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THE KOSZUL PROPERTY IN AFFINE SEMIGROUP RINGS[1]

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Abstract: We investigate the Koszul property for quotients of affine semigroup rings by semigroup ideals. Using a combinatorial and topological interpretation for the Koszul property in this context, we recover known results asserting that certain of these rings are Koszul. In the process, we prove a stronger fact, suggesting a more general definition of Koszul rings. This more general definition of Koszulness turns out to be satisfied by all Cohen-Macaulay rings of minimal multiplicity.

1. INTRODUCTION

The main point in this paper is an interplay between topological combinatorics and commutative algebra. Our goal is to relate the homological properties of certain rings to the topology of certain poset complexes.

We begin by describing the rings to be studied. Let Λ be an associative commutative finitely generated semigroup with a neutral element and satisfying the cancellation rule: $\lambda_1\lambda = \lambda_2\lambda$ implies $\lambda_1 = \lambda_2$. We further assume that no element of Λ is invertible. We call such a semigroup *affine* since it may be identified with a semigroup generated by a finite set of monomials in a polynomial ring, or, in additive notation, as a subsemigroup of $\mathbf{N}^d$ for some d. We can write the elements of an affine semigroup Λ in either additive or multiplicative notation, once we identify an element $\lambda = (\lambda_1, \cdots, \lambda_d) \in \mathbf{N}^d$ with the monomial $\mathbf{z}^\lambda = z_1^{\lambda_1} \cdots z_d^{\lambda_d}$. Addition of vectors corresponds to multiplication of monomials and we write 0 or 1 for the neutral element of Λ. Consider the semigroup ring $k[\Lambda]$ of Λ over a field k. The *toric ideal* I_Λ is the kernel of the map $\phi: k[x_1, \ldots, x_n] \to k[z_1, \ldots, z_d]$ sending x_i to $\mathbf{z}^{\alpha_i} := z_1^{\alpha_{i1}} z_2^{\alpha_{i2}} \cdots z_d^{\alpha_{id}}$, where the α_i's are minimal generators of the semigroup. Consequently, we have

$$k[x_1, \ldots, x_n]/I_\Lambda \cong k[\Lambda] \subset k[z_1, \ldots, z_d].$$

Let J be an ideal in $k[\Lambda]$ that is generated by semigroup elements, i.e. a *semigroup ideal.* We denote by R the quotient ring $k[\Lambda]/J$.

In the spirit of [BH] we extend a result in [LS], [PRS] showing that the computation of $\mathrm{Tor}^R_\cdot(k, k)$ is equivalent to the computation of the relative homology of order complexes

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of the divisor poset of the semigroup and certain combinatorially defined sub-complexes (Theorem 2.1). Section 3 gives a more concrete geometric description (Theorem 3.1) and topological analysis (Lemma 3.3) of these complexes whose proofs use results from the theory of subspace-arrangements and homotopy colimits as in [WZZ],[ZZ].

In Section 4 we apply this topological information to prove that R is *Koszul* (see definition in Section 2) in the cases when either $\Lambda = \mathbf{N}^d$ and J is a quadratic monomial ideal (Corollary 4.1), or $J = \mathfrak{o}$ and Λ has minimal generators $\alpha_1, \ldots, \alpha_n$ for which the toric ideal I_Λ has a quadratic Gröbner basis (Corollary 4.3). In the first case, R was shown to be Koszul by Fröberg [Fr], who explicitly described the generalized Koszul complex providing the linear minimal free resolution of k. In the second case, if I_Λ is graded with respect to the grading with $\deg(x_i) = 1$ then R is again known to be Koszul; this can be proved using the first case along with standard deformation argument (cf. [BHV] or [Ei, Section 15]). However our topological techniques also apply when I_Λ is not graded in the above way, and in particular Theorem 4.2 generalizes some of the results in [PRS] which used the technique of non-pure shellability. In particular, we recover the results of [PRS] giving a rational expression for the Poincaré series of $\mathrm{Tor}_{\cdot}^R(k,k)$ when R is any 2-dimensional normal affine semigroup ring.

Furthermore, Corollary 4.3 inspired the developments of Section 5, in which we generalize the notion of Koszul algebras to local rings and to $\mathbf{N}^d$-graded k-algebras (Definition 5.1). Along with providing an algebraic proof of Corollary 4.3, we also prove (Theorem 5.2) that this new notion of Koszulness applies to Cohen-Macaulay rings of minimal multiplicity. As a consequence we obtain a rational expression for the Poincaré series of such rings, generalizing the case of 2-dimensional normal affine semigroup rings.

2. Computing Betti numbers of rings by simplicial complexes

Computation of Betti numbers of rings using relative homology was introduced by Bruns and Herzog in [BH] for finite resolutions of monomial ideals in semigroup rings. The main result of this section, Theorem 2.1, is in the same spirit, but deals with infinite resolutions. In the case when $J = \mathfrak{o}$ the formula in Theorem 2.1 was proved in [LS] and [PRS]. The idea is to identify the chains in the Bar resolution of k with the chains of certain simplicial complexes, as we now discuss.

The k-algebra R is naturally augmented with augmentation ideal $\mathfrak{m} = \bigoplus_{\lambda \in \Lambda, \lambda \neq 0} k\lambda$.
Note that R is a Λ-graded k-algebra, and $\mathfrak{m}$ a Λ-graded R-module. Indeed, we may define $R_\lambda = k\lambda$ for all $\lambda \in \Lambda$, and similarly for $\mathfrak{m}$. Thus the minimal free resolution of k over R and the Tor-groups $\mathrm{Tor}_i^R(k,k)$ are naturally Λ-graded k-vector spaces, and we will denote

by $\mathrm{Tor}_i^R(k,k)_\lambda$ the λ-graded piece of $\mathrm{Tor}_i^R(k,k)$.

In order to compute the Tor-groups we must construct certain simplicial complexes. First, partially order the semigroup Λ by divisibility: for $\lambda, \mu \in \Lambda$, say μ *divides* λ (written $\mu \leq \lambda$) if $\lambda = \mu + \iota$ for some $\iota \in \Lambda$ Then the closed and open intervals

$$[\lambda, \mu] := \{\rho \in \Lambda : \mu \leq \rho \leq \lambda\}$$
$$(\lambda, \mu) := \{\rho \in \Lambda : \mu < \rho < \lambda\}$$

are partially ordered sets with the induced ordering.

The following picture shows the Hasse diagram for the interval $[(0,0,0),(3,3,3)]$ in the poset where $\Lambda \subset \mathbf{N}^3$ is generated by

$$\{(3,0,0),(0,3,0),(0,0,3),(2,1,0),(2,0,1),(1,2,0),(1,0,2),(0,2,1),(0,1,2)\}.$$

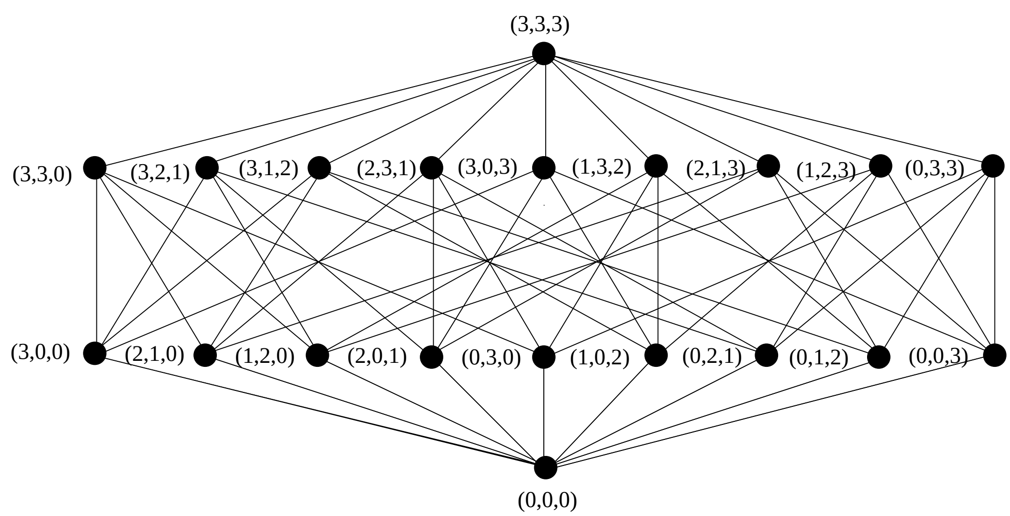

Figure 1: The closed interval $[(0,0,0),(3,3,3)]$ *in* Λ

Associated with a poset P is its *order complex* $\Delta(P)$. The ground set of $\Delta(P)$ is the set of elements of P, and faces are the sub-sets $\{p_1, \ldots, p_k\}$ of P whose elements form a chain $p_1 < p_2 < \cdots < p_k$. Define $\Delta_\lambda := \Delta((0,\lambda))$, i.e Δ_λ is the order complex for the open interval lying strictly between 0 and λ in the poset structure on Λ. Note that the complexes Δ_λ are not pure (all maximal faces of the same dimension) unless the minimal generators of the monoid Λ lie on a hyperplane in $\mathbf{N}^d$.

Let J be an ideal in $k[\Lambda]$ that is generated by semigroup elements, what we call a *semigroup ideal.* We will often abuse notation and use J both to denote the ideal in the ring, and to denote the set $\{\lambda \in \Lambda : \mathbf{z}^\lambda \in J\}$. In the case $\Lambda = \mathbf{N}^d$ the ideal J is called a *monomial ideal.* For $\lambda \in \Lambda$ we denote by $\Delta_{\lambda,J}$ the sub-complex of Δ_λ consisting of all chains $\lambda_1 < \cdots < \lambda_l$ in $(0, \lambda)$ such that if we set $\lambda_0 = 0$ and $\lambda_{l+1} = \lambda$ then there is an index $0 \leq i \leq l$ such that

$$\lambda_{i+1} - \lambda_i \in J.$$

In other words, $\Delta_{\lambda,J}$ is the sub-set of Δ_λ consisting of the chains which contain a "jump" from J. Since J is an ideal, this sub-set is actually a simplicial sub-complex of Δ_λ. The (relative, reduced) *simplicial homology* of the pair $(\Delta_\lambda, \Delta_{\lambda,J})$ with coefficients in k will be denoted by $\tilde{\mathrm{H}}_i(\Delta_\lambda, \Delta_{\lambda,J}; k)$. Furthermore, we denote by $\tilde{C}_\cdot(\Delta_\lambda, \Delta_{\lambda,J})$ the augmented oriented chain complex of the pair $(\Delta_\lambda, \Delta_{\lambda,J})$.

Theorem 2.1. *Let $\Lambda \subseteq \mathbf{N}^d$ be an affine semigroup and let J be a semigroup ideal in $k[\Lambda]$. Set $R = k[\Lambda]/J$. Then*

$$\mathrm{Tor}_i^R(k,k)_\lambda \cong \tilde{\mathrm{H}}_{i-2}(\Delta_\lambda, \Delta_{\lambda,J}; k)$$

for all $\lambda \in \Lambda$ and all $i > 0$.

