QUASIDETERMINANTS, I

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Introduction

A notion of quasideterminants for matrices over a noncommutative skew-field was introduced in [GR], [GR1], [GR2]. It proved its effectiveness for many areas including noncommutative symmetric functions [GKLLRT], noncommutative integrable systems [RS],[EGR], quantum algebras [GR], [GR1], [GR2], [KL], [M], etc.

The main property of quasideterminants is a "heredity principle": let A be a square matrix over a skew-field and (A_{ij}) be its block decomposition into submatrices of A. Consider A_{ij}'s as elements of a matrix X. Then a quasideterminant of matrix X will be a matrix B again, and (under a natural assumption) a quasideterminant of B will be equal to a suitable quasideterminant of A.

This principle is not valid for commutative determinants, because they are not defined for block-matrices.

Quasideterminants are not analogues of commutative determinants but rather a ratio of determinants of $n \times n$-matrices to determinants of their $(n-1) \times (n-1)$-submatrices. In fact, for a matrix over a commutative algebra a quasideterminant is equal to a correspondence ratio.

Many noncommutative areas of mathematics (Ore rings, rings of differential operators, theory of factors, "quantum mathematics", Clifford algebras, etc) were developed separately from each other. Our approach shows an advantage of working with totally noncommutative variables (over free rings and skew-fields). It leads us to a big variety of results, and their specialization to different noncommutative areas implies known theorems with additional information.

The price you pay for this is a huge number of inversions in rational noncommutative expressions measured by their height which cannot be reduced. But this invariant (inversion height) shows us a "degree of noncommutativity" and it is of a great interest by itself.

Our experience shows that dealing with noncommutative objects one should not imitate the classical commutative mathematics, but follow "the way it is" starting with basics. In this paper we consider mainly two such problems: noncommutative Plücker coordinates (as a background of a noncommutative geometry) and noncommutative Bezout and Vieta theorems (as a background of noncommutative algebra). We apply these results to the theory of noncommutative symmetric functions started in [GKLLRT].

The first chapter of this paper contains basic definitions and properties of quasideterminants. We consider Plücker coordinates in Chapter II. Chapter III is devoted to Bezout and Vieta theorems and also noncommutative symmetric functions. In the last Chapter we continue our investigation of noncommutative continued fractions and almost triangular matrices. It turns out that this problem is related with computation of quantum cohomology [G], [FK].

Some results in the paper are given without proofs, since proofs are straightforward or contained in previous publications.

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I. A general theory and main identities

§1.1 Definition

I.1.1. The skew-field of rational functions of free variables.

This theory was developed by Amitsur, Bergman and P. M. Cohn (see, for example [C], [C1]). We consider here skew-fields over fields of characteristic 0, but everything may be generalized for fields of characteristics p. Here we will remind the approach by Amitsur presented by Bergman. Given a set $X = \{x_1, \dots, x_n\}$ we write $\mathbb{F}(X)$ for a the algebra freely generated from the elements of X over rationals $\mathbb{Q}$ by the operations of addition, subtraction, multiplication and taking an inverse.

No relations are imposed, thus e.g. $(x - x)^{-1}$ exists. The algebra $\mathbb{F}(X)$ is called an algebra of rational formulas on X. Let R be a ring with unit. Any map $\alpha : X \to R$ defines a map $\overline{\alpha}$ of a subset of $\mathbb{F}(X)$ into R by the following rules

i) $\overline{\alpha}(m) = m,\ m \in \mathbb{Z}$,
ii) $\overline{\alpha}(x_i) = \alpha(x_i), \quad i = 1, \dots, n$,
iii) if $a = -b$, or $b + c$, or bc and $\overline{\alpha}(b), \overline{\alpha}(c)$ are defined then $\overline{\alpha}(a) = -\overline{\alpha}(b)$, or $\overline{\alpha}(b + c)$, or $\overline{\alpha}(b) \cdot \overline{\alpha}(c)$,
iv) if $a = b^{-1}$ and $\overline{\alpha}(b)$ is defined and invertible in R then $\overline{\alpha}(a) = (\overline{\alpha}(b))^{-1}$.

Let now $\alpha : X \to D$ be a map to a skew-field D. Then $\alpha(a)$ is undefined if and only if a has a subexpression b^{-1} such that $\alpha(b) = 0$. For each $\alpha : X \to D$ one can consider a subset $E(\alpha)$ of $\mathbb{F}(X)$, the domain of $\overline{\alpha}$, consisting of the expressions which can be evaluated for α. Similarly with each $f \in \mathbb{F}(X)$ one can associate its domain dom$f$, a subset of D^n consisting of the points $(\alpha = (\alpha_i), i = 1, \dots n)$ such that$f \in E(\alpha)$ is defined; f is called *nondegenerate* if dom$f \neq \emptyset$.

One can show [C],[C1] that if D is a skew-field which is an algebra over an infinite field k and f and g are nondegenerate then dom$f \cap$dom$g \neq \emptyset$.

Given $f, g \in \mathbb{F}(X)$, let us put $f \sim g$ if f, g are nondegenerate and f, g have the same value at each point of dom $f \cap$dom$g$. This is an equivalence.

We call the classes of equivalence the rational functions on a skew-field D. Then we have

Theorem 1.1.1 [C], [C1].

i) Let D be a skew-field with a center k of characteritic 0. Then the equivalence classes of rational formulas of variables $x_1, \dots, x_n$ form a skew-field $F_D(X)$.

ii) If D is infinite dimensional over k then $F_D(X)$ does not depend on the skew-field D.

Under assumptions of i) and ii) we will identify all $F_D(X)$ and use a notation $F(X)$. For example, if $\#X = 1$, i.e. $X = x$ then $F(X) = \mathbb{Q}(x)$.

This skew-field $F(X)$ is called *the free skew-field.*

All the equalities in this chapter are considered as the equalities of rational functions.

Remark. As shown by P.M. Cohn [C], [C1] the free field is universal in the following sense: consider the ring $k\langle x = (x_1, \dots, x_n)\rangle$ of noncommutative polynomials over the commutative field k, and the category whose objects are homomorphisms $k\langle x\rangle \to P, P$ is a skew-field; the morphisms between two such objects $\phi_i : k\langle x\rangle \to P_i, i = 1, 2$ are *specializations* $s : P_1 \to P_2$, i.e. homomorphisms from a local subring of P_1 into P_2 such that $s\phi_1 = \phi_2$ (see [C], [C1] for exact definitions). Then the free skew-field is an initial object in this category.

We remind the following definition (see, for example, [R]).

Definition 1.1.2. The inversion height of a rational expression is the maximal number of nested inversions in it. The inversion height of an element in the free skew-field $F(X)$ is the smallest inversion height of its rational expressions.

Let D be a skew-field generated by elements $x_1, \dots, x_n$ over a central subfield. If $y \in D$ may be written as a polynomial expression $P(x_1, \dots, x_n)$ in D then y has inversion height zero. If $x_1, \dots, x_n$ are generators of an Ore ring, and D its skew-field of fractions then any element $z \in D$ is a ratio of two polynomial expressions of generators. Then, z has height ≤ 1.

I.1.2. The definition of quasideterminants. We start from a definition for free skew-fields [GR1, GR2]. Let I, J be ordered sets consisting of n elements. Let $A = (a_{ij}), i \in I, j \in J$ be a matrix with formal noncommuting entries a_{ij}. Let us define by induction n^2 rational expression $|A|_{pq}$ of variables $a_{ij}, p \in I, q \in J$ over a free skew-field generated by a_{ij}'s.

We call these expressions the *quasideterminants.*

For $n = 1$ we set $|A|_{ij} = a_{ij}$.

For a matrix $A = (a_{ij}), i \in I, j \in J$ of order n we denote $A^{\alpha\beta}, \alpha \in I, \beta \in J$ the matrix of order $n - 1$ constructed by deleting the row with the index α and the column with the index β in the matrix A. Suppose that for the given $p \in I, q \in J$ expressions $|A^{pq}|_{ij}^{-1}, i \in I, j \in J, i \neq p, j \neq q$ are defined and set

$$|A|_{pq} = a_{pq} - \sum a_{pj}|A^{pq}|_{ij}^{-1}a_{iq} \tag{1.1.1}$$

Here the sum is taken over all $i \in I \setminus \{p\}, j \in J \setminus \{q\}$.

Definition 1.1.3. We call the expression $|A|_{pq}$ the quasideterminant of indices p and q of the matrix A.

The following definition of quasideterminants for matrices over general rings was given in [GR].

Definition 1.1.4. *For a matrix A over a ring with unit the quasideterminant $|A|_{pq}$ is defined if the matrix A^{pq} is invertible. In this case*

$$|A|_{pq} = a_{pq} - \sum_{\substack{i \in I \setminus \{p\} \\ j \in J \setminus \{q\}}} a_{pj}b_{ji}a_{iq}, \tag{1.1.2}$$

where b_{ji} are the entries of the matrix $(A^{pq})^{-1}$.

It is known [GR,GR1] that both definitions of a quasideterminant coincide over free skew-fields.

Examples.

