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The homology representations of the symmetric group on Cohen-Macaulay subposets of the partition lattice

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Proposed running head: Homology representations

0. Introduction

The action of the symmetric group S_n on the top homology $\tilde{H}_{n-3}(\Pi_n)$ of the partition lattice Π_n has been studied from various viewpoints in recent years, beginning with a character computation by Hanlon ([H1]). Stanley (1982) used this computation to show that up to sign, this representation is isomorphic to the one obtained by inducing a faithful character of a cyclic group of order n up to S_n. The work of Witt and Brandt, (see also a more recent result of Klyachko (1974)), implies that this in turn is isomorphic to the S_n-representation on the (1^n)-graded piece of the free Lie algebra on n generators. Orlik and Solomon observed that the Whitney homology (whose definition we recall in Section 1) of Π_n is S_n-isomorphic to the cohomology of the complement $M_n(\mathbf{C})$ of the thick diagonal in $\mathbf{C}^n$. (This is the Orlik-Solomon algebra for the root system A_{n-1}). Stanley's observation and Hanlon's computation are recovered in the 1986 paper of Lehrer and Solomon, where the S_n-module structure of the cohomology of $M_n(\mathbf{C})$ (i.e., the complement of the hyperplanes for the root system A_{n-1}) is computed explicitly. An equivalent formulation of the main theorem of [LS] is that as an S_n-module, the Orlik-Solomon algebra for A_{n-1} coincides, up to tensoring by the sign, with the S_n-module structure on the (1^n)-graded component of the exterior algebra on the free Lie algebra (see Remark 1.8.1). This result is placed in a more general homological setting via spectral sequences in a recent paper of Schechtman and Varchenko.

Following questions raised by Stanley ([St2]), in this paper we consider the S_n-representation on the homology of certain Cohen-Macaulay subposets of Π_n. In Section 1, we present a general technique for manipulating these homology modules. The unique properties of the partition lattice allow further simplification of these formulas, culminating in plethystic generating functions which, by recursive computation, yield the Frobenius characteristic of the representation. We illustrate our technique by giving simple derivations of three known formulas:

1. a formula for the plethystic inverse of the sum of the cycle indicators of the symmetric groups; this is essentially equivalent to Cadogan's formula;
2. a plethystic formula which determines the characteristic of the homology representation on the lattice Π_n^d of partitions of a set of size nd all of whose blocks are multiples of d; this formula was originally derived in a paper of Calderbank, Hanlon and Robinson;
3. a plethystic generating function identity, also due to these authors, which determines the homology representation on the subposet $\Pi_n^{(1,2)}$ of partitions of n elements into blocks of size congruent to 1 mod 2.

The idea behind our basic observation, Lemma 1.1, is implicitly used to calculate the Möbius functions of fixed-point posets by Calderbank, Hanlon and Robinson ([CHR], Theorem 2.2). Our approach has the advantage of being more conceptual from the representation-theoretic point of view: we avoid intricate Möbius function computations by manipulating virtual homology modules, and by exploiting the considerable machinery of symmetric function theory as presented by Macdonald.

It is known that rank-selected subposets of a Cohen-Macaulay poset are also Cohen-Macaulay (see [Ba2, Theorem 6.4], [Mu, Corollary 6.6] and [St2, Theorem 4.3]). Section 2 contains our most general result, a recursive plethystic formula for the characteristic of the homology representation on an arbitrary rank-selected subposet of Π_n, Theorem 2.13. Our methods show that the plethysm operation is inextricably linked to the partition lattice. As one specific example, we mention the case when the ranks are selected in arithmetic progression: here we obtain intriguing formulas (Theorems 2.8-2.9) which generalise the plethystic equation (see Example 1.6(i)) whose solution gives the homology representation of Π_n.

Using these formulas we can extract a considerable amount of concrete information about the representation.

The same idea allows us, in Section 3, to give recursive formulas for the permutation representation on the rank-selected chains of Π_n. In particular we determine completely the orbit decomposition of the representation on the maximal chains of Π_n. The multiplicities we obtain turn out to enumerate a known refinement of the Euler numbers. In fact we show that the action of S_n on the maximal chains is closely related to an action of S_2^{n-1} on the set of $n!$ permutations, defined by Foata and Strehl ([FS1], [FS2]) in their enumerative work on the Euler numbers.

