Hankel Matrices in Coding Theory and Combinatorics

I. Introduction

A Hankel matrix (or persymmetric matrix) is a matrix (a_{ij}) in which for every r the entries on the diagonal $i+j=r$ are the same, i.e., $a_{i,r-i}=c_r$ for some c_r.

For a sequence $c_0, c_1, c_2, \ldots$ of real numbers we consider the collection of Hankel matrices $A_n^{(k)}$, $k=0,1,\ldots$, $n=1,2,\ldots$, where

$$A_n^{(k)} = \begin{pmatrix} c_k & c_{k+1} & c_{k+2} & \cdots & c_{k+n-1} \\ c_{k+1} & c_{k+2} & c_{k+3} & \cdots & c_{k+n} \\ c_{k+2} & c_{k+3} & c_{k+4} & \cdots & c_{k+n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{k+n-1} & c_{k+n} & c_{k+n+1} & \cdots & c_{k+2n-2} \end{pmatrix}. \qquad (1)$$

So the parameter n denotes the size of the matrix and the $2n-1$ successive elements $c_k, c_{k+1}, \ldots, c_{k+2n-2}$ occur in the diagonals of the Hankel matrix.

We shall further denote the determinant of a Hankel matrix by

$$d_n^{(k)} = \det(A_n^{(k)}).$$

Hankel matrices occur in the Berlekamp - Massey algorithm for the decoding of BCH - codes and they found recent applications in Combinatorics motivated by the proof of the refined alternating sign matrix conjecture on the one hand and by the derivation of combinatorial identities for their determinants on the other hand.

One such identity concerns the Catalan numbers $\frac{1}{2m+1}\binom{2m+1}{m}$, namely it is well known that the Catalan numbers are the unique sequence such that $\det(A_n^{(0)}) = \det(A_n^{(1)}) = 1$, cf. e. g. [36], p. 232. Furthermore, Mays and Wojciechowski [23] gave a combinatorial interpretation of this determinant.

Theorem 1 ([23]): If the c_m's are Catalan numbers, $c_m = \frac{1}{2m+1}\binom{2m+1}{m}$, then $\det(A_n^{(k)})$ is the number of n–tuples $(\gamma_0, \ldots, \gamma_{n-1})$ of vertex – disjoint paths in the integer lattice $\mathbb{Z}\times\mathbb{Z}$ (with directed vertices from (i,j) to either $(i,j+1)$ or to $(i+1,j)$) never crossing the diagonal $x=y$, where the path γ_r is from $(-r,-r)$ to $(k+r,k+r)$. Especially, $\det(A_n^{(2)}) = n+1$ and $\det(A_n^{(3)}) = \frac{(n+1)(n+2)(2n+3)}{6}$.

We shall derive a formula for $\det(A_n^{(k)})$ and all $n\geq 1$, $k\geq 0$ in case that the entries c_m are Catalan numbers. Further, a similar formula holds for Hankel determinants formed of binomial coefficients $\binom{2m+1}{m}$.

Theorem 2:

a) For the sequence $c_m = \frac{1}{2m+1}\binom{2m+1}{m}$, $m=0,1,\ldots$ of Catalan numbers it is

$$d_n^{(0)} = d_n^{(1)} = 1, \qquad d_n^{(k)} = \prod_{1\leq i\leq j\leq k-1} \frac{i+j+2n}{i+j} \qquad \text{for } k\geq 2, n\geq 1. \qquad (2)$$

b) For the binomial coefficients $c_m = \binom{2m+1}{m}$, $m=0,1,\ldots$

$$d_n^{(0)} = 1, \qquad d_n^{(k)} = \prod_{1\leq i\leq j\leq k} \frac{i+j-1+2n}{i+j-1} \qquad \text{for } k,n\geq 1. \qquad (3)$$

The proof is based on the following identity for Hankel determinants.

$$d_n^{(k+1)} \cdot d_n^{(k-1)} - d_{n-1}^{(k+1)} \cdot d_{n+1}^{(k-1)} - [d_n^{(k)}]^2 = 0. \tag{4}$$

This identity is an immediate consequence of Dodgson's algorithm for the evaluation of determinants (see Section 4) and can already be found in the book by Polya and Szegö [28], Ex. 19, p. 102.

We are going to derive Theorem 2 in the next Section, where we shall also apply recursion (4) to further Hankel determinants, for instance to those studied by Aigner in a series of papers [1], [2], [4] for the cases that the entries $c_0, c_1, \ldots$ are Motzkin numbers, Bell numbers and Catalan – like numbers.

Let us recall some properties of Hankel matrices. Of special importance is the equation

$$\begin{pmatrix} c_0 & c_1 & c_2 & \ldots & c_{n-1} \\ c_1 & c_2 & c_3 & \ldots & c_n \\ c_2 & c_3 & c_4 & \ldots & c_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{n-1} & c_n & c_{n+1} & \ldots & c_{2n-2} \end{pmatrix} \cdot \begin{pmatrix} a_{n,0} \\ a_{n,1} \\ a_{n,2} \\ \vdots \\ a_{n,n-1} \end{pmatrix} = \begin{pmatrix} -c_n \\ -c_{n+1} \\ -c_{n+2} \\ \vdots \\ -c_{2n-1} \end{pmatrix}. \tag{5}$$

Setting $A_n = A_n^{(0)}$, by Cramer's rule it is immediate that for $j = 0, \ldots, n-1$

$$a_{n,j} = \frac{\det(A_{n,j})}{\det(A_n)}. \tag{6}$$

where $A_{n,j}$ is the matrix obtained from A_n by replacing the j–th column with the vector $\begin{pmatrix} -c_n \\ -c_{n+1} \\ \vdots \\ -c_{2n-1} \end{pmatrix}$.