1) For a matrix $a = (a_{ij}), i, j = 1, 2$ there exist four quasideterminants

$$|A|_{11} = a_{11} - a_{12} \cdot a_{22}^{-1} \cdot a_{21}, \qquad |A|_{12} = a_{12} - a_{11} \cdot a_{21}^{-1} \cdot a_{22},$$
$$|A|_{21} = a_{21} - a_{22} \cdot a_{12}^{-1} \cdot a_{11}, \qquad |A|_{22} = a_{22} - a_{21} \cdot a_{11}^{-1} \cdot a_{12}.$$

2) For a matrix $A = (a_{ij}), i, j = 1, 2, 3$ there exist 9 quasideterminants but we will write here only

$$|A|_{11} = a_{11} - a_{12}(a_{22} - a_{23}a_{33}^{-1}a_{32})^{-1}a_{21} \quad -a_{12}(a_{32} - a_{33} \cdot a_{23}^{-1}a_{22})^{-1}a_{31}$$
$$-a_{13}(a_{23} - a_{22}a_{32}^{-1}a_{33})^{-1}a_{21} \quad -a_{13}(a_{33} - a_{32} \cdot a_{22}^{-1}a_{23})^{-1}a_{31}$$

3) If in formulas (1.1), or (1.2) variables a_{ij} commute each other then

$$|A|_{pq} = (-1)^{p+q}\frac{\det A}{\det A^{pq}}.$$

Remark. If any a_{ij} is an invertible morphism $V_j \to V_i$ in an additive category then the quasideterminant $|A|_{pq}$ is also a morphism from the object V_q to the object V_p.

One can use formulas (1.1) or (1.2) to define a quasideterminant over any skew-field. We recall here that according to Amistsur's theory a rational function f is defined over a ring if at least one of the rational expressions for the function f is defined over this ring. In particular, the quasideterminant $|A|_{pq}$ can be defined if some of $(n-1)$-quasiminors of the matrix A are not defined and expression (1.1) makes no sense (but an equivalent expression does!).

The following theorem was conjectured by I. Gelfand and V. Retakh, and proved by Reutenauer [R].

Theorem I.1.5. *Quasideterminants of the $n \times n$-matrix with formal entries have the inversion height $n-1$ over the free skew-field generated by its entries.*

§1.2 General properties of quasideterminants

I.2.1. Two involutions. For a square matrix $A = (a_{ij})$ denote by $IA = A^{-1}$ inverse matrix, and by $HA = (a_{ji}^{-1})$ its Hadamard inverse. It is evident that if IA, or HA are defined then $I^2A = A$,respectively $H^2A = A$.

Let $A^{-1} = (b_{ij})$. According to [GR, GR1] $b_{ij} = |A|_{ji}^{-1}$. This formula could be written in the following form.

Theorem 1.2.1. *For a square matrix A with formal entries*

$$HI(A) = (|A|_{ij}) \tag{1.2.1}$$

I.2.2. Homological relations [GR, GR1, GR2]. The ratio of two quasideterminants of the same square matrix is the ratio of two rational functions of a smaller height. It follows from the following *homological* relations:

Theorem 1.2.3.

a) Row homological relations:

$$-|A|_{ij} \cdot |A^{i\ell}|_{sj}^{-1} = |A|_{i\ell} \cdot |A^{ij}|_{s\ell}^{-1} \qquad \forall s \neq i$$

b) Column homological relations:

$$-|A^{kj}|_{it}^{-1} \cdot |A|_{ij} = |A^{ij}|_{kt}^{-1} \cdot |A|_{kj} \qquad \forall r \neq j$$

I.2.3 Heredity (special case). Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ be a decomposition of $A = (a_{ij}), i, j = 1, \dots, n$ into a block-matrix. Let A_{11} be a $k \times k$-matrix and the matrix A_{22} be invertible.

Theorem 1.2.3. $|A|_{ij} = |A_{11} - A_{12}A_{22}^{-1} \cdot A_{21}|_{ij}$ *for* $i, j = 1. \dots, k$. *In other words, the quasideterminant* $|A|_{ij}$ *of an* $n \times n$ *matrix can be computed in two steps: first, considering the quasideterminant* $|\tilde{A}|_{11} = A_{11} - A_{12} \cdot A_{22}^{-1} \cdot A_{21}$ *of a* 2×2*-matrix* $\tilde{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$*; and second, considering the corresponding quasideterminant of the* $k \times k$*-matrix* $|\tilde{A}|_{11}$.

I.2.4. General Heredity. Let $A = (a_{ij}), i, j = 1, \dots, n$ be a matrix and

$$A = \begin{pmatrix} A_{11} & \dots & A_{1s} \\ & & \\ A_{s1} & \dots & A_{ss} \end{pmatrix}$$

its block decomposition. Denote by $\tilde{A} = (A_{ij}), i, j = 1, \dots, s$ the matrix with A_{ij}'s as entries.

Suppose that

$$A_{pq} = \begin{pmatrix} a_{k\ell} & \dots & a_{k,\ell+m} \\ & \dots & \\ a_{k+m,\ell} & \dots & a_{k+m,\ell+m} \end{pmatrix}.$$

is a square matrix and that $|\tilde{A}|_{pq}$ is defined.

Theorem 1.2.4.

$$|A|_{k'\ell'} = \left||\tilde{A}|_{pq}\right|_{k'\ell'} \quad \textit{for} \begin{array}{l} k' = k, \dots, k+m \\ \ell' = \ell, \dots, \ell+m \end{array}.$$

I.2.5 Elementary properties of quasideterminants.

i) The quasideterminant $|A|_{pq}$ does not depend of the permutation of rows and columns in the matrix A if the p-*th* row and the q-*th* column are not changed;

ii) *The multiplication of rows and columns.* Let the matrix B be constructed from the matrix A by multiplication of its i-*th* row by a scalar λ from the left. Then

$$|B|_{kj} = \begin{cases} \lambda |A|_{ij} & \text{if } k = i \\ |A|_{kj} & \text{if } k \neq i \text{ and } \lambda \text{ is invertible.} \end{cases}$$

Let the matrix C is constructed from the matrix A by multiplication of its j-*th* column by a scalar μ from the right. Then

$$|C|_{i\ell} = \begin{cases} |A|_{ij}\mu & \text{if } \ell = j \\ |A|_{i\ell} & \text{if } \ell \neq j \text{ and } \mu \text{ is invertible.} \end{cases}$$

iii) *The addition of rows and columns.* Let the matrix B is constructed by adding to some row of the matrix A its k-*th* row multiplied by a scalar λ from the left. Then

$$|A|_{ij} = |B|_{ij}, \quad i = 1, \dots k-1, k+1, \dots n, j = 1, \dots, n.$$

Let the matrix C is constructed by addition to some column of the matrix A its ℓ-th column multiplied by a scalar λ from the right. Then

$$|A|_{ij} = |C|_{ij},\ j = 1, \dots, \ell - 1, \ell + 1, \dots n, i = 1, \dots, n.$$

More generally, let X, Y be any $n \times n$-matrices such that the $i - th$ row of the matrix X consists of the elements δ_{ik} and the $j - th$ column of the matrix Y consists of the elements $\delta_{j\ell}$(here δ is the symbol of Kronecker). Then

$$\begin{aligned} |XA|_{ij} &= |A|_{ij} \quad i = k, j = 1, \dots, n; \\ |AY|_{ij} &= |A|_{ij} \quad j = \ell, i = 1, \dots, n. \end{aligned}$$

¿From these identities we have

Proposition 1.2.5. *Let the quasideterminant $|A|_{ij}$ be defined. Then the following statements are equivalent.*

i) $|A|_{ij} = 0$
ii) *the i-th row of the matrix A is a left linear combination of the other rows of this matrix;*
iii) *the j-th column of the matrix A is a right linear combination of the other columns of this matrix.*

There exists a notion of linear dependence of rows and columns for matrices over a skew-field if rows are multiplied by the elements of this skew-field from the left and columns from the right. So there exist notions of the row rank and the column rank and these ranks are equal [C, GR, GR1, GR2]. It also follows from Proposition 1.2.5

By definition, a r-quasiminor of a square matrix A is a quasideterminant of its $r \times r$-submatrix.

Proposition 1.2.6. *The rank of the matrix A over a skew field is greater or equal to r if and only if at least one of the r-quasiminors of the matrix A is defined and is not equal to zero.*

As it was pointed out in [KL], homological relations lead immediately to the following analogue of the classical expansion of a determinant by a row or a column.

Proposition 1.2.7. *For any $k \neq p$ and any $\ell \neq q$*

$$|A|_{pq} = a_{pq} - \sum_{j \neq q} a_{pj} (|A^{pq}|_{kj})^{-1} |A^{pj}|_{kq}$$

$$|A|_{pq} = a_{pq} - \sum_{i \neq p} |A^{iq}|_{pi} (|A^{pq}|_{i\ell})^{-1} a_{iq}$$

if all terms in these expressions are defined.