A result of Stanley ([St3]) states that when all the rank-selected homologies of the partition lattice are combined into one S_n-module, the multiplicity of the trivial representation in this direct sum is the Euler number E_{n-1} of alternating permutations in S_{n-1}. Our computations enable us to describe combinatorially the multiplicity of the trivial representation in the homology module in some specific cases of rank-selection, although it is as yet unclear how these multiplicities enumerate a refinement of the alternating permutations. By examining the restrictions of these S_n-homology modules to S_{n-1}, we obtain a second (different) refinement of the Euler number E_n into nonnegative integers, also indexed by subsets. These results are collected in Section 4.

The formula of Theorem 2.13 is easily adapted to handle rank-selection in the posets Π_n^d and $\Pi_n^{(1,2)}$. As a by-product of our study we can show that the S_{nd} (respectively S_n)-module structure of the Whitney homology of Π_n^d is a direct generalisation of the corresponding result for the ordinary partition lattice. This suggests the existence of an appropriate generalisation of the Orlik-Solomon *algebra* for A_{n-1}. For the posets $\Pi_n^{(1,2)}$ the existence of such an algebra structure has been confirmed constructively, by Hanlon and Wachs. The lattice Π_n^d is discussed in Section 5.

Since we are interested primarily in group actions, all homology will usually be taken over the complex field. However, Lemma 1.1 and Theorem 1.2 are stated more generally, with integer coefficients. All posets in this paper are assumed to be finite.

1. General methods

Let P be a poset with least element $\hat{0}$. Following Baclawski, let $D_i(P)$ be the free abelian group generated by all $(i-1)$-chains $a_1 < \ldots < a_i$ of elements of $P - \{\hat{0}\}$. Set $D_{-1}(P) = 0$. Also $D_0(P)$ is the free abelian group of rank one. For $i > 0$ and $D_i(P) \neq 0$, define a differential $d_i^W : D_i(P) \longrightarrow D_{i-1}(P)$ by

$$d_i^W(a_1, \ldots, a_i) = \sum_{j=1}^{i-1} (-1)^j (a_1, \ldots, \hat{a}_j, \ldots, a_i);$$

define $d_i^W = 0$ otherwise. (As usual the hat over an element in the chain denotes suppression of that element).

The homology of the algebraic complex $(D(P), d^W)$ is the *Whitney* homology of P. It was defined and studied first by Baclawski ([Ba1]) and then by Björner ([Bj2]), who related it to the usual order homology of the poset P. We write $WH_i(P)$ for the ith Whitney homology group, that is, $WH_i(P) = \ker d_i^W / \mathrm{im}\ d_{i+1}^W$. Note that if P has a least element $\hat{0}$ and a greatest element $\hat{1}$, the Whitney homology in the highest degree coincides with the top homology of the order complex of $P - \{\hat{0}, \hat{1}\}$.

Our first basic result is:

Lemma 1.1 *Let P be a poset with least element $\hat{0}$ and greatest element $\hat{1}$; let G be the automorphism group of P. Assume that the Whitney homology groups are free in all degrees. Then each Whitney homology is a G-module. If r is the maximum length of a chain from $\hat{0}$ to $\hat{1}$ in P, then as a virtual sum of G-modules, one has*

$$WH_r(P) - WH_{r-1}(P) + \ldots + (-1)^{r-1} WH_1(P) + (-1)^r WH_0(P) = 0.$$

Since $WH_r(P) = \tilde{H}_{r-2}(P)$, ($P$ has a greatest element $\hat{1}$), one has, equivalently, the equality of G-modules

$$\tilde{H}_{r-2}(P) = WH_{r-1}(P) - WH_{r-2}(P) + \ldots + (-1)^{r-1} WH_0(P).$$

Here $WH_0(P)$ is the trivial G-module.