Further it is known (cf. [9], p. 246) that the polynomials

$$t_n(x) := x^n + a_{n,n-1}x^{n-1} + a_{n,n-2}x^{n-2} + \ldots a_{n,1}x + a_{n,0}. \tag{7}$$

form a sequence of monic orthogonal polynomials with respect to the linear operator T mapping $T(x^k) = c_k$, i. e.

$$T(t_n(x) \cdot t_m(x)) = 0 \text{ for } n \neq m. \tag{8}$$

Moreover it is also clear from (5) see also [9], p. 246, that (8) is equivalent to

$$T(x^l \cdot t_n(x)) = 0 \text{ for } l = 0, \ldots n-1. \tag{9}$$

This orthogonality can be exploited to find the eigenvalues of A_n. Firstly, there exists a decomposition of the matrix $A_n = V_n D_n V_n^t$ as a product of a Vandermonde matrix V_n, its transpose V_n^t and a diagonal matrix D_n. Here the parameters in the Vandermonde matrix are essentially the roots of the polynomial $t_n(x)$. This decomposition was known already to Prony [29] in 1795, cf. also [8].

A second way to diagnolize the matrix A_n (with D_n the diagonal matrix with the eigenvalues of A_n on its main diagonal) is via the product

$$A_n = T_n \cdot D_n \cdot T_n^t. \tag{10}$$

where

$$T_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ a_{1,0} & 1 & 0 & \dots & 0 & 0 \\ a_{2,0} & a_{2,1} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n-1,0} & a_{n-1,1} & a_{n-2,2} & \dots & a_{n-1,n-2} & 1 \end{pmatrix}. \tag{11}$$

is the triangular matrix whose entries are the coefficients of the polynomials $t_m(x)$, $m = 1, \dots, n-1$.

In Section 3 we shall discuss the Berlekamp – Massey algorithm for the decoding of BCH–codes, where Hankel matrices of syndromes resulting after the transmission of a code word over a noisy channel have to be studied, and a further fast algorithm for the triangularization of a Hankel matrix. Applications of these algorithms to combinatorial identities, especially to the three – term recurrence of the orthogonal polynomials $t_n(x)$ in (7), will be discussed. Finally, in Section 4, the theory of alternating sign matrices is briefly sketched, where Hankel matrices occur in several places.

II. Combinatorial Identities for The Determinants of Hankel Matrices

We shall first derive Theorem 2.

Proof of Theorem 2: We shall derive both results simultaneously. The proof will proceed by induction on $n+k$.

It is well known, e. g. [36], that for the Hankel matrices $A_n^{(k)}$ with Catalan numbers as entries it is $d_n^{(0)} = d_n^{(1)} = 1$. For the induction beginning it must also be verified that $d_n^{(2)} = n+1$ and that $d_n^{(3)} = \frac{(n+1)(n+2)(2n+3)}{6}$ is the sum of squares, cf. [23], which can also be easily seen by application of recursion (4).

Furthermore, for the matrix $A_n^{(k)}$ whose entries are the binomial coefficients $\binom{2k+1}{k}$, $\binom{2k+3}{k+1}$, ... it was shown by Aigner [2] that $d_n^{(0)} = 1$ and $d_n^{(1)} = 2n+1$. Application of (4) shows that $d_n^{(2)} = \frac{(n+1)(2n+1)(2n+3)}{3}$, i. e., the sum of squares of the odd positive integers.

Also, it is easily seen by comparing successive quotients $\frac{c_{k+1}}{c_k}$ that for $n = 1$ the product in (2) yields the Catalan numbers and the product in (3) yields the binomial coefficients $\binom{2k+1}{k+1}$, cf. also [12].

Now it remains to be verified that (2) and (3) hold for all n and k, which will be done by checking recursion (4). The sum in (4) is of the form (with either $d = 0$ for (2) or $d = 1$ for (3) and shifting k to $k+1$ in (2))

$$\prod_{i,j=1}^{k} \frac{i+j-d+2n}{i+j-d} \cdot \prod_{i,j=1}^{k-2} \frac{i+j-d+2n}{i+j-d} - \prod_{i,j=1}^{k} \frac{i+j-d+2(n+1)}{i+j-d} \cdot \prod_{i,j=1}^{k-2} \frac{i+j-d+2(n-1)}{i+j-d} -$$
$$- \left[\prod_{i,j=1}^{k-1} \frac{i+j-d+2n}{i+j-d} \right]^2$$