§1.3 Basic identities

I.3.1 Basic identities. The following noncommutative analogue of Sylvester's identity is closely related with the heredity property. Let $A = (A_{ij}), i, j = 1, \dots, n$

be a matrix over a noncommutative skew-field. Suppose that its submatrix $A_0 = (a_{ij}), i, j = 1, \ldots . k$ is invertible. For $p, q = k+1, \ldots, n$ set

$$b_{pq} = \begin{vmatrix} & A_0 & & a_{1q} \\ & & & \vdots \\ & & & a_{kq} \\ & & & \ldots \\ a_{p1} \ldots a_{pk} & & \ldots & \boxed{a_{pq}} \end{vmatrix}_{pq} ;$$

$$B = (b_{pq}), p, q = k+1, \ldots, n.$$

We call the submatrix A_0 *a pivot* for matrix B.

Theorem 1.3.1. *For $i, j = k+1, \ldots, n$*

$$|A|_{ij} = |B|_{ij}$$

For example, a quasideterminant of an $n \times n$-matrix A is equal to the corrsesponding quasideterminant of the 2×2-matrix consisting of $(n-1)$- quasiminors, or to the quasideterminant of $(n-1) \times (n-1)$-matrix consisting of 2-quasiminors of the matrix A. One can use any of these procedures for a definition of a quasideterminants.

In a special case when $A_0 = (a_{ij}), i, j = 2, \ldots, n-1$, Theorem 1.3.1 is an analogue of a well-known commutative indentity which goes under name of "Lewis Carrol identity".

Corollary 1.3.2. *(Sylvester): Let $A = (a_{ij}), i, j = 1, \ldots, n$ be a matrix over a commutative ring. Suppose that its submatrix $A_0 = (a_{ij}), i, j = 1, \ldots, k$ is invertible. For $p, q = k+1, \ldots, n$ set*

$$\tilde{b}_{pq} = \det \begin{pmatrix} & A_0 & & a_{1q} \\ & & & \vdots \\ & & & a_{kq} \\ & & & \ldots \\ a_{p1} \ldots a_{pk} & & & \ldots a_{pq} \end{pmatrix},$$

$$\tilde{B} = (\tilde{b}_{pq}), \ \ p, q = k+1, \ldots, n.$$

Then

$$\det A = \frac{\det \tilde{B}}{(\det A_0)^{n-k-1}}.$$

I.3.2. A noncommutative version of Jacobi theorem (involution for quasiminors). For a matrix $A = (a_{ij}), i \in I, j \in J, P \subset I, Q \subset J$ denote by A_{PQ} the submatrix:

$$A_{PQ} = (a_{\alpha\beta}), \alpha \in P, \beta \in Q.$$

Let $|I| = |J|$ and $B = A^{-1} = (b_{rs})$. Suppose that $|P| = |Q|$.

Theorem 1.3.3. *Let $k \notin P, \ell \notin Q$. Then*

$$|A_{P\cup\{k\},Q\cup\{\ell\}}|_{k\ell} \cdot |B_{I\setminus Q,J\setminus P}|_{\ell k} = 1.$$

Set $P = I \setminus \{k\}, Q = J \setminus \{\ell\}$. Then this theorem leads to the already mentioned identity

$$|A|_{k\ell} \cdot b_{\ell k} = 1.$$

I.3.3. A generalization of the homological relations.

For a matrix $A = (a_{ij}), i \in I, j \in J$ and two subsets $L \subset I, M \subset J$ denote by $A^{L,M}$ the submatrix of the matrix A obtained from A by deleting the rows with the indexes $\ell \in L$ and the columns with the indexes $m \in M$.

Let A be a square matrix, $L = (\ell_1, \dots, \ell_k), M = (m_0, \dots, m_k)$. Set $M_i = M \setminus \{m_i\}, i = 0, \dots, k$.

Theorem 1.3.4. *[GR1, GR2] For $p \notin L$*

$$\sum_{i=0}^{k} |A^{L,M_i}|_{pm_i} \cdot |A|^{-1}_{\ell m_i} = \delta_{p\ell},$$

$$\sum_{i=0}^{k} |A|^{-1}_{m_i\ell} \cdot |A^{M_i,L}|_{m_i p} = \delta_{\ell p}.$$

I.3.4 Multiplicative properties of quasideterminants.

Let $X = (x_{pq}), Y = (y_{rs})$ be $n \times n$-matrices.

Theorem 1.3.5.

$$|XY|^{-1}_{ij} = \sum_{p=1}^{n} |Y|^{-1}_{pj} |X|^{-1}_{pi}.$$

§1.4 Noncommutative linear algebra

Here we recall some results from [GR], [GR1], [GR2].

I.4.1. Solutions of systems of linear equations.

Theorem 1.4.1. *If the system*

$$\begin{cases} a_{11}x_1 + \dots & +a_{1n}x_n = \xi_1 \\ \dots & \\ a_{n1}x_1 + \dots & +a_{nn}x_n = \xi_n \end{cases}$$

is given and for the matrix $A = (a_{ij})$ the quasideterminants of A are defined and invertible, then

$$x_i = \sum_{j=1}^{n} |A|^{-1}_{ji} \xi_j.$$

for $i = 1, \dots, n$.

I.4.2 Cramer rule. Let $A_\ell(\xi)$ be the $n \times n$-matrix constructed by replacing in the matrix A its $\ell - th$ column by the column $(\xi_1, \dots, \xi_n)$.

Theorem 1.4.2. *If the quasideterminants $|A|_{ij}$ and $|A_j(\xi)|_{ij}$ are defined then*

$$|A|_{ij}x_j = |A_j(\xi)|_{ij}.$$

I.4.3. Cayley - Hamilton theorem. Let t be a formal variable and $A = (a_{ij}), i,j = 1,\dots,n$ be *a* matrix over a skew field. We do not suppose that t commutes with variables $a_{ij}, i,j = 1,\dots,n$. Let $\mathbb{I}_n$ be $n \times n$-unit matrix and $f_{ij}(t) = |t \cdot \mathbb{I}_n - A|_{ij}$. Consider the expression $\tilde{f}_{ij}(t)$ as a matrix function of a matrix variable changing in $f_{ij}(t)$ elements a_{ij} by matrices $\tilde{a}_{ij} = a_{ij} \cdot \mathbb{I}_n$. For example, if $n = 2$ then $\tilde{f}_{11}(t) = t - \tilde{a}_{11} - \tilde{a}_{12}(t - \tilde{a}_{22})^{-1}\tilde{a}_{21}$.

The following theorem was formulated in [GR1], [GR2]. An independent proof (similar to the original one) was given by C. Reutenauer in his Montreal lectures.

Theorem 1.4.3. *For all $i,j = 1,\dots,n$*

$$f_{ij}(A) = 0.$$

II. Noncommutative Plücker and Flag Coordiniates

§2.1 Definition and basic properties of quasi-Plücker coordinates

II.1.1. Definition of left quasi-Plücker coordinates. Let $A = (a_{pq}),\ p = 1,\dots,k,\ q = 1,\dots,n, k < n$ be a matrix over a skew-field F. Fix
$1 \le i, j, i_1, \dots, i_{k-1} \le n$ such that $i \notin I = \{i_1,\dots,i_{k-1}\}$. For $1 \le s \le k$ set

$$p^I_{ij}(A) = \begin{vmatrix} a_{1i}a_{1i_1} & \dots & a_{1i_{k-1}} \\ & \dots & \\ a_{ki}a_{ki_1} & \dots & a_{ki_{k-1}} \end{vmatrix}^{-1}_{si} \cdot \begin{vmatrix} a_{1j}a_{1i_1} & \dots & a_{1i_{k-1}} \\ & \dots & \\ a_{kj}a_{ki_1} & \dots & a_{ki_{k-1}} \end{vmatrix}_{sj}.$$

Proposition 2.1.1.
i)$p^I_{ij}(A)$ does not depend of s;
ii)$p^I_{ij}(gA) = p^I_{ij}(A)$ for any invertible $k \times k$ matrix g.

Proof. Use homological relations.

Definition 2.1.2. We call $p^I_{ij}(A)$ left quasi-Plücker coordinates of the matrix A.

One may consider p^I_{ij} as quasi-Plücker coordinates of a k-dimensional left subspace generated in the left F-linear space F^n by rows of the matrix A.

In commutative case $p^I_{ij} = \frac{p_{jI}}{p_{iI}}$, where $p_{\alpha_1\dots\alpha_k}$ is the standard Plücker coordinate.

II.1.2 Generating identities. The following properties of p^I_{ij}'s trivially follow from their definition:

i) p^I_{ij} does not depend of reordering of elements in I;
ii) $p^I_{ij} = 0$ if $j \in I$;
iii) $p^I_{ii} = 1$ and $p^I_{ij} \cdot p^I_{jk} = p^I_{ik}$.

Proposition 2.1.3. *(Skew-Symmetry). Let N be a set of indices, $|N| = k + 1, i, j, m \in N$. Then*

$$p^{N\smallsetminus\{i,j\}}_{ij} \cdot p^{N\smallsetminus\{j,m\}}_{jm} \cdot p^{N\setminus\{m,i\}}_{mi} = -1.$$

Proposition 2.1.4. *(Plücker relations). Fix $M=(m_1,\dots,m_{k-1})$, $L=(\ell_1,\dots,\ell_k)$. Let $i\notin M$. Then*

$$\sum_{j\in L} p_{ij}^{M}\cdot p_{ji}^{L\setminus\{j\}}=1.$$

Examples. Suppose that $k=2$.

