$

here $h_m = h_m(x+y+\cdots) = \sum_{i+j+\cdots=m} h_i(x)h_j(y)\cdots$, and similarly for h_n^*.

Fomin (1992) has used more general commutation relations to count paths in graphs.

Acknowledgment: I would like to thank Adriano Garsia for helpful discussions regarding the operators h_n^* and e_n^*.

References

1. M.-P. Delest, S. Dulucq, and L. Favreau (1988), *An analogue to Robinson-Schensted correspondence for oscillating tableaux*, Actes 20e Séminaire Lotharingien, Publ. I.M.R.A., 372/S-20, Strasbourg, pp. 57–78.
2. S. Fomin (1992), *Schur operators and Knuth correspondences*, preprint.
3. I. M. Gessel (1990), *Symmetric functions and P-recursiveness*, J. Combin. Theory Ser. A **53**, 257–285.
4. D. E. Knuth (1970), *Permutations, matrices, and generalized Young tableaux*, Pacific J. Math. **34**, 709-727.
5. A. Lascoux and P. Pragacz (1988), *S-function series*, J. Phys. A.: Math. Gen. **21**, 4105–4114.
6. I. G. Macdonald (1979), *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford.
7. J. B. Remmel (1984), *The combinatorics of (k,l)-hook Schur functions*, Combinatorics and Algebra, (C. Greene, Ed.) Contemporary Mathematics, Vol. 34, American Mathematical Society, Providence, pp. 253–288.
8. T. W. Roby, *Applications and Extensions of Fomin's Generalization of the Robinson-Schensted Correspondence to Differential Posets*, Ph. D. Thesis, Massachusetts Institute of Technology.

9. B. E. Sagan and R. P. Stanley (1990), *Robinson-Schensted algorithms for skew tableaux*, J. Combin. Theory Ser. A **55**, 161–193.
10. R. P. Stanley (1988), *Differential posets*, J. Amer. Math. Soc. **1** (1988), 919–961.
11. R. P. Stanley (1990), *Variations on differential posets*, Invariant Theory and Tableaux, ed. D. Stanton, The IMA Volumes in Mathematics and Its Applications, Vol 19, Springer-Verlag, New York, pp. 145–165.
12. S. Sundaram (1986), *On the Combinatorics of Representations of* $Sp(2n, \mathbf{C})$, Ph. D. Thesis, Massachusetts Institute of Technology.
13. S. Sundaram (1990), *The Cauchy identity for* $Sp(2n)$, J. Combin. Theory Ser. A **53**, 209–238.