Proof: Consider the *bar resolution* $\mathbf{B}$ of k over $k[\Lambda]$, see [Ma, Ch. 10,§2],

$$\mathbf{B} : \cdots \to B_i \to B_{i-1} \to \cdots \to B_0 = k$$

This is a particular free resolution of k as an R-module, which is far from being minimal. In our case, the i^{th} term B_i in $\mathbf{B}$ may be identified with the free R-module with basis $\{[\lambda_1|\cdots|\lambda_i]\}$, indexed by ordered i-tuples of non-zero elements of $\Lambda \backslash J$. The differential $d_i : B_i \to B_{i-1}$ acts by the rule

$$d_i[\lambda_1|\cdots|\lambda_i] \quad = \quad z^{\lambda_1}[\lambda_2|\cdots|\lambda_i] + \sum\nolimits_{1\leq j\leq i-1} (-1)^j \cdot [\lambda_1|\cdots|\lambda_j + \lambda_{j+1}|\cdots|\lambda_i]$$

in which $[\lambda_1|\cdots|\lambda_i]$ is interpreted as 0 whenever any λ_j is in J. To compute $\mathrm{Tor}_i^R(k,k)$, we must tensor this resolution with the trivial R-module $k(= R/\mathfrak{m})$. This yields the complex of k-vector spaces

$$\mathbf{B} \otimes k : \cdots \to B_i \otimes k \to B_{i-1} \otimes k \to \cdots \to B_0 \otimes k = k$$

with differential

$$d_i[\lambda_1|\cdots|\lambda_i] \quad = \quad \sum\nolimits_{1\leq j\leq i-1} (-1)^j \cdot [\lambda_1|\cdots|\lambda_j + \lambda_{j+1}|\cdots|\lambda_i].$$

By definition, $\mathrm{Tor}_i^R(k,k)$ is the homology of this complex. Note that the differential preserves the sum $\lambda = \lambda_1 + \cdots + \lambda_i$ of the entries in each bracket, that is, the Bar complex is the direct sum of its finite-dimensional graded components $(\mathbf{B} \otimes k)_\lambda$. If we identify $[\lambda_1|\lambda_2|\cdots|\lambda_i]$ with the chain $\lambda_1 < \lambda_1 + \lambda_2 < \cdots < \lambda_1 + \cdots + \lambda_{i-1}$ in the open interval

$(0, \lambda)$, then the differential of the Bar complex $(\mathbf{B} \otimes k)_\lambda$ is precisely the boundary map in the augmented relative chain complex $\tilde{C}_{\cdot-2}(\Delta_\lambda, \Delta_{\lambda,J})$. ■

Before closing this section, we give a brief discussion of Koszul rings in this context. When the generators of Λ all lie on some hyperplane in $\mathbf{N}^d$, the toric ideal I_Λ is homogeneous with respect to the $\mathbf{N}$-grading defined by $\deg(x_i) = 1$ for all i, and R and $\mathrm{Tor}^R_\cdot(k, k)$ inherit this $\mathbf{N}$-grading. For any $\mathbf{N}$-graded k-algebra R, we let $\mathrm{Tor}^R_i(k, k)_j$ denote the j^{th}-graded component of $\mathrm{Tor}^R_i(k, k)$ with respect to this grading, and it is easy to see that $\mathrm{Tor}^R_i(k, k)_j = 0$ for $j < i$. We say that R is *Koszul* if $\mathrm{Tor}^R_i(k, k)_j = 0$ for all $j \neq i$. Recall that $\dim_k \mathrm{Tor}^R_i(k, k)_j$ equals the number of basis elements of degree j in the i^{th} free R-module occurring in a minimal resolution of k as a trivial R-module. Because of this interpretation, Koszulness of R is equivalent to this resolution being *linear*, i.e. all matrix entries for the boundary maps are either 0 or of degree 1. Koszulness has important consequences for R (see for example, Lemma 5.5), and in [PRS, Theorem Corollary 2.2] it was observed that Theorem 2.1. has the following consequence: in the case $J = \mathfrak{o}$, R is Koszul if and only if Λ is a *Cohen-Macaulay* poset (see [BGS] for information on Cohen-Macaulay posets).

It follows from work of Fröberg [Fr] that when $\Lambda = \mathbf{N}^d$, R is Koszul if and only if the monomial ideal J is generated by quadratic monomials. At present, we know of no example where $J = \mathfrak{o}$ and I_Λ is generated by quadrics such that R is not Koszul. On the other hand, the following example shows that a non-Koszul ring R which is a quotient of quadrics can appear in our set-up, if we allow $J \neq \mathfrak{o}$.

Example 2.2. Let Λ be the monoid generated by $x = (0, 2), y = (1, 1), z = (2, 0)$ in $\mathbf{N}^2$. Taking the quotient by the semigroup ideal $J = \langle xy, z^2 \rangle$ and the toric ideal $I_\Lambda = \langle y^2 - xz \rangle$ we obtain the ring $R = k[x, y, z]/\langle y^2 - xz, xy, z^2 \rangle$. Consider the multidegree $\lambda = (4, 4)$, which lies in degree 4 of the $\mathbf{N}$-grading. One can compute, for example with the computer algebra package MACAULAY, that $\mathrm{Tor}^R_3(k, k)_{(4,4)} \cong \tilde{\mathrm{H}}_1(\Delta_{(4,4)}, \Delta_{(4,4),\langle xy,z^2\rangle}; k)$ does not vanish for any field k, so that $\mathrm{Tor}^R_3(k, k)_4 \neq 0$ and R is not Koszul. By [BF], up to an isomorphism, this is the only example of a non-Koszul ideal generated by quadratic binomials in three variables.

We close this section with the following simple consequence of Theorem 2.1.

Proposition 2.3. *Let R be a quotient of an affine semigroup ring $k[\Lambda]$ by a semigroup ideal J. Then the minimal free resolution $(F_\cdot, d_\cdot)$ of k over R is monomial, meaning that for $p \geq 0$ we can choose basis for F_p so that the matrix for the differential $d_p : F_p \to F_{p-1}$ has monomial entries.*

Proof: We know that the basis elements for the modules F_p can be chosen to be homogeneous with respect to the Λ-grading, and also the maps can be chosen to preserve this

Λ-grading. But this then implies that each entry in the map is a monomial, since the ring $R = k[\Lambda]/J$ has (up to multiplication by constants) a unique element in each multidegree and this element is represented by a monomial. ■

3. Topological analysis

The goal of this section is to provide a more concrete geometric understanding of the pair $(\Delta_\lambda, \Delta_{\lambda,J})$ in the case where $\Lambda = \mathbf{N}^d$. In this case we have $d = n$ and $k[\Lambda] = k[x_1, \ldots, x_n]$, so that $J \subseteq k[x_1, \ldots, x_n]$ is a monomial ideal, and an element $\lambda \in \Lambda$ is simply a monomial m in $k[x_1, \ldots, x_n]$. We will present a detailed topological analysis of the complexes Δ_m and $\Delta_{m,J}$ in terms of subspace arrangements.

An *arrangement of subspaces* $\mathcal{A}$ in $\mathbf{R}^n$ is a finite collection of linear subspaces. We identify $\mathcal{A}$ with its *union* $\bigcup_{H \in \mathcal{A}} H$. Here we consider the real n-space $\mathbf{R}^n$ with coordinates given by $u_1, \ldots, u_n$ (since we are already using the naturals variables $x_1, ..., x_n$ as polynomial ring variables).

Let $J \subset k[x_1, \ldots, x_n]$ be a monomial ideal, and among its minimal generators, let the *square-free* monomials be $m_1, \ldots, m_l$. Given a monomial m, let U_m denote the subspace which is the intersection of all hyperplanes $u_p = u_q$ for monomials $x_p x_q$ dividing m. Then the *canonical arrangement* associated to J is the subspace arrangement inside the hyperplane $u_1 + \cdots + u_n = 0$ consisting of its intersection with the union of all linear subspaces U_{m_i}. For example, if $J = (x_1^2 x_2, x_1 x_3, x_2^4, x_2 x_3 x_4) \subset k[x_1, x_2, x_3, x_4]$, then $\mathcal{A}_J$ is the union of the subspaces

$$\{u_1 = u_3\} \cup \{u_2 = u_3 = u_4\}$$

inside the hyperplane $u_1 + u_2 + u_3 + u_4 = 0$.

Theorem 3.1. *Let J be a monomial ideal and m a square-free monomial in $k[x_1, \ldots, x_n]$, say without loss of generality, $m = x_1 \cdots x_e$. Then the pair $(\Delta_m, \Delta_{m,J})$ is homeomorphic to the pair $(\mathbf{S}^{e-2}, \mathbf{S}^{e-2} \cap \mathcal{A}_{J^e})$, where $\mathbf{S}^{e-2}$ is the unit sphere in the hyperplane $u_1 + \cdots + u_e = 0$ in real e-space and $\mathcal{A}_{J^e}$ is the canonical arrangement corresponding to $J^e = J \cap k[x_1, \ldots, x_e]$ in $\mathbf{R}^e$.*

Proof: The proof consists of identifying particular geometric realizations of Δ_m and $\Delta_{m,J}$. The complex Δ_m is easily seen to be the order complex of the proper part of the *Boolean lattice* B_e on an e-element set, i.e. it is the simplicial complex of chains of proper sub-sets of an e-element set. To see this, identify a monomial with the set of indices of the occurring variables. Thus Δ_m is the simplicial complex associated to the barycentric subdivision of the boundary of an $(e-1)$-simplex (sometimes known as the *Coxeter complex* for type

A_{e-1}). In particular, Δ_m is homeomorphic to $\mathbf{S}^{e-2}$. If we embed this $(e-1)$-simplex so that its vertices are the standard basis vectors in $\mathbf{R}^e$, then it will naturally lie in the affine hyperplane $u_1 + \cdots + u_e = 1$. A face of Δ_m corresponds to a chain of sub-sets

$$\emptyset = A_0 \subset A_1 \subset \cdots \subset A_r \subset A_{r+1} = \{1, 2 \ldots, e\}.$$

One can check that geometrically the relative interior of this face is the set of points in the affine hyperplane $u_1 + \cdots + u_e = 1$ satisfying the conditions
(a) $u_i = u_{i'}$ if i, i' lie together in some set $A_{j+1} - A_j$, and
(b) $u_i < u_{i'}$ if $i \in A_j$ and $i' \in A_{j'}$ for some $j < j'$.

We obtain a homeomorphic picture if we first translate the entire $(e-1)$-simplex by the vector $\frac{-1}{e}(1, \ldots, 1)$ so that the simplex now lies in the hyperplane $u_1 + \cdots + u_e = 0$, and then map each point of the boundary of the simplex radially to the point on the unit sphere $\mathbf{S}^{e-2}$ in $u_1 + \cdots + u_e = 0$ which lies on the same ray from the origin. Now the relative interior of the above face is the set of points on the unit sphere satisfying the same conditions (a) and (b) above. The advantage to this situation is that now every subspace of the form $u_{i_1} = \cdots = u_{i_r}$ has its intersection with the sphere $\mathbf{S}^{e-2}$ triangulated as a sub-complex of the Coxeter complex Δ_m.

By the definition of $\Delta_{m,J}$, it will be the sub-complex consisting of all such chains in which there is an index j for which $\prod_{i \in A_{j+1} - A_j} x_i$ is divisible by a square-free monomial in J. Comparing this with the description in (a) and (b) above, these are exactly the faces in the barycentric subdivision which triangulate the intersection of the canonical arrangement $\mathcal{A}_{J^e}$ and the unit sphere $\mathbf{S}^{e-2}$. ■

The next two results give a closer topological analysis of the pair $(\Delta_m, \Delta_{m,J})$ in the case where J is a quadratic monomial ideal. To state these results, we need to establish some notation.