1) From Proposition 2.1.3 it follows that

$$p_{ij}^{\{\ell\}}\cdot p_{j\ell}^{\{i\}}\cdot p_{\ell i}^{\{j\}}=-1.$$

In the commutative case one has $p_{ij}^{\{\ell\}}=\frac{p_{j\ell}}{p_{i\ell}}$ and $p_{ij}=-p_{ji}$, so this identity is valid.

2) From Proposition 2.1.4 it follows that for any i,j,ℓ,m

$$p_{ij}^{\{\ell\}}\cdot p_{ji}^{\{m\}}+p_{im}^{\{\ell\}}\cdot p_{mi}^{\{j\}}=1.$$

In commutative case this identity implies the standard identity

$$p_{ij}\cdot p_{\ell m}-p_{i\ell}\cdot p_{jm}+p_{im}\cdot p_{\ell j}=0.$$

Remark. In fact the products $p^{\{\ell\}}\}_{ij}p^{\{m\}}\}_{ji}$ (which are equal to $\frac{p_{j\ell}}{p_{i\ell}}\cdot\frac{p_{im}}{p_{jm}}$ in the commutative case) are noncommutative cross-ratio's.

We will consider different specializations of the relations of quasi-Plücker coordinates in a separate paper.

To prove propositions 2.1.3 and 2.1.4 we need the following lemma. Let $A=(a_{ij}), i=1,\dots,k, j=1,\dots,n, k<n$ be a matrix over a skew-field. Denote by $A_{j_1,\dots,j_k}$ its submatrix $(a_{ij}), i=1,\dots,k, j=j_1,\dots,j_k$. Consider $n\times n$-matrix

$$X=\begin{pmatrix} A_{1\dots k} & A_{k+1\dots n}\\ 0 & E_{n-k}\end{pmatrix},$$

where E_m is a $m\times m$-unit matrix.

Lemma 2.1.5. *Let $j<k<i$. If $p_{ij}^{1\dots\hat{j}\dots k}(A)$ is defined then $|X|_{ij}$ is defined and*

$$|X|_{ij}=-p^{1\dots\hat{j}\dots k}(A) \tag{2.1}$$

Proof. We have to prove

$$|X|_{ij}=-|A_{1\dots\hat{j}\dots ki}|_{si}^{-1}\cdot|A_{1\dots k}|_{sj} \tag{2.2}$$

if the right-hand side is defined. We will prove it by induction over $\ell=n-k$. Suppose that formula (2.2) is valid for $l=m$ and prove it for $\ell=m+1$. Without loss of generality we may set $j=1, i=k+1$. By Proposition 1.2.7

$$|X|_{k+1,1}=-|X^{k+1,1}|_{s,k+1}^{-1}\cdot|X^{k+1,k+1}|_{s1}$$

for an appropriate $1 \le s \le k$. Here

$$X^{k+1,1} = \begin{pmatrix} A_{2\dots k+1} & A_{k+2\dots n} \\ 0 & E_{n-k-1} \end{pmatrix},$$

$$X^{k+1,k+1} = \begin{pmatrix} A_{1\dots k} & A_{k+2\dots n} \\ 0 & E_{n-k-1} \end{pmatrix}.$$

By the induction assumption

$$|X^{k+1,1}|_{s,k+1} = -|A_{23\dots kk+2}|^{-1}_{s,k+2} \cdot |A_{23\dots k+1}|_{s,k+1},$$

$$|X^{k+1,k+1}|_{s1} = -|A_{23\dots kk+2}|^{-1}_{s,k+2} \cdot |A_{1\dots k}|_{s1}$$

and $|X|_{k+1,1} = -p^{23\dots k}_{k+1,1}$. $\square$

To prove Proposition 2.1.4 we apply Theorem 1.3.4 ii) to the matrix

$$X = \begin{pmatrix} A_{1\dots k} & A_{k+1\dots n} \\ 0 & E_{n-k} \end{pmatrix}$$

for $M = (k+1, \dots, n)$ and any L such that $|L| = n-k-1$. By Lemma 2.1.5

$|X|_{m_i \ell} = -p^{1\dots \hat{\ell} \dots k}(A)$, $|X^{M_i,L}|_{m_i q} = -p^{1\dots n \setminus L}_{m_i q}(A)$, and Proposition 2.1.4 follows from Theorem 3.4.ii) $\square$.

To prove Proposition 2.1.3 one have consider matrix X for $n = k+1$ and use homological relations.

Theorem 2.1.6. *Let $A = (a_{ij}), i = 1, \dots, k, j = 1, \dots, n$ be a matrix with formal entries, and $f(a_{ij})$ be a rational function over a free skew-field F generated by $(a_{ij}$. Let f be invariant for all transformations:*

$$A \to gA$$

where g is an invertible $k \times k$-matrix over f. Then f is a rational function of quasi-Plücker coordinates.

Proof. Let $b_{ij} = a_{ij}$ for $i, j = 1, \dots k$. Consider matrix $B = (b_{ij})$. Then $B^{-1} = (|B|^{-1}_{ji})$.

Set $C = (c_{ij}) = B^{-1}A$. Then

$$c_{ij} = \begin{cases} \delta_{ij} & j \le k \\ p^{1\dots \hat{i} \dots k}_{ij}(A) & j > k \end{cases}$$

By invariance we have $f(A) = f(1, C)$. $\square$

II.1.4. Right quasi-Plücker coordinates. Consider a matrix $B = (b_{ij}), i = 1, \dots, n; j = 1, \dots, k, k < n$ over a skew-field F. Fix $1 \le i, j, i_1, \dots, i_{k-1} \le n$ such that $j \notin I = (i_1, \dots, i_{k-1})$. Fix also $1 \le t \le k$ and set

$$r^I_{ij}(B) = \begin{vmatrix} b_{i1} & \dots & b_{ik} \\ b_{i_1 1} & \dots & b_{i_1 k} \\ & \dots & \\ b_{i_{k-1} 1} & \dots & b_{i_{k-1} k} \end{vmatrix}_{it} \cdot \begin{vmatrix} b_{j1} & \dots & b_{jk} \\ b_{i_1 1} & \dots & b_{i_1 k} \\ & \dots & \\ b_{i_{k-1} 1} & \dots & b_{i_{k-1} k} \end{vmatrix}^{-1}_{jt}.$$

Proposition 2.1.7.
i) $r^I_{ij}(B)$ *does not depend of* t*;*
ii) $r^I_{ij}(Bg) = r^I_{ij}(B)$ *for any invertible* $k \times k$*-matrix* g *over* F

Definition 2.1.8. *We call* $r^I_{ij}(B)$ *right quasi-Plücker coordinates of the matrix* B*.*

Proposition 2.1.7 ii) shows that $r^I_{ij}(B)$ are quasi-Plücker coordinates of the right k-dimensional subspace generated by columns of B in the right vector space F^n.

II.1.5 Generating identities for right quasi-Plücker coordinates.. Generating identities for r^I_{ij}'s are dual to generating identities for the left quasi-Plücker coordinates p^I_{ij}.

II.1.6 The duality between quasi-Plücker coordinates.

Let $A = (a_{ij}), i = 1, \dots, k; j = 1, \dots, n$; and $B = (b_{rs}), r = 1, \dots, n; s = 1, \dots, n-k$. Suppose that $AB = 0$. It is equivalent to the statement that the subspace generated by rows of A in the left linear space is orthogonal to the subspace generated by columns of B in the right linear space. Fix $1 \leq i, j \leq n, I \subset [1, n], i \notin I, |I| = k-1$. Set $J = [1, n] \setminus I \setminus \{i, j\}$

Theorem 2.1.9.

$$p^I_{ij}(A) + r^J_{ij}(B) = 0.$$

II.1.7 Quasi-Plücker coordinates for $k \times n$-matrices for different k.

Let $A = (a_{\alpha\beta}), \alpha = 1, \dots, k; \beta = 1, \dots, n$ be any $k \times n$ - matrix over a noncommutative skew-field and A' be a $(k-1) \times n$-submatrix of A. Fix $1 \leq i, j, m, j_1, \dots, j_{k-2} \leq n$ such that $i, m \notin J = \{j_1, \dots, j_{k-2}\}$ and $i \neq m$.

Proposition 2.1.10.

$$p^J_{ij}(A') = p^{J \cup \{m\}}_{ij}(A) + p^J_{im}(A') \cdot p^{J \cup \{i\}}_{mj}(A).$$

§2.2 Applications of quasi-Plücker coordinates

II.2.1 Row and column expansion of a quasideterminant.

Some of the results obtained in [GR], [GR1], [GR2] and partially listed in Chapter I can be rewritten in terms of quasi-Plücker coordinates. Let $A = (a_{ij}), i, j = 1, \dots, n$ be a matrix over a skew-field.