Proof The proof follows from a routine application of the Hopf trace formula to the Whitney complex $(D(P), d^W)$. Denote by $C_i(P)$ the group of i-chains in the augmented chain complex of $P \backslash \{\hat{0}, \hat{1}\}$, that is, in the order complex of P. First note that the boundary maps, in the case of both the simplicial and the Whitney complex, are group-equivariant. It is clear that as G-modules $D_i(P) = C_i(P) \oplus C_{i-1}(P)$, $1 \leq i \leq r-1$, and

$D_0(P) = C_0(P)$, $D_r(P) = C_{r-1}(P)$. Let $g \in G$. By the Hopf trace formula applied to the Whitney complex, we have (with $\mathrm{tr}(g, *)$ denoting the trace of g)

$$\begin{aligned}\sum_{i=0}^{r}(-1)^i \mathrm{tr}(g, WH_i(P)) &= \sum_{i=0}^{r}(-1)^i \mathrm{tr}(g, D_i(P)) \\ &= (-1)^r \mathrm{tr}(g, C_{r-1}(P)) + \sum_{i=1}^{r-1}(-1)^i \left(\mathrm{tr}(g, C_i(P)) + \mathrm{tr}(g, C_{i-1}(P))\right) + \mathrm{tr}(g, C_0(P)) \\ &= 0,\end{aligned}$$

as required. □

For a finite poset P with $\hat{0}$, write $WH(P)$ for the direct sum of all the Whitney homology groups of P. A theorem of Orlik and Solomon states that $WH(\Pi_n)$ is isomorphic to the cohomology ring of the complement of the thick diagonal in $\mathbf{C}^n$, and consequently has the structure of a graded anti-commutative algebra, in this case the Orlik-Solomon algebra for the root system A_{n-1} ([OS], Corollary 5.6). In fact Orlik and Solomon show that for any geometric lattice, the associated graded anti-commutative algebra satisfies the acyclicity property of Lemma 1.1. (See [OS], Lemma 2.18). Our Lemma 1.1 points out that this is true in the more general context of the Whitney homology of any poset. Orlik and Solomon also show that for any geometric lattice L, the Whitney homology $W(L)$ does indeed have the structure of a graded anti-commutative algebra.

A fundamental theorem is that, when P is Cohen-Macaulay, the homology groups of the complex $(D(P), d^W)$ are free. This was proved by Baclawski for geometric lattices ([Ba1]), and follows in general from the following theorem of Björner ([Bj2]).

Theorem 1.2 *([Bj2, Theorem 5.1]) Let P be any poset with least element $\hat{0}$. Then the Whitney homology of P is related to the usual reduced order homology of P by isomorphisms establishing*

$$WH_i(P) = \bigoplus_{x \in P} \tilde{H}_{i-2}(\hat{0}, x), \ 0 \le i \le r,$$

where r is the length of the longest chain in P.
Furthermore it follows from Björner's proof that the isomorphism commutes with any automorphism of P.

If G is a finite group of automorphisms of P, in general this gives an expression for $WH_i(P)$ as a sum of homology modules which are induced G-modules. In addition, when P is Cohen-Macaulay, the only terms which contribute to the homology on the right-hand side come from elements x in P at rank i. Now suppose that P belongs to a family of Cohen-Macaulay posets P_r of rank r with the property that for x at rank i, the lower interval $(\hat{0}, x)$ is isomorphic to P_i. (The Boolean lattice of subsets of an n-element set, for instance, has this property). In that case the two preceding results can be combined to obtain a recursive formula for the homology of P_r in terms of the homology of P_i, for $i < r$.

We illustrate this technique by computing the homology representation for a simple well-known poset.

Example 1.3 The Boolean lattice B_n of subsets of an n-element set, ordered by inclusion. Take G to be the symmetric group S_n, the automorphism group of B_n. Let β_n denote the representation of S_n on the top homology of B_n. By Lemma 1.1 and Björner's theorem, we have that, as S_n-modules,

$$\beta_n = \bigoplus_{i=0}^{n-1}(-1)^{n-1-i} WH_i(B_n),$$

where, since for every subset x of size i, the order complex of the open interval $(\hat{0}, x)$ is homotopy equivalent to the unit sphere S^{i-1} in i-dimensional space, we have

$$WH_i(B_n) = \bigoplus_{x \in B_n : |x| = i} \tilde{H}_{i-2}(\hat{0}, x).$$

Note that $WH_0(B_n)$ is the trivial S_n-module.