Say that a non-commutative monomial in $k\langle y_1, \ldots, y_n\rangle$ has *content* m (or that it lies in the *non-commutative fiber of* m) if it maps to m under the map

$$k\langle y_1, \ldots, y_n\rangle \to k[x_1, \ldots, x_n]$$

defined by $y_i \mapsto x_i$. Here we use the notation $k\langle \cdots \rangle$ and $k[\cdots]$ to denote non-commutative and commutative polynomial rings, and all ideals in the non-commutative ring $k\langle y_1, \ldots, y_n\rangle$ are assumed to be two-sided. Given a quadratic monomial ideal J in $k[x_1, \ldots, x_n]$, let its *complementary* ideal J' be the ideal generated by the set of all quadratic monomials in $k[x_1, \ldots, x_n]$ that do *not* lie in J. Given two non-commutative monomials, say that they are *J'-commuting equivalent* if they have the same image under the quotient map

$$k\langle y_1, \ldots, y_d\rangle \to k\langle y_1, \ldots, y_d\rangle / (y_i y_j - y_j y_i : x_i x_j \in J')$$

Lastly, say that a non-commutative monomial is *J'-non-stuttering* if no rearrangement in its J'-commuting class contains two *adjacent* letters $y_i y_i$ with $x_i^2 \in J'$. For a (commutative) monomial m we denote by $\mathcal{C}(m)$ the set of J'-non-stuttering J'-commuting classes of non-commutative monomials with content λ.

The next result further analyzes the case where m is square-free and J is *quadratic*.

Lemma 3.2. *Let J, m be as in Theorem 3.1. If in addition J is a quadratic monomial ideal then, $\mathcal{A}_{J^e}$ is an arrangement of hyperplanes, and the set of chambers cut out by $\mathcal{A}_{J^e}$ is in bijection with the set $\mathcal{C}(m)$.*

Proof: When J is a quadratic monomial ideal, it is clear from the definition that $\mathcal{A}_{J^e}$ is the arrangement of hyperplanes (inside $\sum_i u_i = 0$) which is the union of all $u_i = u_j$ with $x_i x_j \in J$. So without loss of generality, it only remains to identify the chambers of $\mathcal{A}_J$ in case J is an ideal generated by square-free quadratic monomials.

In this case, $\mathcal{A}_J$ is a sub-arrangement of the *braid arrangement* $\mathcal{A}_{e-1}$ which has all hyperplanes of the form $u_i = u_j$ with $1 \leq i < j \leq n$. Notice that the chambers cut out by the braid arrangement are of the form

$$u_{\sigma_1} > \cdots > u_{\sigma_e}$$

for some permutation $\sigma : i \mapsto \sigma_i$ of $\{1, 2, \ldots, e\}$, corresponding to the maximal faces of the Coxeter complex Δ_m. Note that these chambers also biject with the non-commutative monomials of content m. Since $\mathcal{A}_J$ is a sub-arrangement of $\mathcal{A}_{e-1}$, its chambers will be unions of chambers from $\mathcal{A}_{e-1}$, i.e. they will correspond to certain sub-sets of non-commutative monomials, which we must identify.

The key observation is that $\mathcal{A}_J$ is a *graphic arrangement* [OT, §2.4]: To a graph G (with no loops or multiple edges) on e vertices one associates the arrangement $\mathcal{A}_G \subseteq \mathcal{A}_{e-1}$ consisting of all hyperplanes $u_i = u_j$ for all edges $\{i, j\}$ in G. In particular, $\mathcal{A}_{e-1} = \mathcal{A}_{K_e}$ for K_e the complete graph on e vertices. The graphic arrangements $\mathcal{A}_G$ are exactly the arrangements $\mathcal{A}_J$ for ideals J generated by square-free quadratic monomials. It is known [GZ] that the chambers of a graphic arrangement $\mathcal{A}_G$ are in bijection with the *acyclic orientations* of the edges of G (i.e., the directed graphs containing no directed cycles which are obtained by picking a direction for each of G). One can also view such an acyclic orientation Ω as defining a poset structure P_Ω on the vertices of G, where $v > v'$ in P_Ω if there is a directed edge from v to v'.

If G is the graph associated to the square-free monomial ideal J, meaning $\{i, j\}$ is an edge of G exactly when $x_i x_j \in J$, it is easy to see that these acyclic orientations biject to the set $\mathcal{C}(m)$ of all J'-commuting equivalence classes of permutations ω, under the correspondence which sends Ω to the set of all of *linear extensions* of the poset P_Ω. The inverse correspondence which produces a poset P (and hence an acyclic orientation) from a J'-commuting equivalence class of permutations ω was first described by Cartier and Foata [CF] (see also [Sta, Exercise 3.48, p.163]). Also note that in this case since m is square-free no element of $\mathcal{C}(m)$ can J'-stutter, so the set of J'-commuting classes of permutations ω is the same as the set $\mathcal{C}(m)$. Therefore the chambers of $\mathcal{A}_J$ and the acyclic orientations of G are both in bijection with $\mathcal{C}(m)$. ■

The next result analyzes the case where m is not square-free.

Lemma 3.3. *Let J be a monomial ideal, and m a monomial of degree e in $k[x_1, \ldots, x_n]$.*
(1) If m is not square-free then Δ_m is homeomorphic to a polytopal ball $\mathbf{B}^{e-2}$.
(2) If in addition J is generated by quadratic monomials then the sub-complex $\Delta_{m,J}$ is homotopy equivalent to a one-point wedge of $(e-3)$-spheres indexed by $\mathcal{C}(m)$, i.e. to

$$\bigvee_{\omega \in \mathcal{C}(m)} \mathbf{S}^{e-3}.$$

The proof of this lemma will occupy the remainder of this section.
Proof: Let $m = x_1^{e_1} \cdots x_n^{e_n}$ be a not necessarily square-free monomial of degree e. We polarize m to obtain a square-free monomial m^{pol}:

$$m^{pol} = x_1^{(1)} x_1^{(2)} \cdots x_1^{(e_1)} \cdots x_n^{(1)} x_n^{(2)} \cdots x_n^{(e_n)}.$$

Consider the braid arrangement A_{e-1} associated to m^{pol} as in the proof of the previous lemma, lying in the space having coordinates

$$u_1^{(1)}, u_1^{(2)}, \ldots, u_1^{(e_1)}, \ldots, u_n^{(1)}, u_1^{(2)}, \ldots, u_n^{(e_n)}.$$

We claim that the simplicial complex Δ_m is then isomorphic to the sub-complex of the Coxeter complex $\Delta_{m^{pol}}$ which covers the intersection of $\mathbf{S}^{e-2}$ (the unit sphere in the hyperplane $\sum_i u_i = 0$) with the convex cone defined by

$$\begin{gathered} u_1^{(1)} \geq u_1^{(2)} \cdots \geq u_1^{(e_1)} \\ \vdots \\ u_n^{(1)} \geq u_n^{(2)} \cdots \geq u_n^{(e_n)}. \end{gathered} \tag{3.1}$$

To see this claim, recall that a maximal face in Δ_m corresponds to a non-commutative monomial $y_{i_1} \cdots y_{i_e}$ having content m. Replacing the j^{th} occurrence of y_i in the monomial by $y_i^{(j)}$, we get another non-commutative monomial $y_{i_1}^{(j_1)} \cdots y_{i_e}^{(j_e)}$, and this corresponds to the maximal face in $\Delta_{m^{pol}}$ lying inside the above cone and cut out by the stronger set of inequalities

$$u_{i_1}^{(j_1)} \geq \cdots \geq u_{i_e}^{(j_e)}.$$

As one varies over the non-commutative monomials of content m, these smaller cones cover the cone defined by (3.1), proving the claim.

The previous argumentation shows that Δ_m is homeomorphic to a ball $\mathbf{B}^{e-2}$. We can visualize this as a *polytopal* ball in the following way. If we assume (without loss of generality) that $e_1 \geq 2$, then the radial projection of the intersection of the cone and sphere onto the affine hyperplane $u_1^{(1)} - u_1^{(2)} = 1$ is a homeomorphism of Δ_m onto the convex polytope defined by the inequalities (3.1) inside this affine hyperplane. Assertion (1) of Lemma 3.3 is now proved.

We move on to proving assertion (2), so from now on assume that J is generated by quadratic monomials. We polarize the monomial ideal J as follows. Consider the map

$$k[x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(e_1)}, \ldots x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(e_n)}] \to k[x_1, \ldots, x_n]$$

which sends $x_i^{(j)}$ to x_i for each i and j, and let J^{pol} be the ideal in the domain which is generated by all square-free monomials that map to a (quadratic) monomial which is a minimal generator of J in $k[x_1, \ldots, x_n]$. It is clear that the sub-arrangement $\mathcal{A}_{J^{pol}}$ of $\mathcal{A}_{e-1}$ gives a polytopal decomposition of the polytopal ball Δ_m. Note however that some of the subspaces in $\mathcal{A}_{J^{pol}}$ may intersect only the boundary but not the interior of this ball. So these subspaces do not participate in the decomposition.

The maximal cells in this decomposition will correspond to chambers in $\Delta_{m^{pol}}$ lying in the convex cone described by the inequalities (3.1), and these chambers exactly correspond to the set of J'-commuting equivalence classes ω of non-commutative monomials of content m, as in the proof of Lemma 3.2 (see also Figure 2). For each such class ω, let Δ_ω denote the sub-complex of Δ_m which triangulates this chamber. We can then define a fairly trivial *diagram of spaces* $\mathcal{D}^0$ which is the *covering diagram* for the covering of Δ_m by the maximal cells Δ_ω in the above polytopal decomposition, having an artificial $\hat{1}$ (corresponding to the empty space) at the top of its intersection poset P.

Now consider the sub-complex $\Delta_{m,J}$, which we may identify with the arrangement $\mathcal{A}_{J^{pol}}$ intersected with the ball Δ_m. For each maximal cell Δ_ω in the polytopal decomposition of Δ_m, let $\Delta_{\omega,J}$ denote the intersection $\Delta_\omega \cap \Delta_{m,J}$. See Figure 2.