Fix $1 \leq \alpha, \beta \leq n$. Let $B = (a_{ij}, i \neq \alpha), C = (a_{ij}, j \neq \beta)$ be its $(n-1) \times n$ and $n \times (n-1)$ submatrices respectively. For any $j \neq \beta$ and $i \neq \alpha$ set:

$$p_{j\beta} = p^{1 \dots \hat{j} \dots \hat{\beta} \dots n}_{j\beta}(B),$$
$$r_{\alpha i} = r^{1 \dots \hat{\alpha} \dots \hat{i} \dots n}_{\alpha i}(C).$$

Proposition 2.2.1.
i) $|A|_{\alpha\beta} = a_{\alpha\beta} - \sum_{j \neq \beta} a_{\alpha j} p_{j\beta}$*,*
ii) $|A|_{\alpha\beta} = a_{\alpha\beta} - \sum_{i \neq \alpha} r_{\alpha i} a_{i\beta}$
if terms in the right-hand sides are defined.

II.2.2. Homological relations.

Proposition 2.2.2. *In notations of II.2.1*
i) $|A|_{ij}^{-1} \cdot |A|_{i\ell} = -p_{j\ell}$ *(row relations)*
ii) $|A|_{ij} \cdot |A|_{kj}^{-1} = -r_{ik}$ *(column relations).*

Corollary 2.2.3. *In notations of II.2.1, let* $(i_1, \dots, i_s), (j_1, \dots, j_t)$ *be sequences of indices such that* $i \neq i_1, i_1 \neq i_2, \dots, i_{s-1} \neq i_s; j \neq j_1, j_1 \neq j_2, \dots, j_{t-1} \neq j_t$.
Then

$$|A|_{i_s j_t} = p_{i_s i_{s-1}} \dots p_{i_2 i_1} p_{i_1 i} \cdot |A|_{ij} \cdot r_{j j_1} r_{j_1 j_2} \dots r_{j_{t-1} j_t}.$$

Example.

For a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$a_{21} \cdot a_{11}^{-1} \cdot |A|_{11} \cdot a_{22}^{-1} \cdot a_{22} = |A|_{22},$$
$$a_{12} a_{22}^{-1} a_{21} a_{11}^{-1} |A|_{11} a_{21}^{-1} a_{22} a_{12}^{-1} a_{11} = |A|_{11}$$

II.2.3 Matrix multiplication. The following formula has been already used in the proof of Theorem 2.1.6. Let $A = (a_{ij}), i = 1, \dots, n; j = 1, \dots, m, n < m; B = (a_{ij}), i = 1, \dots, n, j = 1, \dots, n; C = (a_{ik}), i = 1, \dots, n; k = n+1, \dots, m$.

Proposition 2.2.4. *Let matrix* B *is invertible. Then* $p_{ik}^{1 \dots \hat{i} \dots n}(A)$ *are defined for* $i = 1, \dots n, k = n+1, \dots m$ *and*

$$B^{-1} C = (p_{ik}^{1 \dots \hat{i} \dots n}(A)), i = 1, \dots, n; k = n+1, \dots, m.$$

II.2.5 Quasideterminant of a product. Let $A = (a_{ij}), B = (b_{ij}), i, j = 1, \dots n$ be matrices over a skew-field. Fix $1 \leq k \leq n$. Consider $(n-1) \times n$-matrix $A' = (a_{ij})$, $i \neq k$, and $n \times (n-1)$-matrix $B'' = (b_{ij})$, $j \neq k$.

Proposition 2.2.5.

$$|B|_{kk} \cdot |AB|_{kk}^{-1} |A|_{kk} = 1 + \sum_{\alpha \neq k} r_{k\alpha} \cdot p_{\alpha k},$$

where

$$r_{k\alpha} = r_{k\alpha}^{1 \dots \hat{\alpha} \dots n}(B'') - \textit{ right quasi-Plücker coordinates}$$
$$p_{\alpha k} = p_{\alpha k}^{1 \dots \hat{\alpha} \dots n}(A') - \textit{ left quasi-Plücker coordinate}$$

if all the expressions are defined.

The proof follows from the multiplicative property of quasideterminants and Proposition 2.2.2.

II.2.6. Gauss decomposition. Consider matrix $A = (a_{ij}), i, j = 1, \dots, n$ over a skew-field. Let $A_k = (a_{ij}), i, j = k, \dots n$, $B_k = (a_{ij}), i = 1, \dots n, j = k, \dots n$, and $C = (a_{ij}), i = k, \dots n, j = 1, \dots n$ be its submatrices of sizes $(n-k+1) \times (n-k+1)$, $n \times (n-k+1)$, and $(n-k+1) \times n$ respectively.

Suppose that quasideterminants

$$y_k = |A_k|_{kk}, \; k = 1, \dots, n$$

are defined and invertible. From [GR1], [GR2] it follows

Theorem 2.2.5.

$$A = \begin{pmatrix} 1 & & x_{\alpha\beta} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} y_1 & & 0 \\ & \ddots & \\ 0 & & y_n \end{pmatrix} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ z_{\beta\alpha} & & 1 \end{pmatrix},$$

where

$$x_{\alpha\beta} = r_{\alpha\beta}^{\beta+1\dots n}(B_\beta),\ 1 \le \alpha < \beta \le n$$
$$z_{\beta\alpha} = p_{\beta\alpha}^{\beta+1\dots n}(C_\beta).\ 1 \le \alpha < \beta \le n$$

II.2.7 Flag coordinates. We remind now a definition of noncommutative flag coordinates introduced in [GR1], [GR2]. Let $A = (a_{ij}), i = 1, \dots, k; j = 1, \dots, n$ be a matrix over a skew-field R. Let $\mathcal{F} = (F_1 \subset F_2 \subset \dots \subset F_k)$ be a flag in the left vector space R^n, F_p is generated by the first p rows of A. Put

$$f_{j_1\dots j_k}(\mathcal{F}) = \begin{vmatrix} a_{1j_1} & \dots & a_{1j_k} \\ & \dots & \\ a_{kj_1} & \dots & a_{kj_k} \end{vmatrix}_{kj_1}.$$

Proposition. *[GR1], [GR2] Functions $f_{j_1\dots j_m}(\mathcal{F})$ do not change under multiplication of A by upper triangular matrix with units as its diagonal entries from the left.*

In [GR1], [GR2] functions $f_{j_1\dots j_m}(\mathcal{F})$ were called flag coordinates of $\mathcal{F}$.

It is easy to see that

$$p_{ij}^{i_1\dots i_{k-1}}(A) = \left(f_{ii_1\dots i_{k-1}}(\mathcal{F})\right)^{-1} \cdot f_{ji_1\dots i_{k-1}}(\mathcal{F}).$$

In [GR1],[GR2] the relations between flag coordinates for $k \times n$- and $(k-1) \times n$)-matrices were considered. Our Proposition 2.1.9 may be deduced from these relations.

III. Noncommutative Symmetric Functions, Bezout Theorem, and Vieta Theorem

§3.1 Bezout and Vieta Theorems

III.1.1 Vandermonde quasideterminants. In this section we will discuss analogues of Bezout and Vieta Theorem for polynomials over a noncommutative skew-field. We will prove Vieta Theorem which was formulated in [GR3] using our noncommutative form of Sylvester identity. The first prove of this theorem using differential operators has appeared in [EGR]. Another noncommutative version of Vieta theorem based on notions of traces and determinants was given by A. Connes and A. Schwarz in [CS].

Let $x_1, x_2, \dots, x_k$ be a set of elements of a skew-field F. For $k > 1$ define a *Vandermonde* quasideterminant

$$V(x_1, \dots, x_k) = \begin{vmatrix} x_1^{k-1} & \dots & x_k^{k-1} \\ & \dots & \\ x_1 & \dots & x_k \\ 1 & \dots & 1 \end{vmatrix}_{1k}.$$

We say that a sequence $x_1, \dots, x_n \in F$ is *independent* of for any $k = 2, \dots, n$ quasideterminants $V(x_1, \dots, x_k)$ are defined and invertible. For independent sequences $x_1, \dots, x_n$ and $x_1, \dots, x_{n-1}$, where $x_1, \dots, x_n, z \in F$ set

$$y_1 = x_1,\ z_1 = z$$
$$y_k = V(x_1, \dots, x_k) x_k V^{-1}(x_1, \dots, x_k), k \geq 2$$
$$z_k = V(x_1, \dots, x_{k-1}, z) z V^{-1}(x_1, \dots, x_{k-1}, z), k \geq 2.$$

In the commutative case $y_k = x_k$, and $z_k = z$ for $k = 1, \dots, n$.

III.1.2 Bezout and Vieta decomposition. We will prove here

Theorem 3.1.1. *(Bezout decomposition of a Vandermonde quasideterminant). Suppose that sequences $x_1, \dots, x_n$ and $x_1, \dots, x_{n-1}, z$ are independent. then*

$$V(x_1, \dots, x_n, z) = (z_n - y_n)(z_{n-1} - y_{n-1}) \dots (z_1 - y_1). \tag{3.1}$$

Note that if z commutes with $x_i, i = 1, \dots n$ then

$$V(x_1, \dots, x_n, z) = (z - y_n)(z - y_{n-1}) \dots (z - y_1).$$

Theorem 3.1.2. *(Vieta decomposition of a Vandermonde quasideterminant). For a independent sequence $x_1, \dots, x_n, z$*

$$V(x_1, \dots, x_n, z) = z^n + a, z^{n-1} + \dots + a_{n-1} z + a_n, \tag{3.2}$$

where

$$a_k = (-1)^k \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} y_{i_k} \cdot y_{i_{k-1}} \dots y_1. \tag{3.3}$$

In particular

$$a_1 = -(y_1 + \dots + y_n),$$
$$a_2 = \sum_{1 \leq i < j \leq n} y_j y_i,$$
$$\dots$$
$$a_n = (-1)^n y_n \dots y_1.$$

Theorem 3.1.2 follows from Theorem 3.1.1 by induction.