It is clear that the stabiliser of the action of S_n on $WH_i(B_n)$ is $S_i \times S_{n-i}$, and that S_{n-i} acts trivially, while S_i acts like the representation β_i. Hence as S_n-modules, $WH_i(B_n)$ is isomorphic to the induced module $(\beta_i \otimes 1_{S_{n-i}}) \uparrow_{S_i \times S_{n-i}}^{S_n}$. If we let β_i also stand for the Frobenius characteristic of the representation β_i, in the language of symmetric functions, the β_i satisfy the following identity:

$$\beta_n - \beta_{n-1}\, h_1 + \beta_{n-2}\, h_2 - \ldots + (-1)^n h_n = 0,$$

for all $n \geq 2$, and $\beta_1 = h_1$.

But the elementary symmetric functions e_i satisfy the well-known identity

$$\sum_{i=0}^{n} (-1)^{n-i} e_{n-i} h_i = 0,$$

with the same initial conditions. It follows that $\beta_n = e_n$, that is, the action of S_n on the top homology of B_n is the sign representation.

Note that in this particular case it is of course easy to write down a fundamental cycle for the rank one homology module, and from this it is immediate that S_n acts like the sign. However the preceding computation is a good example of our technique.

As a final remark concerning this example, the preceding discussion shows that the ith Whitney homology of the Boolean lattice B_n affords the ith exterior power of the natural representation of S_n on $\mathbf{C}^n$. Hence the sum $\oplus_{i=0}^{n} WH_i(B_n)$ is the exterior algebra over $\mathbf{C}^n$.

We review briefly some basic facts about the partition lattice Π_n. As a poset it consists of the set partitions of $\{1, 2, \ldots, n\}$ ordered by refinement, the finest being the n-block partition in which every element is in its own block, and the coarsest being the single-block partition in which all the elements are in one block. The rank function on Π_n is $\mathrm{rank}(x) = n -$ (the number of blocks of x), so that the rank of the lattice is $n-1$. For every set partition x in Π_n we define the **type** of x to be the integer partition λ of n, consisting of m_i parts of size i, if the set partition x has m_i blocks of size i. Clearly the action of S_n on Π_n is type-preserving.

Our representation-theoretic results are described most conveniently by using the theory of symmetric functions (see [M]). Recall that the complete homogeneous symmetric function h_n is the Frobenius characteristic of the trivial representation of the symmetric group S_n, while the elementary symmetric function e_n is the characteristic of the sign representation. We shall write π_n for the characteristic of the top homology module $\tilde{H}_{n-3}(\Pi_n)$ of the partition lattice. Following [M], denote by ω the involution on the ring of symmetric functions which takes h_n to e_n. A well-known result on the homology of Π_n, which follows from Hanlon's computation of the character values of π_n and earlier results of Witt and Brandt, states that $\pi_n = \omega(\ell_n)$, where ℓ_n is the characteristic of the S_n-representation on the nth graded component of the free Lie algebra. Joyal gives a direct functorial proof of this in [J], while Barcelo shows in [Bc] that up to sign, and dual bases, there is an S_n-equivariant bijection between the Lyndon basis for the free Lie algebra and Björner's NBC basis for homology.

The plethysm operation occurs naturally in connection with the partition lattice; for symmetric functions f and g we write $f[g]$ for the plethysm of f with g.

We now apply this method to derive the Hanlon-Stanley formula ([H1], [St3]) for the homology representation of the symmetric group S_n on the partition lattice Π_n. For the partition lattice, it is the upper order ideals $(x, \hat{1})$ which have the property of being isomorphic to smaller partition lattices Π_r, where r is the number of blocks of x. We shall therefore apply Lemma 1.1 to the dual poset Π_n^*. (Although the dual is no longer a geometric lattice, it is still Cohen-Macaulay). We need to compute the S_n-module structure of the Whitney homology $WH_r(\Pi_n^*)$. This calculation is crucial in everything that follows, since it generalises to arbitrary rank-selected subposets of the partition lattice.

At this stage it is convenient to set up some notation, which will be used frequently throughout the rest of the paper.