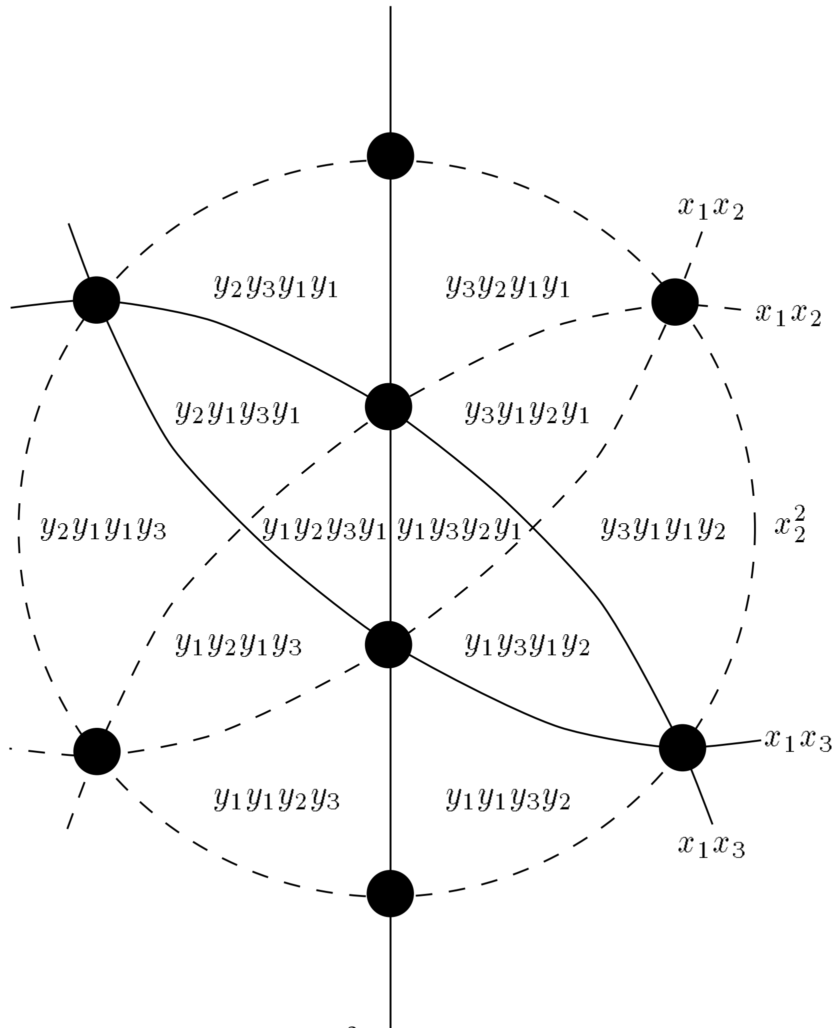

Figure 2: For $J = (x_1x_3, x_2x_3)$ and $m = x_1^2x_2x_3$, the polytopal decomposition of the complex Δ_m (shown as the upper hemisphere of the unit sphere $\mathbf{S}^2$) by the hyperplanes corresponding to quadratic monomials in J (shown darker). The chambers Δ_ω cut out by these hyperplanes correspond to J'-commuting equivalence classes ω of monomials in the non-commutative fiber of m.

Clearly the $\{\Delta_{\omega,J}\}$ form a cover of $\Delta_{m,J}$, and we can again create the diagram of spaces $\mathcal{D}$ corresponding to this covering.

Lemma 3.4. *In the covering of Δ_m by the maximal cells $\{\Delta_\omega\}$, every proper intersection lies in the sub-complex $\Delta_{m,J}$.*

Proof: Since the covering of Δ_m by these maximal cells was the polytopal decomposition cut out by the arrangement $\mathcal{A}_{J^{pol}}$, all proper intersections of the maximal cells lie inside some hyperplane of this arrangement, and hence inside $\Delta_{m,J}$. ■

As a consequence of this lemma, the diagrams of spaces $\mathcal{D}$ and $\mathcal{D}^0$ share the same intersection poset, and differ only in the spaces D_p assigned to minimal elements p in the poset, where $\mathcal{D}^0$ has the spaces Δ_ω and $\mathcal{D}$ has the spaces $\Delta_{\omega,J}$.

Recall that if J is a quadratic monomial ideal in $k[x_1, \ldots, x_e]$ then J' is the complementary quadratic monomial ideal, generated by the complementary set of quadratic monomials to those in J.

Lemma 3.5. *For any J'-commuting class ω of non-commutative monomials with content m, the space $\Delta_{\omega,J}$ is either a sphere $\mathbf{S}^{e-3}$ or ball $\mathbf{B}^{e-3}$, depending upon whether ω is J'-non-stuttering or J'-stuttering.*

Proof: Since $\Delta_{\omega,J} = \Delta_\omega \cap \Delta_{m,J}$, it is a sub-complex of the boundary of the polytope Δ_ω. It is not hard to see that the walls of the chamber Δ_ω in the arrangement $\mathcal{A}_{J^{pol}}$ correspond to all possible

- adjacent pairs of repeated letters, or
- adjacent pairs of letters corresponding to a quadratic monomial in J

occurring among the non-commutative monomials belonging to the class ω. Therefore, if ω is J'-non-stuttering then every such wall corresponds to a pair of letters that comes from a quadratic monomial in J, so every wall lies in $\Delta_{\omega,J}$. This implies $\Delta_{\omega,J}$ is the *entire* boundary sphere $\mathbf{S}^{e-3}$ of the polytopal ball Δ_ω.

On the other hand, assume some word in ω stutters on some elements of J'. Classify the walls of the polytopal ball Δ_ω according to whether they lie in $\Delta_{\omega,J}$ or not. We claim that the walls which do not lie in $\Delta_{\omega,J}$ have non-empty common intersection F, which must therefore be some boundary face F of Δ_ω. To see this claim, note that each such wall not in $\Delta_{\omega,J}$ corresponds to an occurrence of a pair of adjacent equal letters $y_i y_i$ with x_i^2 in J' in some representative of the class ω. It is not hard to see (and follows easily from Cartier-Foata theory [CF]) that there is some representative of ω which has *all* possible such pairs of J'-stutterings, and then the intersection of the J'-stuttering facets of the simplex corresponding to this particular representative will be non-empty and lie in the desired intersection F. Given this claim, we can describe $\Delta_{\omega,J}$ as the sub-complex of the boundary of the polytope Δ_ω which is the union of all facets not containing the face F. This implies $\Delta_{\omega,J}$ is a ball by Lemma 6.4. This completes the proof of Lemma 3.5. ■

To complete the proof of assertion (2) in Lemma 3.3, we need to show that $\Delta_{m,J}$ is homotopy equivalent to the one-point wedge of spheres

$$\bigvee_{\omega \in \mathcal{C}(m)} \mathbf{S}^{e-3}.$$

To see this, we apply the Wedge Lemma 6.1 to the diagram $\mathcal{D}$. This diagram satisfies the necessary hypotheses because all of the spaces D_p for p non-minimal in P coincide with the corresponding spaces D_p^0 in the diagram $\mathcal{D}^0$ (which came from a polytopal decomposition of a polytopal ball). Note that in $\mathcal{D}^0$, for any p in P with $p \neq \hat{1}$, the space D_p^0 (which is the same as D_p) is contractible because it is a polytope. Also note that the order complex $\Delta(P_{<\hat{1}})$ is contractible because $P_{<\hat{1}}$ is the face poset for the polytopal decomposition of the ball Δ_m, and hence it triangulates the *barycentric subdivision* of this polytopal decomposition.

Therefore $\Delta_{m,J}$ is homotopy equivalent to the wedge

$$\bigvee_{p \in P} \Delta(P_{<p}) * D_p$$

By the previous discussion, the only non-contractible contributions to the wedge come from the spaces D_p corresponding to the minimal elements p in the diagram. But these spaces D_p are by definition the spaces $\Delta_{\omega,J}$. Lemma 3.5 then gives exactly what we want, and the proof of Lemma 3.3 is complete. ■

4. Topological proofs of Koszulness

In this section we specifically study the following two situations and give topological proofs of Koszulness of R:

- $\Lambda = \mathbf{N}^d$ (so $R = k[x_1, \ldots, x_d]$) and J is a quadratic monomial ideal, or
- $J = \emptyset$ and Λ has minimal generators $\alpha_1, \ldots, \alpha_n$ for which the toric ideal I_Λ has a quadratic Gröbner basis. We refer the reader to [CLO] for notions and terminology related to Gröbner bases.

Let S be an $\mathbf{N}^d$-graded finitely generated associative k-algebra. The *Hilbert series* of S is the generating function

$$\mathrm{Hilb}(R; z_1, \ldots, z_d) \;=\; \sum\nolimits_{\lambda \in \mathbf{N}^d} \dim_k(S_\lambda)\mathbf{z}^\lambda.$$

although we will occasionally use other variables than $z_1, \ldots, z_n$ as power series variable in the generating function. In fact, we will even abuse notation by sometimes using the variables x_i or z_i as generating function variables when S is a sub-algebra or quotient of $k[x_1, \ldots, x_n]$ or $k[z_1, \ldots, z_d]$. We hope that this creates no confusion.

As was mentioned earlier, the groups $\mathrm{Tor}_i^R(k,k)$ are $\mathbf{N}^d$-graded or Λ-graded as well. Let $\mathrm{Tor}_i^R(k,k)_\lambda$ be the λ-graded part of $\mathrm{Tor}_i^R(k,k)$. The *Poincaré series* of k is

$$\mathrm{Poin}_R^k(t, z_1, \ldots, z_d) \;=\; \sum_{i \geq 0} \sum_{\lambda \in \mathbf{N}^d} \dim_k \mathrm{Tor}_i^R(k,k)_\lambda \, t^i \mathbf{z}^\lambda \,.$$

Let $R = k[\Lambda]/J$ be an affine semigroup ring for a semigroup $\Lambda \subseteq \mathbf{N}^d$ modulo a semigroup ideal J. Then clearly

$$\mathrm{Hilb}(R;\mathbf{z}) := \sum_{\lambda \in \Lambda} \dim_k R_\lambda \mathbf{z}^\lambda = \sum_{\lambda \in \Lambda \setminus J} \mathbf{z}^\lambda .$$

We first recover Fröberg's result:

Corollary 4.1. **[Fr]** *Let $R = k[x_1, \ldots, x_n]/J$ for a quadratic monomial ideal J. Then R is Koszul for all fields k, and*

$$\mathrm{Poin}_R^k(t,\mathbf{x}) = \mathrm{Hilb}(k\langle y_1, \ldots, y_n\rangle/(\{y_i y_j - y_j y_i\}_{x_i x_j \notin J}), \{y_i^2\}_{x_i^2 \notin J}; \mathbf{x}) \tag{4.1}$$

$$= \left[\frac{1}{\mathrm{Hilb}(R;\mathbf{x})}\right]_{x_i \mapsto -t x_i} \tag{4.2}$$

where the non-commutative variable y_i is considered to have the same multidegree as x_i, and here x_i is used both as a variable in the polynomial ring and as an indeterminate in the Hilbert series.

Proof: Equation (4.2) follows from Koszulness of R (see Lemma 5.5). Koszulness of R and equation (4.1) will follow from Theorem 2.1, once we establish that for any monomial m in R of degree e we have

$$\tilde{\mathrm{H}}_{i-2}(\Delta_m, \Delta_{m,J}; k) = \begin{cases} k^{|\mathcal{C}(m)|} & \text{if } i = e \\ 0 & \text{else.} \end{cases}$$

To see this in the case where m is square-free of degree e, by Theorem 3.1 we need to show that

$$\tilde{\mathrm{H}}_{i-2}(\mathbf{S}^{e-2}, \mathbf{S}^{e-2} \cap \mathcal{A}_{J^e}; k) = \begin{cases} k^{|\mathcal{C}(m)|} & \text{if } i = e \\ 0 & \text{else} \end{cases}$$

However, this follows from Lemma 3.2, as we now explain. Note that the relative homology of $(\mathbf{S}^{e-2}, \mathbf{S}^{e-2} \cap \mathcal{A}_{J^e})$ is the same as the (absolute) homology of the quotient space $\mathbf{S}^{e-2}/(\mathbf{S}^{e-2} \cap \mathcal{A})$, because the two spaces form a relative pair of finite CW-complexes. On the other hand, for any hyperplane arrangement $\mathcal{A}$ in an $(e-1)$-dimensional space, the quotient space $\mathbf{S}^{e-2}/(\mathbf{S}^{e-2} \cap \mathcal{A})$ will be the one-point wedge of spheres $\bigvee_{\omega \in \mathcal{C}} \mathbf{S}^{e-2}$ indexed by the set of chambers $\mathcal{C}$ cut out by $\mathcal{A}$. Therefore Lemma 3.2 gives the assertion.