III.1.3 Proof of Theorem 3.1.2. For $n = 1$ one has $V(x_1, z) = z - x_1$ and formulas (3.2) are valid. Suppose that these formulas are valid for $m = n - 1$. We have to prove that formula (3.2) can be written in the form

$$V(x_1, \dots, x_n, z) = (z_n - y_n) V(x_1, \dots, x_{n-1}, z) =$$
$$= V(x_1, \dots, x_{n-1}, z) \cdot z - y_n \cdot V(x_1, \dots, x_{n-1}, z).$$

By induction,

$$V(x_1, \dots, x_{n-1}, z) = z^{n-1} + b_1 z^{n-2} + \dots + b_{n-1},$$

where

$$b_1 = -(y_1 + \dots + y_{n-1}),$$
$$\dots$$
$$b_{n-1} = (-1)^n y_{n-1} \cdot \dots \cdot y_1.$$

So,

$$V(x_1, \dots, x_n, z) = z^n + (b_1 - y_n) z^{n-1} + (b_2 - y_n b_1) z^{n-2} + \dots - y_n b_n =$$
$$= z^n + a_1 z^{n-1} + \dots + a_n,$$

where $a_1, \dots, a_n$ are given by formulas (3.3). $\square$

III.1.4. To prove Theorem 3.1.1 we need the following

Lemma 3.1.3.

$$V(x_1,\dots,x_n,z)=V(\hat{x}_2,\dots,\hat{x}_n,\hat{z})(z-x_1),$$

where

$$\hat{x}_k=(x_k-x_1)x_k(x_k-x_1)^{-1}, k=2,\dots,n$$
$$\hat{z}=(z-x_1)z(z-x_1)^{-1}.$$

Proof. By definition

$$V(x_1,\dots,x_n,z)=\left|\begin{matrix} x_1^n & x_2^n & \dots & z^n \\ x_1^{n-1} & x_2^{n-1} & \dots & z^{n-1} \\ & & \dots & \\ x_1 & x_2 & \dots & z \\ 1 & 1 & \dots & 1 \end{matrix}\right|_{1n+1}.$$

Multiply k-th row by x_1 from the left and subtract it from $(k-1)$-th row for $k=2,\dots,n$. The quasideterminant will not change, and so

$$V(x_1,\dots,x_n z)=\left|\begin{matrix} 0 & x_2^n-x_1x_2^{n-1} & \dots & z^n-x_z^{n-1} \\ 0 & x_2^{n-1}-x_1x_2^{n-2} & & z^{n-1}-x_1z^{n-2} \\ \vdots & \vdots & & \vdots \\ 0 & x_2-x_1 & & z-x_1 \\ 1 & 1 & & 1 \end{matrix}\right|_{1n+1}=$$

$$=\left|\begin{matrix} 0 & (x_2-x_1)x_2^{n-1} & \dots & (z-x_1)z^{n-1} \\ \vdots & \vdots & & \vdots \\ 0 & x_2-x_1 & & z-x_1 \\ 1 & 1 & & 1 \end{matrix}\right|_{1n+1}.$$

Applying Sylvester theorem to the last quasideterminant with the element of index $(n1)$ as a pivot one has

$$V(x_1,\dots,x_n,z)=\left|\begin{matrix} (x_2-x_1)x_2^{n-1} & \dots & (z-x_1)z^{n-1} \\ \dots & & \dots \\ x_2-x_1 & & z-x_1 \end{matrix}\right|_{1n}.$$

Multiply k-th column by $(x_{k+1}-x_1)$ for $k=1,\dots,n-1$ and the last column by $(z-x_1)$. According to elementary properties of quasideterminants,

$$V(x_1,\dots,x_n,z)=$$

$$=\left|\begin{matrix} (x_2-x_1)x_2^{n-1}(x_2-x_1)^{-1} & \dots & (z-x_1)z^{n-1}(z-x_1)^{-1} \\ \vdots & & \vdots \\ 1 & & 1 \end{matrix}\right|_{1n}(z-x_1)=$$

$$=\left|\begin{matrix} \hat{x}_2 & \dots & \hat{z}^{n-1} \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{matrix}\right|_{1n}\cdot(z-x_1)=V(\hat{x}_2,\dots,\hat{x}_n,\hat{z})\cdot(z-x_1).$$

□

III.1.5 Proof of Theorem 3.1.1. Use induction. By Lemma 3.1.3 Theorem 3.1.1 is valid for $n = 2$. Also by Lemma 3.3.1

$$V(x_1, \dots, x_n, z) = V(\hat{x}_2, \dots, \hat{x}_n, \hat{z})(z - x_1). \tag{3.4}$$

Suppose that our theorem is valid for $m = n - 1$.

Then

$$V(\hat{x}_2, \dots, \hat{x}_n, \hat{z}) = (z'_n - y'_n) \dots (z'_2 - y'_2),$$

where

$$\begin{aligned} z'_2 &= \hat{z}, \\ y'_2 &= \hat{x}_2, \\ z'_k &= V(\hat{x}_2, \dots, \hat{x}_{k-1}, \hat{z})\hat{z}V^{-1}(\hat{x}_2, \dots, \hat{x}_{k-1}, \hat{z}), \\ y'_k &= V(\hat{x}_2, \dots, \hat{x}_k)\hat{x}_k V^{-1}(\hat{x}_2, \dots, \hat{x}_k) \text{ for } k = 3, \dots, n. \end{aligned}$$

It is enough to show that

$$z'_k = z_k, \text{ and } y'_k = y_k \text{ for } k = 2, \dots, n.$$

For $k = 2$ it is obvious. By Lemma 3.3.1

$$V(\hat{x}_2, \dots, \hat{x}_{k-1}, \hat{z}) = V(x_1, \dots, x_{k-1}, z)(z - x_1)^{-1},$$

and by definition $\hat{z} = (z - x_1)z(z - x_1)^{-1}$. So,

$$\begin{aligned} z'_k = \{V(x_1, \dots, x_{k-1}, z)(z - x_1)^{-1}\}(z - x_1)z(z - x_1)^{-1}\cdot \\ \cdot\{(z - x_1)V^{-1}(x_1, \dots, x_{k-1}, z)\} = z_k \text{ for } k = 3, \dots, n. \end{aligned}$$

Similarly, $y'_k = y_k$ for $k = 3, \dots, n$ and from (3.4) we have

$$V(x_1, \dots, x_n, z) = (z_n - y_n) \dots (z_2 - y_2)(z_1 - y_1)$$

□

III.1.6 Another expression for coefficients. Another expression for coefficients $a_1, \dots a_n$ in Vieta decomposition of $V(x_1, \dots, x_n, z)$ may be obtained from Proposition 1.3.7.

Theorem 3.1.4. *[GKLLRT]:*

$$V(x_1, \dots, x_n, z) = z^n + a_1 z^{n-1} + \dots + a_n,$$

where for $k = 1, \dots, n$

$$a_k = - \begin{vmatrix} x_1^n & \dots & x_n^n \\ x_1^{n-k+1} & \dots & x_n^{n-k+1} \\ x_1^{n-k-1} & \dots & x_n^{n-k-1} \\ & \dots & \\ 1 & \dots & 1 \end{vmatrix}_{1n} \cdot \begin{vmatrix} x_1^{n-1} & \dots & x_n^{n-1} \\ & \dots & \\ x_1^{n-k} & \dots & x_n^{n-k} \\ & \dots & \\ 1 & \dots & 1 \end{vmatrix}_{kn}^{-1}. \tag{3.5}$$

¿From Theorem 3.1.4 we will get Bezout and Vieta formulas expressing coefficients of the equation

$$z^n + a_1 z^{n-1} + \dots + a_n = 0 \tag{3.6}$$

§3.2 Vieta theorem and Bezout theorem

III.2.1 Formulas for coefficients.

Lemma 3.2.1. *Suppose that $x_1, \dots, x_n$ is an independent set of roots of the equation (3.6). Then coefficients $a_1, \dots, a_n$ may be written in the form (3.5)*

Proof.

Consider a system of right linear equations

$$x_i^n + a_1 x_i^{n-1} + \dots + a_{n-1} x_i + a_n = 0, i = 1, \dots, n$$

with unknowns $a_1, \dots, a_n$ and use Cramer rules. □

III.2.2 Bezout and Vieta Theorems.

Theorem 3.2.2. *(Noncommutative Bezout Theorem). Let $x_1, \dots, x_n$ be an independent set of roots of equation (3.6). In notations of Theorem 3.1.1*

$$z^n + a_1 z^{n-1} + \dots + a_n = (z_n - y_n) \dots (z - y).$$

□

Proof. Use Lemma 3.2.1, Theorem 3.1.4 and Theorem 3.1.1.