$$= \left[\prod_{i,j=1}^{k-1} \frac{i+j-d+2n}{i+j-d} \right]^2 \cdot$$

$$\cdot \left(\frac{\prod_{j=1}^{k}(k+j-d+2n)}{\prod_{j=1}^{k}(k+j-d)} \cdot \frac{\prod_{j=1}^{k-1}(k-1+j-d)}{\prod_{j=1}^{k-1}(k-1+j-d+2n)} - \frac{\prod_{j=0}^{k-1}(j-d+2n)}{\prod_{j=1}^{k}(k+j-d)} \cdot \frac{\prod_{j=1}^{k-1}(k-1+j-d)}{\prod_{j=1}^{k-1}(1+j-d+2n)} - 1 \right)$$

$$= \left[\prod_{i,j=1}^{k-1} \frac{i+j-d+2n}{i+j-d} \right]^2 \cdot$$

$$\cdot \left(\frac{(2n+2k-d)(2n+2k-1-d)(k-d)}{(2n+k-d)(2k-d)(2k-1-d)} - \frac{(2n-d)(2n+1-d)(k-d)}{(2n+k-d)(2k-d)(2k-1-d)} - 1 \right).$$

This expression is 0 exactly if

$$(2n+2k-d)(2n+2k-1-d)(k-d)-(2n-d)(2n+1-d)(k-d)-(2n+k-d)(2k-d)(2k-1-d) = 0. \tag{12}$$

In order to show (2), now observe that here $d = 0$ and then it is easily verified that

$$(n+k)(2n+2k-1) - n(2n+1) - (2n+k)(2k-1) = 0.$$

In order to show (3), we have to set $d = 1$ and again the analysis simplifies to verifying

$$(2n+2k-1)(n+k-1) - (2n-1)n - (2n+k-1)(2k-1) = 0.$$

□

Remark:

1) The identity $\det(A_n^{(0)}) = 1$, when the c_m's are Catalan numbers or binomial coefficients $\binom{2m+1}{m}$ can already be found in [25], pp. 435 – 436.

2) Observe that in (2) and (3) it is $\frac{d_n^{(4)}}{d_{n-1}^{(4)}} = \frac{d_{n+1}^{(3)}}{d_{n-1}^{(3)}}$. So, for instance in (2) the next sequence is given by

$$d_n^{(4)} = \frac{d_{n+1}^{(3)} \cdot d_n^{(3)}}{5} = \frac{n(n+1)^2(n+2)(2n+1)(2n+3)}{180}.$$

3) Formula (2) was also studied by Desainte–Catherine and Viennot [12] in the analysis of disjoint paths in a bounded area of the integer lattice and perfect matchings in a certain graph as a special Pfaffian. Another interpretation of the determinant $d_n^{(k)}$ in (2) giving the number of k–tuples of disjoint positive lattice paths is found in [23]. However, an explicit formula is given only for $k = 0, 1, 2, 3$. For the case $k = 2$ formula (2) was also derived as the number of pairs of noncrossing positive lattice paths [18]. Indeed, the result in [23] follows from a more general determinant identity for disjoint paths in graphs (see [19]). The use of determinants in the enumeration of disjoint paths is, of course, well known, cf. [15] or [3]. One might further investigate conditions under which the arising determinant is a Hankel determinant. One such situation will be discussed in Section 4.

Aigner evaluated the determinants in case that the entries in the Hankel matrices are the Motzkin numbers ([1]) and the Bell numbers ([4]).

Theorem 3 ([1]): Choosing the sequence $(c_m)_{m=0,1,2,\ldots}$ as the Motzkin numbers it is

$$\det(A_n^{(0)}) = 1, \quad \det(A_n^{(1)}) = \begin{cases} 1 & \text{if } n \equiv 0,1 \bmod 6 \\ 0 & \text{if } n \equiv 2,5 \bmod 6 \\ -1 & \text{if } n \equiv 3,4 \bmod 6 \end{cases}$$

Theorem 4 ([4]): The Bell numbers are the unique sequence $(c_m)_{m=0,1,2,\ldots}$ such that

$$\det(A_n^{(0)}) = \det(A_n^{(1)}) = \prod_{k=0}^{n} k!, \qquad \det(A_n^{(2)}) = r_{n+1} \prod_{k=0}^{n} k!,$$

where $r_n = 1 + \sum_{l=1}^{n} n(n-1)\cdots(n-l+1)$ is the total number of permutations of n things.

Aigner [4] used an approach via generating functions in order to derive $d_n^{(2)} = \det(A_n^{(2)})$ in Theorem 4. Let us demonstrate the use of recursion (4) by setting $d_n^{(2)} = r_{n+1} \cdot \prod_{k=0}^{n} k!$. Then using (4) one obtains the recurrence

$$r_{n+1} = (n+1) \cdot r_n + 1, r_2 = 5,$$

which just characterizes the total number of permutations of n things, cf. [30], p. 16, and hence can derive $\det(A_n^{(2)})$ from $\det(A_n^{(0)})$ and $\det(A_n^{(1)})$ also this way.