In the case m is not square-free, Δ_m is a ball by Lemma 3.3. Thus the long exact sequence for homology gives

$$\begin{aligned}\tilde{\mathrm{H}}_{i-2}(\Delta_m, \Delta_{m,I}; k) &\cong \tilde{\mathrm{H}}_{i-3}(\Delta_{m,I}; k) \\ &\cong \tilde{\mathrm{H}}_{i-3}\left(\bigvee_{\omega \in \mathcal{C}(m)} \mathbf{S}^{e-3}; k\right) \\ &= \begin{cases} k^{|\mathcal{C}(m)|} & \text{if } i = e \\ 0 & \text{else.} \end{cases}\end{aligned}$$

where the second isomorphism also follows from Lemma 3.3 ■

For the rest of this section, we consider the case when the toric ideal I_Λ has a quadratic initial ideal with respect to some monomial ordering $\prec$ on $k[x_1, \ldots, x_n]$. Let J be the initial ideal $J = \mathrm{init}_\prec \mathrm{I}_\Lambda$.

Extending the notation of §3, given λ in Λ, say that a monomial m in $k[x_1, \ldots, x_n]$ is in the *commutative fiber* of λ if it maps to $\mathbf{z}^\lambda$ under $x_i \mapsto \mathbf{z}^{\alpha_i}$, and say that a monomial in $k\langle y_1, \ldots, y_n\rangle$ is in the *non-commutative fiber* of λ if it maps to an element of the commutative fiber of λ under $y_i \mapsto x_i$. Note that the *facets* (maximal faces) of Δ_λ correspond to maximal chains in the open interval poset $(0, \lambda)$ and therefore naturally biject with the monomials λ in the non-commutative fiber of λ.

Say that the monomial ordering $\prec$ *reversely grades* Λ if whenever m, m' lie in the commutative fiber of some $\lambda \in \Lambda$ and $m \prec m'$ we have $\deg(m) \geq \deg(m')$.

As in Section 3, for any ideal J generated by quadratic monomials, we have a notion of J'-commuting equivalence classes and J'-non-stuttering for non-commutative monomials. Denote by $\mathcal{C}(\lambda)$ the set of J'-non-stuttering J'-commuting equivalence classes in the non-commutative fiber of λ.

Theorem 4.2. *Let $\Lambda \subseteq \mathbf{N}^d$ be an affine semigroup. Assume that either*

- *I_Λ is homogeneous with respect to* $\deg(x_i) = 1$ *and there is a term order for which it has a Gröbner basis of quadratic binomials, or*
- *I_Λ is not homogeneous with respect to the above grading and there is a term order $\prec$ which reversely grades Λ and for which I_Λ has a quadratic initial ideal.*

Then the simplicial complex Δ_λ is homotopy equivalent to the wedge of spheres

$$\bigvee_{\omega \in \mathcal{C}(\lambda)} \mathbf{S}^{|\omega|-2},$$

where $|\omega|$ is the length of any of the non-commutative monomials in its equivalence class.

Proof: For each monomial m in the commutative fiber of λ, let Δ_m be the sub-complex of Δ_λ generated by the maximal faces corresponding to non-commutative monomials λ of content m. It follows from Theorem 3.1 and Lemma 3.3 that Δ_m is either a sphere $\mathbf{S}^{\deg(m)-2}$ or a ball $\mathbf{B}^{\deg(m)-2}$ depending upon whether or not m is square-free.

Order the monomials $m_1, m_2, \ldots, m_r$ in the commutative fiber of λ according to $\prec$, i.e.

$$m_1 \prec \cdots \prec m_r.$$

Let J be the initial ideal of I_Λ with respect to $\prec$. For a fixed $s \leq r$, if we let $X = \bigcup_{i=1}^{s-1} \Delta_{m_i}$ and $Y = \Delta_{m_s}$, it is then easy to check that $X \cap Y$ may be identified with the sub-complex $\Delta_{m_s,J}$ in the notation of Theorem 3.1. By induction on s, we have that X is homotopy equivalent to $\bigvee_\omega \mathbf{S}^{|\omega|-2}$ as λ ranges over the J'-non-stuttering J'-commuting classes of monomials having content in $\{m_1, \ldots, m_{s-1}\}$. Now we consider the change in topology from X to $X \cup Y$ in the inductive step when we add in $Y = \Delta_{m_s}$. We will see below that the intersection $X \cap Y$ is always homotopy equivalent to a wedge of spheres of codimension 1 in Y, and then the assumption that $\prec$ reversely grades Λ insures that the hypotheses in Lemma 6.2 about dimensions are satisfied. In order to analyze which spheres occur in the wedge, we break up into two cases depending on whether m_s is square-free.

If m_s is square-free of degree e, then $Y = \mathbf{S}^{e-2}$ and $X \cap Y \cong \mathbf{S}^{e-2} \cap \mathcal{A}_{J^e}$. By Lemma 6.3 and Lemma 3.2, we have

$$X \cap Y = \bigvee_{j=1}^{|\mathcal{C}(m_s)|-1} \mathbf{S}^{e-3}.$$

Hence by induction on s and Lemma 6.2 we have

$$X \cup Y = \bigvee_\omega \mathbf{S}^{|\omega|-2}$$

as ω ranges over the J'-non-stuttering J'-commuting classes of monomials having content in $\{m_1, \ldots, m_s\}$. This completes the inductive step in this case.

If m_s is not square-free, then $Y \cong \mathbf{B}^{\deg(m_s)-2}$ and Lemma 3.3 says that $X \cap Y$ is homotopy equivalent to $\bigvee_{j=1}^{|\mathcal{C}(m_s)|} \mathbf{S}^{\deg(m_s)-2}$. Hence by induction and Lemma 6.2 we have

$$X \cup Y = \bigvee_\omega \mathbf{S}^{|\omega|-2}$$

as ω ranges over the J'-non-stuttering J'-commuting classes of monomials with content in $\{m_1, \ldots, m_s\}$. This completes the inductive step in the remaining case, so the theorem is proved. ∎

Corollary 4.3. *With Λ, R, I_Λ, J as in the previous result, we have*

$$\begin{aligned}\mathrm{Poin}_R^k(t,\mathbf{z}) &= \mathrm{Hilb}(k\langle y_1,\ldots,y_n\rangle/(\{y_iy_j - y_jy_i\}_{x_ix_j\notin J}, \{y_i^2\}_{x_i^2\notin J}); t, \mathbf{z}) \\ &= \left[\frac{1}{\mathrm{Hilb}(k[x_1,\ldots,x_n]/J;\mathbf{x})}\right]_{x_i\mapsto -t\mathbf{z}^{\alpha_i}} .\end{aligned} \tag{4.3}$$

Furthermore,

- *if I_Λ is homogeneous with respect to $\deg(x_i) = 1$, then R is Koszul and all of the complexes Δ_λ are Cohen-Macaulay for any field k, i.e. Λ is a Cohen-Macaulay poset.*
- *if I_Λ is not homogeneous and there is a term order $\prec$ which reversely grades Λ and for which I_Λ has a quadratic initial ideal, then all of the complexes Δ_λ are sequentially Cohen-Macaulay for any field k (see [Wa] for a definition and facts about sequential Cohen-Macaulayness).*

Proof: By Theorems 2.1 and 4.2 we have that

$$\begin{aligned}\mathrm{Poin}_R^k(t,\mathbf{z}) &= \sum_{\lambda\in\Lambda, i\geq 0} \dim_k \mathrm{Tor}_i^R(k,k)_\lambda \; t^i\mathbf{z}^\lambda \\ &= \sum_{\lambda\in\Lambda, i\geq 0} \dim_k \tilde{\mathrm{H}}_{i-2}(\Delta_\lambda; k) \; t^i\mathbf{z}^\lambda \\ &= \sum_{\lambda\in\Lambda, i\geq 0} \dim_k \tilde{\mathrm{H}}_{i-2}\left(\bigvee_{\omega\in\mathcal{C}(\lambda)} \mathbf{S}^{|\lambda|-2}; k\right) t^i\mathbf{z}^\lambda \\ &= \sum_{\lambda\in\Lambda}\sum_{\omega\in\mathcal{C}(\lambda)} t^{|\omega|} z^\lambda \\ &= \mathrm{Hilb}(k\langle y_1,\ldots,y_n\rangle/(\{y_iy_j - y_jy_i\}_{x_ix_j\notin J}, \{y_i^2\}_{x_i^2\notin J}); t, \mathbf{z})\end{aligned}$$

This proves the first equality in the corollary. Equality (4.3) then follows from equation (4.2).

In the homogeneous case, Koszulness follows from Theorem 4.2 since every λ in $\mathcal{C}(\lambda)$ has the same length $|\lambda|$. Cohen-Macaulayness of Δ_λ follows from Theorem 4.2 and the fact that all links of its faces are joins of complexes of the form Δ_μ for μ in Λ.

In the inhomogeneous case, sequential Cohen-Macaulayness again follows from Theorem 4.2 and [Wa, Theorem 1.3] along with the fact that all links of faces in Δ_λ are joins of complexes of the same form. ■

One of the motivating special cases for these results are the *2-dimensional normal affine semigroup rings* whose Poincaré series were studied in [LS] and further studied in [PRS]. In this case Λ is the set of points in $\mathbf{N}^2$ which lie in a closed 2-dimensional cone bounded by two rays of rational slope emanating from the origin (see proof of Proposition 5.3). If one numbers the minimal generators $\alpha_1, \ldots, \alpha_n$ of Λ as they are traversed in counterclockwise order from the x-axis to the y-axis, then it is easy to check that the *reverse lexicographic order* $\prec$ reversely grades Λ and gives a quadratic initial ideal J.

We remark that in the next section, it will be shown that in the inhomogeneous case of Corollary 4.3, R is Koszul in a more generalized sense to be defined.

5. Koszulness

In this section we extend the notion of a Koszul algebra to local and $\mathbf{N}^d$-graded k-algebras (Definition 5.1). Our motivation for this definition was to understand algebraically equation (4.3) in the previous section, and this is achieved in Theorem 5.4 and the results following it. Theorem 5.4 shows that for rings R which are Koszul in our more general sense, one can again relate the Poincaré series to the Hilbert series of its associated graded ring, and thereby obtain a rational expression for the Poincaré series. One application is that this more general notion of Koszulness is shown to apply to all Cohen-Macaulay rings of minimal multiplicity (Theorem 5.2).

Throughout this section $(R, \mathfrak{m})$ denotes either

- a Noetherian local ring R with maximal ideal $\mathfrak{m}$, or
- an $\mathbf{N}^d$-graded k-algebra R with maximal graded ideal

$$\mathfrak{m} := \bigoplus_{\substack{\lambda \in \mathbf{N}^d \\ \lambda \neq 0}} R_\lambda.$$

We will generally state results for both kinds of rings, but often give proofs only in the case of local rings when the proofs for $\mathbf{N}^d$-graded rings are analogous.

Let M be an R-module (respectively, graded R-module). Then we denote by $\mathrm{gr}_\mathfrak{m}(M)$ the *associated graded module* $\bigoplus_{i \geq 0} \mathfrak{m}^i M / \mathfrak{m}^{i+1} M$ with respect to $\mathfrak{m}$. In particular, the *associated graded ring* $\mathrm{gr}_\mathfrak{m}(R)$ is an $\mathbf{N}$-graded k-algebra.

Definition 5.1. We call a ring $(R, \mathfrak{m})$ as above *Koszul*, if $\mathrm{gr}_\mathfrak{m}(R)$ is Koszul in the usual sense for $\mathbf{N}$-graded rings, i.e. if $\mathrm{Tor}_i^{\mathrm{gr}_{\mathfrak{m}}(R)}(k,k)_j = 0$ for $j \neq i$.