Theorem 3.2.3. *(Noncommutative Vieta Theorem) [GR3] Let $x_1, \dots, x_n$ be an independent set of roots of equation (3.6). Then coefficients $a_1, \dots, a_n$ of the equation are given by formulas (3.3).*

Proof.

Use Lemma 3.2.1, Theorem 3.1.4 and Theorem 3.1.2 □

§3.3. Noncommutative symmetric functions

A general theory of noncommutative symmetric functions was developed in [GKLLRT]. In fact, in [GKLLRT] were studied different systems of multiplicative and linear generators in a free algebra $\mathbb{S}ym$ generated by any system of noncommuting variables $\Lambda_i, i = 1, 2, \dots$. In [GKLLRT] these variables were called elementary symmetric functions but the theory was developed independently of the origin of Λ_i's.

In this section we apply the general theory to noncommutative symmetric functions generated by specific Λ_i's. As in the commutative case, they depend of a set of roots of a polynomial equation.

III.3.1 A construction of new variables. We fix n independent indeterminants $x_1, x_2, \dots, x_n$ and construct new variables $y_1, \dots, y_n$ which are *rational* functions in $x_1, \dots, x_n$:

$$y_1 = x_1,$$

$$y_2 = \begin{vmatrix} x_1 & x_2 \\ 1 & 1 \end{vmatrix}_{12} x_2 \begin{vmatrix} x_1 & x_2 \\ 1 & 1 \end{vmatrix}_{12}^{-1},$$

$$\dots$$

$$y_n = \begin{vmatrix} x_1^{n-1} & \dots & x_n^{n-1} \\ x_1^{n-2} & \dots & x_n^{n-2} \\ & \dots & \\ 1 & \dots & 1 \end{vmatrix}_{1n} x_n \begin{vmatrix} x_1^{n-1} & \dots & x_n^{n-1} \\ x_1^{n-2} & \dots & x_n^{n-2} \\ & \dots & \\ 1 & \dots & 1 \end{vmatrix}_{1n}^{-1}.$$

In the commutative case $x_i = y_i, i = 1, \dots n$. In the noncommutative case they are obviously different.

Remark. Consider a free skew-field F generated by $x_1, \dots, x_n$. Define on F differential operators ∂_i given by formulas $\partial_i x_j = \delta_{ij}$ and satisfying Leibnitz rule $\partial_i(fg) = \partial_i(f)g + f\partial_i(g)$, for $i = 1, \dots n$.

It easy to see that $\partial_i y_j \neq \delta_{ij}$. Consider, however, $\partial = \partial_1 + \dots + \partial_n$. Then

$$\partial(V(x_1, \dots, x_k)) = 0, \ \ k = 2, \dots n$$

and

$$\partial(y_i) = \partial(x_i) = 1, \ \ i = 1, \dots, n.$$

III.3.2. Elementary symmetric functions.

Defintion 3.3.1. The functions

$$\Lambda_1(x_1, \dots, x_n) = y_1 + y_2 \dots + y_n,$$

$$\Lambda_2(x_1, \dots, x_n) = \sum_{i<j} y_j y_i,$$

$$\dots$$

$$\Lambda_n(x_1, \dots, x_n) = y_n \dots y_1$$

are called elementary symmetric functions of $x_1, \dots, x_n$.

In commutative case these functions are the ordinary elementary symmetric functions of $x_1, \dots, x_n$.

By Theorem 3.2.3 $\Lambda_i(x_1, \dots, x_n) = (-1)^i a_i, i = 1, \dots, n$ where $x_1, \dots, x_n$ are roots of the equation

$$x^n + a_1 x^{n-1} + \dots + a_{n-1}x + a_n = 0.$$

This implies

Proposition 3.3.2. *Functions $\Lambda_i(x_1, \dots, x_n)$ are symmetric in $x_1, \dots, x_n$.*

Remark. The order of $y_1, \dots, y_n$ is essential in the definition of $\Lambda_i, i = 1, \dots, n$.

For example, $\Lambda_2 = y_2 y_1$ is symmetric in x_1, x_2 but the product $y_1 y_2$ is not symmetric.

To see this let $n = 2$ and set $d = x_2 - x_1$. Then the product $y_1 y_2$ is symmetric in x_1, x_2 if and only if $x_1 d^2 = d^2 x_1$.

III.2.3 Complete symmetric functions. Let t be a formal variable commuting with $x_i, i = 1, \dots n$. Consider the generating function $\lambda(t) = 1 + \Lambda_(x_1, \dots, x_n)t + \dots + \Lambda_n(x_1, \dots, x_n)t^n$.

Following [GKLLRT] define *complete* symmetric functions $S_i(x_1, \dots, x_n), i = 1, 2, \dots$ using a generating function

$$1 + \sum_i S_i(x_1, \dots, x_n)t^i = \lambda^{-1}(-t).$$

According to [GKLLRT]

$$S_k(x_1, \dots, x_n) = \sum_{i_1 \leq i_2 \leq \dots \leq i_k} y_{i_1} \dots y_{i_k}, k = 1, 2, 3, \dots, \tag{3.7}$$

In commutative case S_k's are just complete symmetric functions and so we will call S_k's defined by formula (3.7) complete symmetric functions of $x_1, \dots, x_n$.

Formula (3.7) shows that S_k's are polynomial in $y_1, \dots, y_n$ and

Proposition 3.3.3. *$S_k(x_1,\dots,x_n)$ are symmetric in $x_1,\dots,x_n$.*

Remark. The order of $y's$ in (3.7) is essential: $S_2(x_1,x_2) = y_1^2 + y_1y_2 + y_2^2$ is symmetric in $x_1,\dots,x_n$ and $y_1^2 + y_2y_1 + y_2^2$ is not symmetric. Otherwise, the sum $y_1^2 + y_2^2$ will be also symmetric and for $n = 2$ and $d = x_2 - x_1$ one would have $d^2x_1 = x_1d^2$.

III.3.4 Ribbon Schur Functions. To consider a more general example we need a vocabulary. Let $w = a_{i_1}\dots a_{i_k}$ be a word in ordered letters $a_1 < \dots < a_n$. An integer m is called a descent of w if $1 \le m \le k-1$ and $i_m > i_{m+1}$.

Let $J = (j_i,\dots,j_k)$ be a set of positive integers. Define the *ribbon Schur function* $R_J(x_1,\dots,x_n)$ by formula

$$R_J(x_1,\dots,x_n) = \sum y_{i_1}\dots y_{i_m},$$

where sum is running over all words $w = y_{i_1}\dots y_{i_m}$ whose descent set is exactly equal to $\{j_1, j_1+j_2,\dots,j_1+j_2+\dots+j_{k-1}\}$.

Theorem 3.3.4. *Functions $R_J(x_1,\dots,x_n)$ are symmetric in $x_1,\dots,x_n$ for any J.*

¿From [GKLLRT] it follows

Theorem 3.3.5. *The set of functions $\{R_J\}$ for all J is a $\mathbb{Q}$-linear basis in the free $\mathbb{Q}$-algebra generated by $\Lambda_1(x_1,\dots,x_n),\dots,\Lambda_n(x_1,\dots,x_n)$*

III.3.5 Main Theorem. In a commutative case the main theorem of a theory of commutative functions says that every symmetric polynomial of n variables is a polynomial of (elementary) symmetric functions of these variables. Its analogue for a noncommutative case is given by

Conjecture 3.3.6. *Let a polynomial $P(y_1,\dots,y_n)$ over $\mathbb{Q}$ be symmetric as a function of $x_1,\dots,x_n$ then $P(y_1,\dots,y_n) = Q(\Lambda_1,\dots,\Lambda_n)$ where Q is a noncommutative polynomial of n variables over $\mathbb{Q}$.*

We proved this conjecture for $n = 2$.

IIII Continued fractions and Almost Triangular Matrices

IIII.1 Continued fractions and quasideterminants.

Consider an infinite matrix A over a skew-field

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n}\dots \\ -1 & a_{22} & a_{23} & \dots & a_{2n}\dots \\ 0 & -1 & a_{33} & \dots & a_{3n}\dots \\ 0 & 0 & -1 & \dots & \dots \end{pmatrix}$$

It was pointed in [GR1], [GR2] that its quasideterminant $|A|_{11}$ can be written as a generalized continued fraction

$$|A|_{11} = a_{11} + \sum_{j_1\neq 1} a_{1j_1}\frac{1}{a_{2j_1} + \sum_{j_2\neq 1,j_1} a_{2j_2}\frac{1}{a_{3j_2}+\dots}}.$$

Let

$$A_n = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ -1 & a_{22} & \dots & a_{2n} \\ 0 & -1 & \dots & a_{3n} \\ & & \dots & \\ \dots & 0 & -1 & a_{nn} \end{pmatrix}.$$

The following proposition was formulated in [GR1], [GR2].