Further, in [2] Aigner introduced Catalan – like numbers and considered Hankel determinants for these numbers. For positive reals $a, s_1, s_2, s_3, \ldots$ Catalan – like numbers $C_m^{(a,\vec{s})}$, $\vec{s} = (s_1, s_2, s_3, \ldots)$ can be defined as entries $b(m,0)$ in a two – dimensional array $b(m,k)$, $m = 0,1,2,\ldots$, $k = 0,1,\ldots,m$ with initial conditions $b(m,m) = 1$ for all $m = 0,1,2,\ldots$, $b(0,k) = 0$ for $k > 0$, and recursion

$$\begin{aligned} b(m,0) &= a \cdot b(m-1,0) + b(m-1,1), \\ b(m,k) &= b(m-1,k-1) + s_k \cdot b(m-1,k) + b(m-1,k+1) \text{ for } k = 1,\ldots,m. \end{aligned} \tag{13}$$

The matrices $B_n = (b_{m,k})_{m,k=0,\ldots,n-1}$, obtained from this array, have the property that $B_n \cdot B_n^t$ is a Hankel matrix, which has, of course, determinant 1. Aigner [2] intensively studied Catalan – like numbers with $s_k = s$ for some fixed s denoted here by $C_m^{(a,s)}$. In the example below the binomial coefficients $\binom{2m+1}{m}$ arise as $C^{(3,2)}$.

$$\begin{array}{ccccc} 1 \\ 3 & 1 \\ 10 & 5 & 1 \\ 35 & 21 & 7 & 1 \\ 126 & 84 & 36 & 9 & 1 \end{array}$$

So, by the previous considerations, choosing $c_m = C_m^{(a,\vec{s})}$ we have that the determinant $d_n^{(0)} = 1$ for all n. In [2] it is also computed the determinant $d_n^{(1)}$ via the recurrence

$$d_n^{(1)} = s_{n-1} \cdot d_{n-1}^{(1)} - d_{n-2}^{(1)}. \tag{14}$$

with initial values $d_0^{(1)} = 1$, $d_1^{(1)} = a$.

One might now introduce a new leading element c_{-1} to the sequence $c_0, c_1, c_2, \ldots$ and define the $n \times n$ Hankel matrix $A_n^{(-1)}$ and its determinant $d_n^{(-1)}$ for this new sequence. For instance:

Corollary 1: Let $(c_m = C_m^{(s,s)})_{m=0,1,\ldots}$ be the sequence of Catalan–like numbers with parameters (s,s), $s > 1$ and let $c_{-1} = 1$. Let $A_n^{(k)}$ be the Hankel matrix of size $n \times n$ as under (1) and let $d_n^{(k)}$ denote its determinant. Then

$$d_n^{(-1)} = (s-1)(n-1)+1, \qquad d_n^{(0)} = 1, \qquad d_n^{(1)} = sn+1, \qquad d_n^{(2)} = \sum_{j=1}^{n+1} (sj+1)^2.$$

Proof: $d_n^{(0)}$ and $d_n^{(1)}$ are known from Propositions 6 and 7 in [2]. So the sequences $d_n^{(k)}$ are known for two successive k's, such that the formulae for $d_n^{(-1)}$ and $d_n^{(2)}$ are easily found using recursion (4). □

III. The Berlekamp – Massey Algorithm

Peterson [26] and Gorenstein and Zierler [17] presented an efficient algorithm for the decoding of BCH codes. The most time–consuming task is the inversion of a Hankel matrix A_n ($= A_n^{(0)}$ as in (1)), in which the entries c_i now are special syndromes resulting after the transmission of a codeword over a noisy channel. Matrix inversion, which takes $O(n^3)$ steps was proposed to solve equation (5).

Berlekamp found a way to determine the $a_{n,j}$ in (5) in $O(n^2)$ steps by an approach regarding them as coefficients of a polynomial. Massey [24] gave a variation in terms of a linear feedback shift register. The algorithm is presented by Berlekamp in [5]. We follow here Blahut's book [6], p. 180.

The algorithm consist in constructing a sequence of shift registers $(L_i, u_i(x))$, $i = 1, \ldots,$ $2n-2$, where L_i denotes the length and

$$u_i(x) = b_{i,i}x^i + b_{i,i-1}x^{i-1} + \ldots + b_{i,1}x + 1.$$

the feedback–connection polynomial of the i–th shift register. For an introduction to shift registers see, e. g., [6], pp. 131, The Berlekamp – Massey algorithm will iteratively compute the polynomials $u_i(x)$ as follows using a second sequence of polynomials $v_i(x)$.

Berlekamp – Massey Algorithm (as in [6], p. 180): Let $u_0(x) = 1, v_0(x) = 1$ and $L_0 = 0$. Then for $i = 1, \ldots, 2n-2$ set

$$e_i = \sum_{j=0}^{i} b_{i-1,j} c_{i-1-j}, \tag{15}$$

$$L_i = \delta_i (i - L_{i-1}) + (1-\delta_i) L_{i-1}, \tag{16}$$

$$\begin{pmatrix} u_i(x) \\ v_i(x) \end{pmatrix} = \begin{pmatrix} 1 & -e_i x \\ \delta_i \cdot 1/e_i & (1-\delta_i)x \end{pmatrix} \cdot \begin{pmatrix} u_{i-1}(x) \\ v_{i-1}(x) \end{pmatrix}, \tag{17}$$

where

$$\delta_i = \begin{cases} 1 & \text{if } e_i \neq 0 \text{ and } 2L_{i-1} \leq i-1 \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

We shall assume from now on that all principal submatrices A_j, $j \leq n$ of the Hankel matrix A_n are nonsingular. For this case, Imamura and Yoshida [20] demonstrated that $L_i = L_{i-1} = \frac{i}{2}$ for even i and $L_i = i - L_{i-1} = \frac{i+1}{2}$ for odd i such that δ_i is 1 if i is odd and 0 if i is even. An interpretation of the Berlekamp – Massey algorithm via the theory of Hankel matrices is given in [21].