Note that in the case where R is an $\mathbf{N}$-graded k-algebra, then $\mathrm{gr}_{\mathfrak{m}}(R) \cong R$ as $\mathbf{N}$-graded rings. Thus our definition is an extension of the usual notion of Koszulness.

Before exploring the consequences of Koszulness for R, we exhibit a class of rings which are Koszul in this generalized sense, but we must first recall some basic notions of commutative algebra.

A *regular sequence* for $(R, \mathfrak{m})$ is a sequence $(\theta_1, \ldots, \theta_r)$ of elements in $\mathfrak{m}$ such that θ_{i+1} is a non-zerodivisor in $R/(\theta_1, \ldots, \theta_i)$. We say that $(R, \mathfrak{m})$ is *Cohen-Macaulay* if there exists a regular sequence $(\theta_1, \ldots, \theta_r)$ in R of length r equal to the *Krull dimension* $\dim R$.

When R is a Noetherian $\mathbf{N}$-graded k-algebra, its *Hilbert series*

$$\mathrm{Hilb}(R;t) := \sum_{i \geq 0} \dim_k R_i t^i$$

is known to be a rational function $P(t)/Q(t)$ by a fundamental result of Hilbert and Serre. The *multiplicity* of R is defined to be $e(R) := P(1)$ when P, Q are chosen so that $\gcd(P(t), Q(t)) = 1$. For our local and $\mathbf{N}^d$-graded rings $(R, \mathfrak{m})$, we define more generally the multiplicity $e(R)$ of R as $e(\mathrm{gr}_{\mathfrak{m}}(R))$; see [BH1, p. 181].

Recall (see [Ab]) that if $(R, \mathfrak{m})$ is Cohen-Macaulay, then the multiplicity $e(R)$ is bounded from below as follows:

$$e(R) \geq \dim_k \mathfrak{m}/\mathfrak{m}^2 - \dim(R) + 1$$

where $\dim_k \mathfrak{m}/\mathfrak{m}^2$ is called the *embedding dimension* of $(R, \mathfrak{m})$. In the case of equality one says that the ring has *minimal multiplicity* (see [BH1, Exercise 4.5.14]).

Theorem 5.2. *If Λ is an affine semigroup in $\mathbf{N}^d$ and $k[\Lambda]$ is Cohen-Macaulay with minimal multiplicity then $k[\Lambda]$ is Koszul. In particular, if $k[\Lambda]$ is normal with minimal multiplicity then $k[\Lambda]$ is Koszul in the generalized sense.*

Proof: The second assertion follows from the first, since by a result of Hochster every normal affine semigroup ring is Cohen-Macaulay; see [Theorem 6.3.5 (a), BH1].

By a result of J.D. Sally [Sa], if R is Cohen-Macaulay and of minimal multiplicity then $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay too. It also follows that $\mathrm{gr}_{\mathfrak{m}}(R)$ is of minimal multiplicity, since $\mathrm{gr}_{\mathfrak{m}}(R)$ and R share the same Krull dimension [BH1, Theorem 4.4.6 (b)], and clearly also the same embedding dimension. Therefore, we may replace R by $\mathrm{gr}_{\mathfrak{m}}(R)$ and assume that we have the following situation: R is $\mathbf{N}$-graded, generated by R_1, and Cohen-Macaulay of minimal multiplicity. We may also assume that k is an infinite field, since one can easily check that the Koszul property is independent of flat ring extensions. Hence, there exists a maximal regular sequence $\theta_1, \ldots, \theta_r$ of 1-forms in R such that $\mathfrak{m}^2 = (\theta_1, \ldots, \theta_r)\mathfrak{m}$; see

[Exercise 4.5.14 (c),BH1]. Thus, the square of the maximal ideal of $A = R/(\theta_1, \ldots, \theta_r)$ is zero, which implies that

$$A \cong k[x_1, \ldots, x_m]/(x_i x_j)_{1 \leq i \leq j \leq m}.$$

Therefore, by the result of Fröberg (Corollary 4.1) the ring A is Koszul too. However, it is known that the quotient of a homogeneous k-algebra by a regular sequence of 1-forms is Koszul if and only if the algebra itself is Koszul. This can be easily seen from the fact that if Q is an $\mathbf{N}$-graded k-algebra and θ is a non-zero divisor of degree 1 then

$$\mathrm{Poin}_Q^k(t,z) := \sum_{i,j \geq 0} \dim_k \mathrm{Tor}_i^Q(k,k)_j t^i z^j = \mathrm{Poin}_{Q/(\theta)}^k(t,z) \cdot (1+tz).$$

Hence Q is Koszul if and only if $Q/(\theta)$ is. Applying this fact repeatedly to R and the regular sequence $\theta_1, \ldots, \theta_r$, we conclude that R is Koszul. ■

Since much of this work was originally motivated by the study of 2-dimensional normal affine semigroup rings, it is worth mentioning the following fact:

Proposition 5.3. *All 2-dimensional normal affine semigroup rings are Cohen-Macaulay of minimal multiplicity, and hence Koszul in the generalized sense.*

Proof: Assume $\Lambda \subseteq \mathbf{N}^2$ is normal. We know that $R = k[\Lambda]$ is Cohen-Macaulay because it is normal. By Gordan's Lemma (see Proposition 6.1.2 [BH1]) we can assume that Λ is the set of integral points in the intersection of two rational halfspaces. Recall, that in the given situation the multiplicity $e(R)$ is counted by the normalized volume (=twice the usual volume) of the set-theoretic difference of the convex hull of Λ and the convex hull of $\Lambda \setminus \{\mathbf{0}\}$ (see Chapter 5.3 of [GKZ]). This set is triangulated by triangles whose only integral points are the vertices, which consist of two neighboring minimal generators of Λ and the origin (see Figure 3).

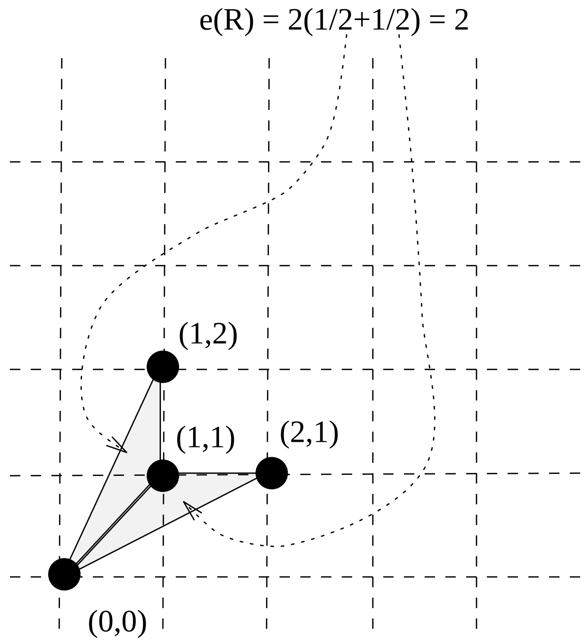

Figure 3: Computing the multiplicity of $k[x^2y, xy, y^2x]$

Each of these triangles has volume $1/2$. The number of triangles equals the number of minimal generators of Λ minus 1. But this implies that $k[\Lambda]$ is of minimal multiplicity. ■

We now proceed to investigate the consequences of Koszulness for $(R, \mathfrak{m})$. Our main result (Theorem 5.4) tells us that R and $\mathrm{gr}_{\mathfrak{m}}(R)$ share the same ($\mathbf{N}^d$-graded) Poincaré series, when $(R, \mathfrak{m})$ is Koszul. In particular, this means that $\mathrm{gr}_{\mathfrak{m}}(R)$ is Koszul in the usual sense, and hence both its own Poincaré series and that of R are rational. Before we can discuss this, however, we must first make clear what is meant by the various gradings, Poincaré series, and Hilbert series for $\mathrm{gr}_{\mathfrak{m}}(R)$ when R is $\mathbf{N}^d$-graded.

Given $(R, \mathfrak{m})$ with a $\mathbf{N}^d$-grading on R, we know that $\mathrm{gr}_{\mathfrak{m}}(R)$ carries a natural $\mathbf{N}$-grading in which the degree of a homogeneous element f is the number i such that $f \in \mathfrak{m}^i \setminus \mathfrak{m}^{i+1}$. We also define a $\mathbf{N}^d$-grading on $\mathrm{gr}_{\mathfrak{m}}(R)$ as follows. An element $g + \mathfrak{m}^{i+1}$, $g \in \mathfrak{m}^i \setminus \mathfrak{m}^{i+1}$ is called $\mathbf{N}^d$-homogeneous of degree $a = (a_1, \ldots, a_d)$, if there exists an $\mathbf{N}^d$ homogeneous element $h \in R$ of degree a such that $g + \mathfrak{m}^i = h + \mathfrak{m}^i$. One easily checks that this is well-defined and induces an $\mathbf{N}^d$-grading on $\mathrm{gr}_{\mathfrak{m}}(R)$ that is compatible with the multiplication and that $\mathrm{gr}_{\mathfrak{m}}(R)$ is the direct sum of its multigraded components. Putting together the two gradings ($\mathbf{N}-$ and $\mathbf{N}^d-$) on $\mathrm{gr}_{\mathfrak{m}}(R)$ gives a $\mathbf{N}^{d+1}$-grading, and we can consider the finely graded Hilbert series $\mathrm{Hilb}(\mathrm{gr}_{\mathfrak{m}}(R), z_0, z_1, \ldots, z_d)$ in which the first variable z_0 keeps track of the $\mathbf{N}$-degree. The Poincaré series $\mathrm{Poin}_R^k(t, z_1, \ldots, z_d)$ and $\mathrm{Poin}^k_{\mathrm{gr}_{\mathfrak{m}}(R)}(t, z_0, z_1, \ldots, z_d)$ are then defined as usual (i.e. with t keeping track of the homological degree). In the local case, there is no ambiguity in defining $\mathrm{Poin}_R^k(t)$, and we define $\mathrm{Hilb}(\mathrm{gr}_{\mathfrak{m}}(R); z), \mathrm{Poin}_{\mathrm{gr}_{\mathfrak{m}}(R)}(t, z)$ similarly to the above.