Proposition 4.1. $|A_n|_{11} = P_n Q_n^{-1}$, *where*

$$P_n = \sum_{1\le j_1<\cdots<j_k<n} a_{1j_1}a_{j_1+1j_2}a_{j_2+1j_3}\dots a_{j_k+1n} \tag{4.1}$$

$$Q_n = \sum_{2\le j_1<\cdots<j_k<n} a_{2j_1}a_{j_1+1j_2}a_{j_2+1j_3}\dots a_{j_k+1n}. \tag{4.2}$$

Proof. From homological relations one has

$$|A_n|_{11}|A_n^{1n}|_{21}^{-1} = -|A_n|_{1n}|A_n^{11}|_{2n}^{-1}.$$

We will apply Definition 1.1.3 for a computation of $|A_n|_{1n}$, $|A_n^{11}|_{2n}$, and $|A_n^{1n}|_{21}$. It is easy to see, that $|A_n^{1n}|_{21} = -1$. To compute two other quasideterminants we have to invert triangular matrices. Setting $P_n = |A_n|_{1n}$ and $Q_n = |A_n^{11}|_{2n}$ we arrive to formulas (4.1), (4.2). □

Remark. In a commutative case Proposition 4.1 is well-known. In this case $P_n = |A_n|_{1n} = (-1)^n \det A_n$ and $Q_n = (-1)^{n-1}\det A_n^{11}$.

¿From formulas (4.1), (4.2) it follows [GR1], [GR2]

Corollary 4.2. *The polynomials P_k for $k \ge 0$ and Q_k for $k \ge 1$ are related via formulas:*

$$P_k = \sum_{s=0}^{k-1} P_s a_{s+1,k} \qquad P_0 = 1, \tag{4.3}$$

$$Q_k = \sum_{s=1}^{k-1} Q_s a_{s+1,k} \qquad Q_1 = 1. \tag{4.4}$$

The following corollary was pointed out to us by A. Berenstein.

Corollary 4.3. *Consider the matrix A_n. Suppose that for any $i \ne j$ and any p, q*

$$a_{ij}a_{pq} = a_{pq}a_{ij}$$

and

$$a_{jj}a_{ii} - a_{ii}a_{jj} = a_{ij} \ 1 \le i < j \le n.$$

Then

$$P_n = |A_n|_{1n} = a_{nn}a_{n-1n-1}\dots a_{11} \tag{4.5}$$

The proof follows from formula (4.3). Formulas similar to (4.5) appeared in [G] and [FK].

Corollary 4.4. *[GR1], [GR2] For a Jacobian matrix*

$$A = \begin{pmatrix} a_1 & 1 & 0 & \dots \\ -1 & a_2 & 1 & \\ 0 & -1 & a_3 & \dots \end{pmatrix}$$

$$|A|_{11} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}},$$

and

$$P_k = P_{k-1} a_k + P_{k-2}, \ k \geq 2; \ P_0 = 1, \ P_1 = a_1,$$

$$Q_k = Q_{k+1} a_k + Q_{k-2}. \ k \geq 3; Q_1 = 1, \ Q_2 = a_2.$$

In this case P_k is a polynomial of $a_1, \dots, a_k$ and Q_k is a polynomial of $a_2, \dots, a_k$.

III.2 Continued fractions and formal series.

In notations of the previous subsection infinite continued fraction $|A|_{11}$ may be written as a ratio of formal series of letters a_{ij} and a_{ii}^{-1}.

Namely, set

$$P_\infty = \sum_{\substack{1 \leq j_1 < j_2 \dots < j_k < r-1 \\ r=1,2,3,\dots}} a_{1j_1} a_{j_1+1 j_2} \dots a_{j_k+1 r} a_{rr}^{-1} \cdot \dots \cdot a_{11}^{-1} =$$

$$= 1 + a_{12} a_{22}^{-1} a_{11}^{-1} + a_{13} a_{33}^{-1} a^{-1} a_{22}^{-1} a_{11}^{-1} + a_{11} a_{23} a_{33}^{-1} a_{22}^{-1} a_{11}^{-1} + \dots,$$

$$Q_\infty = a_{11}^{-1} + \sum_{\substack{2 \leq j_1 < j_2 \dots < j_k < r-1 \\ r=2,3\dots}} a_{2j_1} a_{j_1+1 j_2} \dots a_{j_k+1 r} a_{rr}^{-1} \cdot \dots \cdot a_{11}^{-1} =$$

$$= a_{11}^{-1} + a_{23} a_{33}^{-1} a_{22}^{-1} a_{11}^{-1} + a_{24} a_{44}^{-1} a_{33}^{-1} a_{22}^{-1} a_{11}^{-1} + \dots.$$

It is easy to see that these formal series are correctly defined.

The following theorem was proved in [PPR]. Here we consider another proof.

Theorem 4.5.

$$|A|_{11} = P_\infty \cdot Q_\infty^{-1}.$$

Proof. Consider matrix $B = (b_{ij} = a_{ij} a_{jj}^{-1})$, $i, j = 1, 2, 3, \dots$. According to a property of quasideterminants $|A|_{11} = |B|_{11} a_{11}$.

Applying noncommutative Sylvester Theorem to B with matrix $(b_{ij}, i, j \geq 3)$ as a pivot one has

$$|B|_{11} = 1 + |B^{21}|_{12} |B^{11}|_{22}^{-1} a_{11}^{-1}.$$

It follows

$$|A|_{11} = (a_{11} |B^{11}|_{22} a_{11}^{-1} + |B^{21}|_{12} a_{11}^{-1})(|B^{11}|_{22} a_{11}^{-1})^{-1}. \tag{4.6}$$

¿From [GKLLRT, Proposition 2.4] it follows that the first factor in (4.6) equals to P_∞, and the second one to Q_∞. $\square$

The following application of Theorem 4.5 to Rogers-Ramanujan continued fraction was given in [PPR]. Consider a continued fraction with two formal variables x and y:

$$A(x,y) = \frac{1}{1 + x\frac{1}{1+x\frac{1}{1+\dots}y}y}.$$

It is easy to see that

$$A(x,y) = \begin{vmatrix} 1 & x & & \cdot & & \\ -y & 1 & x & & \cdot & O \\ & -y & 1 & x & & \cdot \\ & & & 1 & & \cdot \\ & O & & \ddots & \ddots & \cdot \end{vmatrix}_{11}^{-1} =$$

$$= \begin{vmatrix} 1 & x & O & & \\ -1 & y^{-1} & xy^{-1} & & \\ O & -1 & y^{-1} & xy^{-1} & \\ & & -1 & y^{-1} & \ddots \end{vmatrix}_{11}$$

Theorem 4.5 implies the following

Corollary 4.6.

$A(x,y) = P \cdot Q^{-1}$, where $Q = yPy^{-1}$ and

$$P = 1 + \sum_{\substack{k\geq 1\\ n_1,\dots,n_k\geq 1}} y^{-n_1}xy^{-n_2}x\dots\, y^{-n_k}xy^{k+n_1+n_2+\dots+n_k}.$$

Following [PPR] suppose that $xy = qyx$, where q commutes with x and y. Set $z = yx$.

Then Corollary 4.6 implies Rogers-Ramanujan continued fraction identity

$$A(x,y) = \frac{1}{1+\frac{qz}{1+\frac{q^2z}{1+\dots}}} =$$

$$= \frac{1+\sum_{k\geq 1}\frac{q^{k(k+1)}}{(1-q)\dots(1-q^k)}z^k}{1+\sum_{k\geq 1}\frac{q^{k^2}}{(1-q)\dots(1-q^k)}z^k}.$$

IIII.3 Determinants of almost triangular matrices.

Consider now a general almost triangualr matrix

$$B = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ b_{21} & a_{22} & \dots & a_{2n} \\ 0 & b_{32} & \dots & a_{3n} \\ & & \dots & \\ \dots & 0 & b_{n-1n} & a_{nn} \end{pmatrix}.$$

In this subsection we suppose that all b_{i+1i}'s are invertible and all a_{ij}'s are free variables.

Denote by $B_k, k = 1, 2, \dots, n-1$ submatrices of B obtained by deleting first k rows and columns from matrix B.

Set

$$D(1,2,\dots,n) = |B|_{11}b_{21}^{-1}|B_1|_{22}b_{32}^{-1}|B_2|_{33}b_{43}^{-1}\dots b_{nn-1}^{-1}a_{nn}.$$

Homological relations imply the following proposition [GR1], [GR2] (cf. subsection IIII.1).

Proposition 4.7.

$$(-1)^{n+1}D(1,2,\dots n) = |B|_{1n} =$$
$$= \sum_{1\le j_1<\dots<j_k<n} (-1)^{k+1} a_{1j_1} b^{-1}_{j_1+1j_1} a_{j_1+1j_2} b^{-1}_{j_2+1j_2} a_{j_2+1j_3} \dots b^{-1}_{j_k+1j_k} a_{j_k+1n}.$$

Remark. When $b_{i+1i} = 1,\ i = 1,\dots,n-1$ and all a_{ij}'s commute with each other, then the expression $D(1,2,\dots,n)$ is the determinant of B.

It is interesting to find out other quasideterminants of an almost triangular matrix.

Proposition 4.8. *[GR1], [GR2] Set $D(\emptyset) = 1$. For $i \le j$*

$$|B|_{ij} = D(1,\dots,i-1)^{-1} D(1,\dots,n) D(j+1,\dots,n)^{-1}.$$

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