With the result of Imamura and Yoshida [20] the algorithm is simplified in (16) and we obtain the recursion

$$\begin{pmatrix} u_{2i}(x) \\ v_{2i}(x) \end{pmatrix} = \begin{pmatrix} 1 - \frac{e_{2i}}{e_{2i-1}}x & -e_{2i-1}x \\ \frac{1}{e_{2i-1}}x & 0 \end{pmatrix} \cdot \begin{pmatrix} u_{2i-2}(x) \\ v_{2i-2}(x) \end{pmatrix}. \tag{19}$$

Also, it is easily seen, that $v_{2i}(x) = \frac{1}{e_{2i-1}} x u_{2i-2}(x)$, such that from (19) we have the following three–term recurrence for $u_{2i}(x)$.

$$u_{2i}(x) = (1 - \frac{e_{2i}}{e_{2i-1}}x)u_{2i-2}(x) - \frac{e_{2i-1}}{e_{2i-3}}x^2 u_{2i-4}(x).$$

Since the Berlekamp - Massey algorithm determines the solution of equation (5) it must be

$$u_{2n}(\frac{1}{x}) = t_n(x).$$

as under (7). By the previous considerations, for $t_n(x)$, $n = 2i$, we have the recurrence

$$t_n(x) = (x - \frac{e_n}{e'_n})t_{n-1}(x) - \frac{e'_n}{e'_{n-1}}t_{n-2}(x), \tag{20}$$

where $e_n = e_{2i}$ and $e'_n = e_{2i-1}$ are obtained as in (15).

From the algorithm it is then clear with T being the linear operator mapping $T(x^k) = c_k$ as in (8) that

$$e_n = e_{2i} = T\left(x^i t_{i-1}(x) - \frac{e_{2i-1}}{e_{2i-3}}x^{i-1}t_{i-2}(x)\right),$$

$$e'_n = e_{2i-1} = T(x^{i-1} \cdot t_{i-1}(x)) = T(t_{i-1}(x) \cdot t_{i-1}(x)), \tag{21}$$

where the last equality follows from (9).

It is not yet obvious, but in the course of the Berlekamp – Massey algorithm we also computed the determinants of the matrices A_j, $j = 0, \ldots, n$. In order to see this, we shall first discuss further fast algorithms yielding the eigenvalues of A_n.

Phillips [27] in 1971 also gave a fast $O(n^2)$ algorithm to find a triangularization of a Hankel matrix A_n. He required that all principal minors A_j, $j \leq n$ are nonsingular. Rissanen [32] gave an algorithm only requiring that A_n is nonsingular. Kung [22] remarked that the Berlekamp – Massey algorithm is related to the Lanczos process in matrix theory. This was further explored by Boley, Lee, and Luk [7], who gave the following method to find a factorization $A_n \cdot U_n = L_n$, where $A_n = A_n^{(0)}$ as in (1) is an $n \times n$ Hankel matrix and

$$L_n = \begin{pmatrix} l_{11} & 0 & 0 & \dots & 0 \\ l_{21} & l_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \dots & l_{nn} \end{pmatrix}, \qquad U_n = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & u_{22} & u_{23} & \dots & u_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & u_{nn} \end{pmatrix}$$

are lower and upper triangular matrices.
The speed of the algorithm is due to the fact that columns No. $j+1$, $\vec{l}_{j+1}$ and $\vec{u}_{j+1}$, in L and U are obtained only using the entries in the previous two columns $\vec{l}_j$, $\vec{l}_{j-1}$ and $\vec{u}_j$, $\vec{u}_{j-1}$, respectively.

Namely, we start with $\vec{l}_1 = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{2n-2} \end{pmatrix}$ consisting of the entries of the Hankel matrix A_n and $\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ being the first unit vector of size $2n-1$.

Having obtained the first j colums in L_n and U_n, respectively, with $Z = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$ of size $(2n-1) \times (2n-1)$ and Z^t its transpose we calculate

$$\vec{l}_{j+1} = Z^t \cdot \vec{l}_j - \frac{l_{jj}}{l_{j-1,j-1}} \cdot \vec{l}_{j-1} - (\frac{l_{j+1,j}}{l_{jj}} - \frac{l_{j,j-1}}{l_{j-1,j-1}})\vec{l}_j \tag{22}$$

and apply to the colums in U the analogous recursion

$$\vec{u}_{j+1} = Z \cdot \vec{u}_j - \frac{l_{jj}}{l_{j-1,j-1}} \cdot \vec{u}_{j-1} - (\frac{l_{j+1,j}}{l_{jj}} - \frac{l_{j,j-1}}{l_{j-1,j-1}})\vec{u}_j. \tag{23}$$

The subvectors of the initial n elements of $\vec{l}_{j+1}$ and $\vec{u}_{j+1}$ then form the $(j+1)$–th column $(j = 2, \dots, n-1)$ of L_n and U_n, respectively. (Of course, in the first step, we only apply $\vec{l}_2 = Z^t \cdot \vec{l}_1 - \frac{l_{2,1}}{l_{11}}\vec{l}_1$ and $\vec{u}_2 = Z \cdot \vec{u}_1 - \frac{l_{2,1}}{l_{11}}\vec{u}_1$.)
Exactly the same recurrence was already known to Chebyshev in his algorithm in [11] in the theory of moments and orthogonal polynomials. His research was motivated by problems concerning continued fractions. The relation between continued fractions and the Berlekamp – Massey algorithm has also been studied in [37]. Applications of continued fractions in Combinatorics are found in Flajolet's paper [14].