We now arrive at the main result of this section:

Theorem 5.4. *If $(R, \mathfrak{m})$ is Koszul and $\mathbf{N}^d$-graded we have for all $\lambda \in \mathbf{N}^d$*

$$\operatorname{Tor}_i^R(k,k)_\lambda \cong \operatorname{Tor}_i^{\operatorname{gr}_{\mathfrak{m}}(R)}(k,k)_\lambda$$

and consequently

$$\operatorname{Poin}_R^k(t, z_1, \dots, z_d) = \left[\operatorname{Poin}^k_{\operatorname{gr}_{\mathfrak{m}}(R)}(t, z_0, z_1, \dots, z_d)\right]_{z_0 \mapsto 1}.$$

If $(R, \mathfrak{m})$ is Koszul and local we have

$$\operatorname{Tor}_i^R(k,k) \cong \operatorname{Tor}_i^{\operatorname{gr}_{\mathfrak{m}}(R)}(k,k)$$

and consequently

$$\operatorname{Poin}_R^k(t) = \left[\operatorname{Poin}^k_{\operatorname{gr}_{\mathfrak{m}}(R)}(t, z)\right]_{z \mapsto 1}.$$

The isomorphism of the Tor-groups asserted in the theorem will be proved in two lemmas (Lemma 5.7 and Lemma 5.8) at the end of this section. But first let us deduce some consequences of Theorem 5.4. The first consequence requires a simple lemma:

Lemma 5.5. *Let Q be a $\mathbf{N}^{d+1}$-graded k-algebra which is Koszul (in the standard sense) with respect to the $\mathbf{N}$-grading induced by the first component of its $\mathbf{N}^{d+1}$-grading. Set*

$$\operatorname{Hilb}(Q; z_0, z_1, \dots, z_d) := \sum_{(a_0, a_1, \dots, a_d) \in \mathbf{N}^{d+1}} \dim_k R_{(a_0, a_1, \dots, a_d)} z_0^{a_0} z_1^{a_1} \cdots z_d^{a_d}$$

and

$$\operatorname{Poin}_Q^k(t, z_1, \dots, z_d) = \sum_{i \geq 0} t^i \sum_{(a_0, a_1, \dots, a_d) \in \mathbf{N}^{d+1}} \dim_k \operatorname{Tor}_i^R(k,k)_{(a_0, a_1, \dots, a_d)} z_1^{a_1} \cdots z_d^{a_d}$$

Then

$$\operatorname{Poin}_Q^k(t, z_1, \dots, z_d) = \left[\frac{1}{\operatorname{Hilb}(Q; z_0, z_1, \dots, z_d)}\right]_{z_0 \mapsto -t}$$

Proof: It is easy to see that one can choose a minimal free resolution of Q in which the basis elements and maps are all $\mathbf{N}^{d+1}$-graded. Interpreting the dimensions of the multigraded components of $\operatorname{Tor}_\cdot^Q(k,k)$ as the multidegrees of the basis elements in the free modules of the resolution, exactness of the resolution gives

$$1 = \sum_{i \geq 0} (-1)^i \sum_{(a_0, \dots, a_d) \in \mathbf{N}^{d+1}} \dim_k \operatorname{Tor}_i^Q(k,k)_{(a_0, \dots, a_d)} z_0^{a_0} z_1^{a_1} \cdots z_d^{a_d} \cdot \operatorname{Hilb}(Q; z_0, z_1, \dots, z_d).$$

Since Q is Koszul with respect to the first grading variable we conclude that only the terms with $i = a_0$ in the above sum are non-zero. Hence

$$1 = \sum_{i \geq 0} (-z_0)^i \sum_{(a_0,\ldots,a_d) \in \mathbf{N}^{d+1}} \dim_k \mathrm{Tor}_i^Q(k,k)_{(a_0,\ldots,a_d)} z_1^{a_1} \cdots z_d^{a_d} \cdot \mathrm{Hilb}(Q; z_0, z_1, \ldots, z_d).$$

The last equation is equivalent to the assertion of the lemma. ■

Corollary 5.6. *Let $(R, \mathfrak{m})$ be Koszul in the generalized sense. Then if R is $\mathbf{N}^d$-graded we have*

$$\mathrm{Poin}_R^k(t, z_1, \ldots, z_d) = \left[\frac{1}{\mathrm{Hilb}(\mathrm{gr}_{\mathfrak{m}}(R); z_0, z_1, \ldots, z_d)} \right]_{z_0 \mapsto -t},$$

and if R is local we have

$$\mathrm{Poin}_R^k(t) = \left[\frac{1}{\mathrm{Hilb}(\mathrm{gr}_{\mathfrak{m}}(R); z_0, z_1, \ldots, z_d)} \right]_{z_0 \mapsto -t}$$

In particular, in either case the Poincaré series for R is rational.

Proof: Apply Theorem 5.4, and then Lemma 5.5 with $Q = \mathrm{gr}_{\mathfrak{m}}(R)$. ■

We are now in a position to deduce equation (4.3) algebraically.

Algebraic proof of Equation (4.3): We use the same notation as in Corollary 4.3. Recall that the toric ideal I_Λ is the kernel of the surjection $k[\mathbf{x}] \to R = k[\Lambda]$ defined by $x_i \mapsto \mathbf{z}^{\alpha_i}$. Similarly, define L_Λ to be the kernel of the surjection $k[\mathbf{x}] \to \mathrm{gr}_{\mathfrak{m}}(R)$ defined by $x_i \mapsto \overline{\mathbf{z}^{\alpha_i}}$. It then follows by definition of $\mathrm{gr}_{\mathfrak{m}}(R)$ that

$$L_\Lambda = in_{\mathfrak{m}}(I_\Lambda) := (in_{\mathfrak{m}}(f) : f \in I_\Lambda)$$

where here $in_{\mathfrak{m}}(f)$ is the sum of all monomials in f having smallest degree in the $\mathbf{N}$-grading induced by powers of $\mathfrak{m}$. Also, since $\prec$ reversely grades Λ,

$$in_\prec(L_\Lambda) = in_\prec(in_{\mathfrak{m}}(I_\Lambda)) = in_\prec(I_\Lambda) =: J. \tag{5.1}$$

As a consequence, since J is assumed to be a quadratic monomial ideal, we conclude that L_Λ has the same quadratic initial ideal J with respect to $\prec$ that I_Λ did. Hence $\mathrm{gr}_{\mathfrak{m}}(R)$ is Koszul and R is Koszul in the generalized sense. We then have the following chain of equalities

$$\begin{aligned}
\mathrm{Poin}_R^k(t,\mathbf{z}) &= \left[\frac{1}{\mathrm{Hilb}(\mathrm{gr}_{\mathfrak{m}}(R); z_0, z_1, \ldots, z_d)}\right]_{z_0 \mapsto -t} \\
&= \left[\frac{1}{\mathrm{Hilb}(k[\mathbf{x}]/L_\Lambda; \mathbf{x})}\right]_{x_i \mapsto -t\mathbf{z}^{\alpha_i}} \\
&= \left[\frac{1}{\mathrm{Hilb}(k[\mathbf{x}]/in_\prec(L_\Lambda); \mathbf{x})}\right]_{x_i \mapsto -t\mathbf{z}^{\alpha_i}} \\
&= \left[\frac{1}{\mathrm{Hilb}(k[\mathbf{x}]/J; \mathbf{x})}\right]_{x_i \mapsto -t\mathbf{z}^{\alpha_i}} .
\end{aligned}$$

The first equality is by Corollary 5.6. The second equality is by definition of L_Λ. The third equality is due to the Gröbner basis fact that for any ideal I and any term order $\prec$, the image in $k[\mathbf{x}]/I$ of the standard monomials (those monomials not in $in_\prec(I)$) form a k-basis of $k[\mathbf{x}]/I$. The fourth equality follows from (5.1). ∎

The next two lemmas prove the isomorphism of the Tor-groups claimed in Theorem 5.4, by describing a relation between the minimal free resolutions of k over R and over $\mathrm{gr}_{\mathfrak{m}}(R)$; this relation is described by a filtration.

Let $G = \bigoplus_{i=1}^{n} Re_i$ be a free R-module of rank n. Define a *standard filtration* with shifts $a_1, \ldots, a_n$ by

$$F_j G = \bigoplus_{i=1}^{n} \mathfrak{m}^{j-a_i} e_i \,.$$

Then $\mathrm{gr}_F(G) = \bigoplus_{i=1}^{n} \mathrm{gr}_{\mathfrak{m}}(R)(-a_i)$.

Let $(G., d.)$ be a resolution of k over R and let F be a standard filtration on $G.$. Assume d_j is compatible with the filtration F, that is, $d_j(F_i G_j) \subset F_i G_{j-1}$. Then there is an obvious induced graded differential

$$\mathrm{gr}_F(d_j)\Big|_{F_i G_j / F_{i+1} G_j} : \quad F_i G_j / F_{i+1} G_j \to F_i G_{j-1} / F_{i+1} G_{j-1} \,.$$

Clearly, $\mathrm{gr}_F(d_j) \circ \mathrm{gr}_F(d_{j+1}) = 0$. Thus $(\mathrm{gr}_F(G.), \mathrm{gr}_F(d.))$ is a graded complex.

Lemma 5.7. *Let $(R, \mathfrak{m})$ be either local or $\mathbf{N}^d$-graded. There is a free (but not necessary minimal) resolution $(G., d.)$ of k over R and a standard filtration F on $G.$, such that*

$(\mathrm{gr}_F(G.), \mathrm{gr}_F(d.))$ *is a minimal graded free resolution of* k *over* $\mathrm{gr}_{\mathfrak{m}}(R)$*. If* R *is* $\mathbf{N}^d$*-graded then* $(\mathrm{gr}_F(G.), \mathrm{gr}_F(d.))$ *is also.*

As a consequence of this lemma, we conclude that when $(R, \mathfrak{m})$ is local we have

$$\dim_k \mathrm{Tor}_i^R(k,k) \leq \dim_k \mathrm{Tor}_i^{\mathrm{gr}_{\mathfrak{m}}(R)}(k,k)$$

for all i, and when $(R, \mathfrak{m})$ is $\mathbf{N}^d$-graded we have

$$\dim_k \mathrm{Tor}_i^R(k,k)_\lambda \leq \dim \mathrm{Tor}_i^{\mathrm{gr}_m(R)}(k,k)_\lambda$$

for any i and any $\lambda \in \mathbf{N}^d$.

Proof of Lemma 5.7: Assume the case where $(R, \mathfrak{m})$ is local, as the proof for the $\mathbf{N}^d$-graded case is similar. We will construct G_j, d_j and F inductively. Let $\mathfrak{m} = (x_1, \ldots, x_n)$ with a minimal set of generators $x_1, \ldots, x_n$. Set $G_0 = R$, $G_1 = \bigoplus_{i=1}^n Re_i$ and $d_1(e_i) = x_i$. Assume we have constructed the complex $(G., d.)$ up to homological degree j together with a standard filtration. Let $v_1, \ldots, v_r$ be a minimal set of homogeneous generators of $\mathrm{Ker}(\mathrm{gr}_F(d_j))$. Then there are $f_1, \ldots, f_r \in G_j$ such that $f_i + F_{l_i+1}G_j = v_i$, where $f_i \in F_{l_i}G_j \setminus F_{l_i+1}G_j$. As $v_i \in \mathrm{Ker}(\mathrm{gr}_F(d_j))$ we have $d_j(f_i) \in F_{l_i+1}G_{j-1}$. By construction the map $d_j : G_j \to \mathrm{Ker}(d_{j-1})$ is an epimorphism of filtered modules. Therefore, there exist elements $h_1, \ldots, h_r \in G_j$ such that $h_i \in F_{l_i+1}G_j$ and $d_j(f_i) = d_j(h_i)$. Then the elements $g_1 = f_1 - h_1, \ldots, g_r = f_r - h_r$ form a (not necessarily minimal) set of generators of $\mathrm{Ker}(d_j)$. Set $a_i = \deg(v_i)$. Define $G_{j+1} = \bigoplus_{i=1}^r Re_i$ with filtration $F_sG_{j+1} = \bigoplus_{i=1}^r \mathfrak{m}^{s-a_i}e_i$ and

$$d_{j+1} : G_{j+1} \to G_j$$
$$d_{j+1}(e_i) = g_i.$$

Then $(\mathrm{gr}_{\mathfrak{m}}(G), \mathrm{gr}_{\mathfrak{m}}(d))$ is a minimal graded free resolution of k over $\mathrm{gr}_{\mathfrak{m}}(R)$. ■

The *natural filtration* on $G.$ with respect to $\mathfrak{m}$ is given by $F_iG_j = \mathfrak{m}^{i-j}G_j$, $j \leq i$ and $F_iG_j = G_j$ for $j > i$. In particular, $\mathrm{gr}_{\mathfrak{m}}(G_j)(-j) \cong \mathrm{gr}_F(G_j) = \bigoplus_{i\geq 0} F_iG_j/F_{i+1}G_j$. Of course, the natural filtration is a standard filtration. Note, that the natural filtration of $G.$ is a filtration of the complex $(G., d.)$ if and only if $d_i(G_i) \subseteq \mathfrak{m}G_{i-1}$.