Since from [7] it is clear, that the matrix U_n is the transpose of the triangular matrix T_n in (11) with the coefficients of the orthogonal polynomials $t_m(x)$, $m \leq n-1$ as entries (and $\det(T_n) = \det(U_n) = 1$), it is $\det(A_n) = \det(L_n)$. So, from (23) it is immediate the following three – term recursion for these polynomials

$$t_n(x) = \left(x - (\frac{l_{n+1,n}}{l_{nn}} - \frac{l_{n,n-1}}{l_{n-1,n-1}})\right) \cdot t_{n-1}(x) - \frac{l_{nn}}{l_{n-1,n-1}} \cdot t_{n-2}(x). \tag{24}$$

By induction it is also clear that the elements on the main diagonal of the lower triangular matrix are

$$l_{11} = c_0, \qquad l_{ii} = \frac{\det(A_i)}{\det(A_{i-1})} \text{ for } i = 2, \ldots, n. \tag{25}$$

Now observe that the two recurrence formulae (21) and (24) yield the same sequence of polynomials $t_n(x)$ (as in (7)), so the coefficients in these recurrences must be the same, thus we have

$$e'_n = l_{nn}, \qquad e_n = (\frac{l_{n+1,n}}{l_{nn}} - \frac{l_{n,n-1}}{l_{n-1,n-1}})/l_{nn}. \tag{26}$$

So we have found two interpretations of the coefficients in the three–term recurrence formula

$$t_n^{(k)}(x) = (x - \mu_n^{(k)})t_{n-1}^{(k)}(x) - \nu_n^{(k)} t_{n-2}^{(k)}(x) \tag{27}$$

for the sequence of orthogonal polynomials

$$t_n^{(k)}(x) = x^n + a_{n,n-1}^{(k)} x^{n-1} + \ldots + a_{n,1}^{(k)} x + a_{n,0}^{(k)},$$

$n = 0, 1, 2, \ldots$ defined as in (7) where the coefficients in $t_n^{(k)}(x)$ yield the solution of the system of linear equations.

$$\begin{pmatrix} c_k & c_{k+1} & c_{k+2} & \cdots & c_{k+n-1} \\ c_{k+1} & c_{k+2} & c_{k+3} & \cdots & c_{k+n} \\ c_{k+2} & c_{k+3} & c_{k+4} & \cdots & c_{k+n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{k+n-1} & c_{k+n} & c_{k+n+1} & \cdots & c_{k+2n-2} \end{pmatrix} \cdot \begin{pmatrix} a_{n,0}^{(k)} \\ a_{n,1}^{(k)} \\ a_{n,2}^{(k)} \\ \vdots \\ a_{n,n-1}^{(k)} \end{pmatrix} = \begin{pmatrix} -c_{k+n} \\ -c_{k+n+1} \\ -c_{k+n+2} \\ \vdots \\ -c_{k+2n-1} \end{pmatrix}.$$

A third formula can be obtained from identity (4).

Proposition 1: Let $d_n^{(k)}$, etc. be the Hankel determinants in (4). Then

$$\mu_n^{(k)} = \frac{d_{n-2}^{(k+1)} d_n^{(k)}}{d_{n-1}^{(k+1)} d_{n-1}^{(k)}} - \frac{d_n^{(k+1)} d_{n-1}^{(k)}}{d_{n-1}^{(k+1)} d_n^{(k)}} \qquad \nu_n^{(k)} = \frac{d_n^{(k)} d_{n-2}^{(k)}}{[d_{n-1}^{(k)}]^2}. \tag{28}$$

Proof: The identity for $\nu_n^{(k)}$ is immediate from (24) and (25). In order to determine the $\mu_n^{(k)}$ observe that for the constant term $a_{n,0}^{(k)}$ in $t_n^{(k)}(x)$ by the three–term recurrence must hold

$$a_{n,0}^{(k)} = \mu_n^{(k)} \cdot a_{n-1,0}^{(k)} + \nu_n^{(k)} \cdot a_{n-2,0}^{(k)}.$$

Now $\mu_n^{(k)}$ can be determined from $\nu_n^{(k)}$ and the fact that by Cramer's rule applied as in (6) it is $a_{n,0}^{(k)} = (-1)^n \frac{d_n^{(k+1)}}{d_n^{(k)}}$. □

Let us now explicitly determine the three – term recursion of some orthogonal polynomials related to Hankel determinants whose entries are Catalan - like numbers $C_m^{(a,\vec{s})}$.

Corollary 2:

a) Let $A_n^{(0)}$ be the Hankel matrix (1) with the Catalan – like numbers $c_m = C_m^{(a,\vec{s})}$, $\vec{s} = (s_1, s_2, \ldots)$ defined by (13) as entries. Then

$$t_n^{(0)}(x) = (x - d_n^{(1)}) \cdot t_{n-1}^{(0)}(x) - t_{n-2}^{(0)}(x), \qquad t_0^{(0)}(x) = 1, \qquad t_1^{(1)}(x) = x - a$$

where $d_n^{(1)} = s_{n-1} \cdot d_{n-1}^{(1)} - s_{n-2} \cdot d_{n-2}^{(1)}$.

b) For the Catalan numbers $c_m = \frac{1}{2m+1}\binom{2m+1}{m}$ additionally

$$t_n^{(0)}(x) = (x-2) \cdot t_{n-1}^{(0)}(x) - t_{n-2}^{(0)}(x), \qquad t_0^{(0)}(x) = 1, \qquad t_1^{(0)}(x) = x - 1,$$

$$t_n^{(1)}(x) = (x-2) \cdot t_{n-1}^{(1)}(x) - t_{n-2}^{(1)}(x), \qquad t_0^{(1)}(x) = 1, \qquad t_1^{(1)}(x) = x - 2,$$

$$t_n^{(2)}(x) = (x - \frac{(n+1)^2 + n^2}{n(n+1)}) \cdot t_{n-1}^{(2)}(x) - \frac{n^2-1}{n^2} t_{n-2}^{(2)}(x), \quad t_0^{(2)}(x) = 1, t_1^{(2)}(x) = x - \frac{5}{2}.$$

Proof: The recursion for the orthogonal polynomials $t_n^{(0)}(x)$ is immediate, since the determinant $d_n^{(0)} = 1$ for all n and hence also $\nu_n^{(0)}$ in (28) must be 1. The recursion for $\mu_n^{(0)}$ then must be the same as (14).

The Catalan numbers $c_m = \frac{1}{2m+1}\binom{2m+1}{m}$ themselves occur as Catalan – like numbers $c_m = C_m^{(1,2)}$ and $c_{m+1} = C_m^{(2,2)}$. Hence, for this choice the recursion for the polynomials $t_n^{(k)}$ are known for $k = 0, 1$. The next recurrence formula for $k = 2$ follows from (28). $\square$

It is well - known that the Chebyshev – polynomials of the second kind

$$u_n(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n-i}{i} (2x)^{n-2i}$$

with recursion

$$u_n(x) = 2x \cdot u_{n-1}(x) - u_{n-2}(x), \qquad u_0(x) = 1, \qquad u_1(x) = 2x$$

come in for Hankel matrices with Catalan numbers as entries as under b) in the previous corollary.

For instance, in this case

$$t_n^{(0)}(x^2) = \frac{1}{x} u_{2n}(\frac{x}{2}), \qquad t_n^{(1)}(x^2) = \frac{1}{x} u_{2n+1}(\frac{x}{2}).$$

It can further be shown that

$$n \cdot t_{n-1}^{(1)}(x) = \frac{d}{dx}(t_n^{(0)}(x) - t_{n-1}^{(0)}(x)), \quad n \cdot t_{n-1}^{(2)}(x) = \frac{d}{dx}(t_n^{(1)}(x) - t_{n-1}^{(1)}(x)).$$

Unfortunately, this generation rule does not continue for $t_n^{(i)}(x)$ with $i > 2$.

IV. Alternating Sign Matrices

An *alternating sign matrix* is a square matrix with entries from $\{0, 1, -1\}$ such that i) the entries in each row and column sum up to 1, ii) the nonzero entries in each row and column alternate in sign.

Robbins and Rumsey discovered the alternating sign matrices in the analysis of Dodgson's algorithm in order to evaluate the determinant of an $n \times n$ – matrix A. Reverend Charles Lutwidge Dodgson, who worked as a mathematician at the Christ College at the University of Oxford is much wider known as Lewis Carroll, the author of [10]. His algorithm, which is presented in [9], pp. 113 – 115, is based on the following identity for any matrix (for a combinatorial proof see [40]).

Theorem 5 [13]:

$$\det\left((a_{i,j})_{i,j=1,\dots,n}\right) \cdot \det\left((a_{i,j})_{i,j=2,\dots,n-1}\right) =$$

$$\det\left((a_{i,j})_{i,j=1,\dots,n-1}\right)\cdot\det\left((a_{i,j})_{i,j=2,\dots,n}\right)-\det\left((a_{i,j})_{i=1,\dots,n-1,j=2,\dots,n}\right)\cdot\det\left((a_{i,j})_{i=2,\dots,n,j=1,\dots,n-1}\right). \tag{29}$$

If $(a_{i,j})_{i,j=1,\dots,n}$ in (29) is a Hankel matrix, then all the other matrices in (29) are Hankel matrices, too. Hence recursion (4) from the introduction is an immediate consequence of Dodgson's result.

From (29) the following algorithm for determinant evaluation is immediate.