Lemma 5.8. *Let* $(R, \mathfrak{m})$ *be either local or* $\mathbf{N}^d$*-graded. Let* $(G., d.)$ *be a minimal free resolution of* k *over* R*, and let* F *be the natural filtration on* $G.$*. Then* R *is Koszul if and only if* $(\mathrm{gr}_F(G.), \mathrm{gr}_F(d.))$ *is acyclic. If these equivalent conditions hold then* $(\mathrm{gr}_{\mathfrak{m}}(G.), \mathrm{gr}_{\mathfrak{m}}(d.))$ *is a minimal graded free resolution of* k *over* $\mathrm{gr}_{\mathfrak{m}}(R)$*. Furthermore, if* R *is a* $\mathbf{N}^d$*-graded* k*-algebra then* $(\mathrm{gr}_{\mathfrak{m}}(G.), \mathrm{gr}_{\mathfrak{m}}(d.))$ *is also multigraded.*

As a consequence of this lemma, in the case when $(R, \mathfrak{m})$ is Koszul, the inequalities which followed from Lemma 5.7 can be sharpened to equalities: if $(R, \mathfrak{m})$ is Koszul and local we have

$$\dim_k \mathrm{Tor}_i^R(k,k) = \dim_k \mathrm{Tor}_i^{\mathrm{gr}_{\mathfrak{m}}(R)}(k,k)$$

for all i, and if $(R, \mathfrak{m})$ is Koszul and $\mathbf{N}^d$-graded we have

$$\dim_k \operatorname{Tor}_i^R(k,k)_\lambda = \dim_k \operatorname{Tor}_i^{\operatorname{gr}_{\mathfrak{m}}(R)}(k,k)_\lambda$$

for all i and all $\lambda \in \mathbf{N}^d$. Thus Theorem 5.4 follows.

Proof of Lemma 5.8: First assume R is Koszul, i.e. that $\operatorname{gr}_{\mathfrak{m}}(R)$ is Koszul in the usual sense. According to Lemma 5.8 we may choose a free resolution $(G., d.)$ of k over R, and a standard filtration F of $G.$ such that $\operatorname{gr}_F(G.)$ is a minimal graded free $\operatorname{gr}_{\mathfrak{m}}(R)$-resolution of k. Since $\operatorname{gr}_{\mathfrak{m}}(R)$ is Koszul, by assumption, we have $\operatorname{gr}_F(G_i) \cong \operatorname{gr}_{\mathfrak{m}}(R)(-i)^{b_i}$ for all i where b_i is the ith Betti number of k as $\operatorname{gr}_{\mathfrak{m}}(R)$-module. It follows from Lemma 5.7 that our standard filtration F on $G.$ is actually the natural filtration. This, in turn, implies that $G.$ has to be a minimal free resolution, since otherwise the natural filtration is not a filtration of the complex (i.e., it is not compatible with the differentials). Thus, the minimal free R-resolution equipped with the natural filtration F has the property that $\operatorname{gr}_F(G.)$ is a minimal free resolution of k as $\operatorname{gr}_{\mathfrak{m}}(R)$-module. This is what we wanted to show.

Next we assume $(\operatorname{gr}_F(G.), \operatorname{gr}_F(d.))$ is acyclic. By construction $\operatorname{gr}_F(G.)$ is a linear complex of free $\operatorname{gr}_{\mathfrak{m}}(R)$-modules with $H_0(\operatorname{gr}_F(G.)) = k$, and hence it is a linear resolution of k as a $\operatorname{gr}_{\mathfrak{m}}(R)$-module. Therefore $\operatorname{gr}_{\mathfrak{m}}(R)$ is Koszul. ∎

6. Appendix: combinatorial topology tools and lemmas

We collect here some of the tools from combinatorial topology that are used in Sections 3 and 4.

If P is a poset then a *diagram over* P is a functor $\mathcal{D} : P \to$ CW–TOP from the poset P (thought of as a category) into the category of CW-complexes. We mention that the concept of "diagrams of spaces" is far more general and has been studied intensively in homotopy theory (see for example the book by Bousfield & Kan [BK]), but we confine ourselves here to the combinatorial setting relevant to the paper. This means that for each $p \in P$ there is a CW-complex D_p and for each order relation $p \leq q$ there is a map $d_{qp} : D_q \to D_p$. If $\mathcal{U} : X = \bigcup_{i \in I} X_i$ is covering of a CW-complex by closed subcomplexes, then this gives rise to a natural diagram $\mathcal{D}^{\mathcal{U}}$ on the poset $P^{\mathcal{U}}$ defined on the set $\{\bigcap_{i \in J} X_j \mid J \subseteq I\}$ of intersections ordered by reverse inclusion. Namely, each $p \in P^{\mathcal{U}}$ is mapped to the intersection corresponding to p and the maps are the inclusion maps. More generally, we call a diagram $\mathcal{D} : P \to$ CW–TOP a *covering diagram* for the space X if

- $X = \bigcup_{p \in P} D_p$,
- for each pair of incomparable elements $q, p \in P$ we have $D_p \cap D_q = \bigcup_{r < p,q} D_q$, and

- all maps are inclusion maps.

Recall, that the *topological join* $X * Y$ of two topological spaces X, Y is defined as the quotient space

$$X * Y := X \times Y \times [0,1]/((x,0) \sim x, (y,1) \sim y).$$

Note that the join $\mathbf{S}^0 * X$ of X with a 0-sphere is the same as the *suspension* $\mathrm{Susp(X)}$. The *wedge* $X \vee Y$ of two topological spaces X and Y is the space $(X \uplus Y)/(x \sim y)$ where two distinguished points $x \in X$ and $y \in Y$ are identified in the disjoint union of X and Y. Note, if X and Y are path-connected then the homotopy type of $X \vee Y$ does not depend on the choice of $x \in X$ and $y \in Y$.

We now formulate a version of the "Wedge Lemma" from [ZZ] (see also [WZZ]) that is adapted to our purposes.

Wedge Lemma 6.1. **[ZZ]** *Let $\mathcal{D} : P \to \mathrm{CW\text{–}TOP}$ be a covering diagram of the space X. Assume that for all $p \in P$ there is a point $c_p \in D_p$ such that all maps $d_{qp} : D_q \to D_p$ are homotopic to the constant map to $D_q \to \{c_p\}$. Then X is homotopy equivalent to the wedge*

$$\bigvee_{p \in P} \Delta(P_{<p}) * D_p$$

in which the wedge identifies the vertex p in $\Delta(P_{<p})$ with the vertex p in $\Delta(P_{<\hat{1}})$, where $\hat{1}$ is the top element of P corresponding to the intersection $\cap_{i \in I} X_i$.

For the proof of Lemma 3.3 we need the following lemma. In the formulation, $X \simeq Y$ means that the topological spaces X and Y are homotopy equivalent.

Lemma 6.2. *Let X, Y be CW-complexes. If X and Y are l-connected and $X \cap Y$ is a sub-complex of dimension at most l, then $X \cup Y \simeq X \vee Y \vee \mathrm{Susp(X \cap Y)}$. In particular, if*

$$X \simeq \bigvee_{i=1}^{r} \mathbf{S}^{d_i^X}, \qquad Y \simeq \bigvee_{j=1}^{s} \mathbf{S}^{d_j^Y}, \qquad X \cap Y \simeq \bigvee_{k=1}^{t} \mathbf{S}^{d_k^{X \cap Y}}$$

and $dim(X \cap Y) < d_i^X, d_j^Y$ for all i, j then $X \cup Y$ is homotopy equivalent to the one-point wedge of spheres

$$X \cup Y \simeq \bigvee_{i=1}^{r} \mathbf{S}^{d_i^X} \vee \bigvee_{j=1}^{s} \mathbf{S}^{d_j^Y} \vee \bigvee_{k=1}^{t} \mathbf{S}^{d_k^{X \cap Y}+1} .$$

Proof: Use a very small diagram of spaces

$$X \hookleftarrow X \cap Y \hookrightarrow Y$$

and the Wedge Lemma 6.1. Observe that this lemma applies since both inclusions are null-homotopic, due to the fact that every map from a space of dimension at most l to an l-connected space is null-homotopic. By the Wedge Lemma, the homotopy type of $X \cup Y$ is given by $X \vee Y \vee \mathrm{Susp}(X \cap Y)$ since the order complex of the poset below $X \cap Y$ is just two points, i.e. the 0-sphere $\mathbf{S}^0$. ∎

The next lemma is needed in the proof of Theorem 4.2.

Lemma 6.3. *Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbf{R}^n$ and $\mathbf{S}^{n-1}$ the unit sphere. Then the intersection $\mathbf{S}^{n-1} \cap \mathcal{A}$ has the homotopy of a wedge of $(n-2)$-spheres, with the number of spheres being one fewer than the number of chambers cut out by $\mathcal{A}$.*

Proof: It is easily seen that intersection $\mathbf{S}^{n-1} \cap \mathcal{A}$ carries the structure of an $(n-2)$-dimensional regular CW-complex, and therefore its homology vanishes in dimensions greater than $n-2$. The quotient space $\mathbf{S}^{n-1}/(\mathbf{S}^{n-1} \cap \mathcal{A})$ is a one-point wedge of as many $(n-1)$-spheres as there are chambers cut out by $\mathcal{A}$ on $\mathbf{S}^{n-1}$. Thus $\widetilde{H}_i(\mathbf{S}^{n-1}/(\mathbf{S}^{n-1} \cap \mathcal{A}))$ is free and concentrated in dimension $n-1$. By the long exact sequence of the pair $(\mathbf{S}^{n-1}, \mathbf{S}^{n-1} \cap \mathcal{A})$, the homology of $\mathbf{S}^{n-1} \cap \mathcal{A}$ is free, concentrated in dimension $n-2$, and has rank one smaller than $H_{n-1}(\mathbf{S}^{n-1}/(\mathbf{S}^{n-1} \cap \mathcal{A}))$. Thus by the Hurewicz and Whitehead Theorems, the intersection $\mathbf{S}^{n-1} \cap \mathcal{A}$ has the homotopy of a wedge of $(n-2)$-spheres, with the number of spheres one smaller than the number of chambers cut out by $\mathcal{A}$. ∎

As an alternative to the previous proof, one could consult the much more general formula by Ziegler & Živaljević in [ZZ] on the homotopy type of $\mathbf{S}^{n-1} \cap \mathcal{A}$ for an arbitrary real subspace arrangement $\mathcal{A}$.

Our last topological lemma is needed in the proof of Lemma 3.5.

Lemma 6.4. *Let P be a d-dimensional polytope and F a face on its boundary. Then the union of all maximal faces of P not containing F is a shellable ball $\mathbf{B}^{d-1}$.*

Proof: Choose a *shelling* of P which shells the maximal faces containing F last, e.g. by picking a linear functional on the vertices of the polar dual P^* which orders the vertices in the dual face F^* last (see [DK] for a definition of shelling and why this choice is possible). Then in this shelling, the union of all facets not containing F gets shelled along the way, and hence is a ball $\mathbf{B}^{d-1}$. ∎

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