Dodgson's algorithm: The algorithm works on pairs of matrices $(A^{(n)}, B^{(n)})$, where $A^{(1)} = A$ is the matrix whose determinant should be found and $B^{(1)}$ is the $(n-1) \times (n-1)$ all–one matrix. Then $(A^{(r+1)}, B^{(r+1)})$ (with $A^{(r)} = (a_{ij}^{(r)})_{i,j=1,\dots,n-r+1}$, $B^{(r)} = (b_{ij}^{(r)})_{i,j=1,\dots,n-r}$ is obtained from $(A^{(r)}, B^{(r)})$ by

$$a_{ij}^{(r+1)} = \frac{1}{b_{ij}^{(r)}}(a_{ij}^{(r)}a_{i+1,j+1}^{(r)} - a_{i,j+1}^{(r)}a_{i+1,j}^{(r)}), \quad i,j = 1,\dots,n-r,$$

$$b_{ij}^{(r+1)} = a_{i+1,j+1}^{(r)}, \quad i,j = 1,\dots,n-r-1.$$

If some $b_{ij}^{(r)}$ in the course of the computation is 0, then the algorithm fails, otherwise the final 1×1 – matrix $A^{(n)}$ contains the determinant of A.

So, only 2×2 determinants have to be calculated in the course of Dodgson's algorithm (which also allows to compute a determinant by hand quite fast). Robbins asked what would happen, if in the algorithm we would replace the determinant evaluation $a_{ij}^{(r)}a_{i+1,j+1}^{(r)} - a_{i,j+1}^{(r)}a_{i+1,j}^{(r)}$ by the prescription $a_{ij}^{(r)}a_{i+1,j+1}^{(r)} + xa_{i,j+1}^{(r)}a_{i+1,j}^{(r)}$, where x is some variable.

It turned out that this yields a sum of monomials in the a_{ij} and their inverses, each monomial multiplied by a polynomial in x. The monomials are of the form $\prod_{i,j=1}^{n} a_{ij}^{b_{ij}}$ where the b_{ij}'s are the entries in an alternating sign matrix. The exact formula can be found in Theorem 3.13 in the book "Proofs and Confirmations: The Story of The Alternating Sign Matrix Conjecture" by David Bressoud [9].

The alternating sign matrix conjecture concerns the total number of $n \times n$ alternating sign matrices, which was conjectured by Mills, Robbins, and Rumsey to be $\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$. The problem was open for fifteen years until it was finally settled by Zeilberger [38]. The development of ideas is described in the book by Bressoud. There are deep relations to various parts of Algebraic Combinatorics, especially to plane partitions, where the same counting function occured, and also to Statistical Mechanics, where the configuration of water molecules in "square ice" can be described by an alternating sign matrix.

As an important step in the derivation of the refined alternating sign matrix conjecture [39], a Hankel matrix comes in, whose entries are $c_m = \frac{1-q^{m+1}}{1-q^{3(m+1)}}$. The relevant orthogonal polynomials in this case are a discrete version of the Legendre polynomials.

Let us conclude with an observation yielding another link to alternating sign matrices. It might be interesting to ask about the values of $\det(A_n)$, when for the entries c_m in the Hankel matrix $A_n = A_n^{(0)}$ we choose generalized Catalan numbers $C_m^{(s)} = \frac{1}{sm+1}\binom{sm+1}{m}$, where $s \geq 2$ is a positive integer.

The following interpretation of the determinant of the Hankel matrix $A_n^{(k)}$ with generalized Catalan numbers $\frac{1}{sm+1}\binom{sm+1}{m}$ as entries c_m is analogous to Proposition 1 and the proof follows the same line as the proof of Proposition 1 in [23], since the generalized Catalan number is the number of paths from $(0,0)$ to $(m,(s-1)m)$, which never cross the diagonal $(s-1)x = y$.

Proposition 2: If the c_m's in (1) are generalized Catalan numbers, $c_m = \frac{1}{sm+1}\binom{sm+1}{m}$, $s \geq 2$ a positive integer then $\det(A_n^{(k)})$ is the number of n–tuples $(\gamma_0, \ldots, \gamma_{n-1})$ of vertex – disjoint paths in the integer lattice $\mathbb{Z} \times \mathbb{Z}$ (with directed vertices from (i,j) to either $(i,j+1)$ or to $(i+1,j)$) never crossing the diagonal $(s-1)x = y$, where the path γ_r is from $(-r,-(s-1)r)$ to $(k+r,(s-1)(k+r))$.

For the choice $s = 3$ (where the sequence of generalized Catalan numbers starts with $1, 1, 3, 12, 55, 273, \ldots$) we observed two identities related to sequences arising in the enumeration of special types of plane partitions and alternating sign matrices. One might ask if the coincidence continues for further elements of the sequences mentioned below.

a) If $c_m = C_m^{(3)}$ for all m, then the first values for the determinant $\det(A_n^{(0)})$ are 1, 2, 11, 170, 7429, 920460. which coincides with the first elements of the sequence of numbers

$$\prod_{i=0}^{n-1} \frac{(3i+1)! \cdot (6i)! \cdot (2i)!}{(4i)! \cdot (4i+1)!}$$

counting cyclically symmetric transpose complement plane partitions as studied in [9], Eq. (6.15), p. 199.

b) The first values of $\det(A_n^{(1)})$ are 1, 3, 26, 646, 45885, 9304650. On these first 6 values the sequence coincides with the sequence $V_{2n+1}, n \geq 1$ defined by $\frac{V_{2n+1}}{V_{2n-1}} = \binom{6n-2}{2n}/2\binom{4n-1}{2n}$ (cf. [9], (6.18) on p. 201) conjectured in [35] to count alternating sign $(2n+1) \times (2n+1)$ matrices which are invariant under a reflection about a vertical axis.

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