Remark 1.4.0

(i) Let ch denote the Frobenius characteristic map. If f_1, f_2, are respectively symmetric functions which are Frobenius characteristics of representations of S_{a_1}, S_{a_2}, we write $f_1 \otimes f_2$ (rather than $ch^{-1}(f_1) \otimes ch^{-1}(f_2)$) for the representation of the direct product $S_{a_1} \times S_{a_2}$ obtained by tensoring the representations f_i. Recall that the characteristic of the induced representation $(f_1 \otimes f_2)\big\uparrow_{(S_{a_1} \times S_{a_2})}^{S_{a_1+a_2}}$ is the ordinary product $f_1\, f_2$ in the ring of symmetric functions. By abuse of notation, when it is clear from the context that we are manipulating the symmetric functions and not the modules themselves, we sometimes interpret

$$(f_1 \otimes f_2)\big\uparrow_{(S_{a_1} \times S_{a_2})}^{S_{a_1+a_2}}$$

to be the product $f_1\, f_2$.
Note that this notation remains meaningful even if the representations $ch^{-1}(f_i)$ are virtual.

(ii) Let $\lambda = \prod_i i^{m_i}$ be an integer partition of n with $(r+1)$ parts, and let G_λ be the subgroup $\times_i S_{m_i}[S_i]$. Here $S_{m_i}[S_i]$ denotes the wreath product group, with S_{m_i} acting on m_i copies of S_i. For any representation γ of S_{r+1}, and any sequence of (possibly virtual) representations R_i of S_i, denote by $\gamma\big\downarrow_{S_{m_1} \times \ldots \times S_{m_i} \times \ldots}[R_1 \otimes \ldots \otimes R_i \otimes \ldots]$ the representation of G_λ defined as follows: Suppose the restriction $\gamma\big\downarrow_{S_{m_1} \times \ldots \times S_{m_i} \times \ldots}$ decomposes into irreducibles for $(S_{m_1} \times \ldots \times S_{m_i} \times \ldots)$ so that the irreducible indexed by the vector of partitions $\underline{\mu} = (\mu^1, \ldots, \mu^i, \ldots)$ appears with multiplicity $c_{\underline{\mu}}$. Then

$$\gamma\big\downarrow_{S_{m_1} \times \ldots \times S_{m_i} \times \ldots}[R_1 \otimes \ldots \otimes R_i \otimes \ldots]$$

denotes the representation

$$\sum_{\underline{\mu}} c_{\underline{\mu}}\; s_{\mu^1}[R_1] \otimes s_{\mu^2}[R_2] \otimes \ldots \otimes s_{\mu^i}[R_i] \otimes \ldots .$$

Note in particular that if the R_i are irreducible representations of S_i, then in the latter sum, each term is an irreducible representation for G_λ.
If R_i is a virtual representation of S_i, this notation is still meaningful if we apply the following formal rule of sings for evaluating plethysms involving non-positive coefficients: $f[-g] = (-1)^{\deg f}\omega(f)[g]$, where ω is the involution in the ring of symmetric functions which takes h_n to e_n, and f is homogeneous (see [M]).

We can now state

Theorem 1.4 *The Frobenius characteristic of the S_n-module $WH_r(\Pi_n^*)$ is given by the degree n term in the plethysm*

$$\pi_{r+1}[h_1 + h_2 + \ldots],\ r = 0, 1, \ldots, n-1.$$

Proof

From Theorem 1.3, and because Π_n^* is Cohen-Macaulay, we have that

$$WH_r(\Pi_n^*) = \bigoplus_{\substack{x \in \Pi_n \\ x \text{ has } (r+1) \text{ blocks}}} \tilde{H}_{r-1}(x, \hat{1}).$$

Note first that the right-hand side decomposes into a direct sum of S_n-submodules W_λ indexed by the type λ of the set partition x, where

$$W_\lambda = \bigoplus_{\substack{x \in \Pi_n \\ type(x) = \lambda}} \tilde{H}_{r-1}(x, \hat{1}).$$

If λ has m_i parts of size i, it is clear that the stabiliser of W_λ is $G_\lambda = S_{m_1}[S_1] \times S_{m_2}[S_2] \times \ldots \times S_{m_i}[S_i] \times \ldots$, where $S_{m_i}[S_i]$ denotes the wreath product group, with S_{m_i} acting on m_i copies of S_i. By considering the action on the chain groups of the order complex of $(x, \hat{1})$, it is also clear that S_n permutes the $(S_n : G_\lambda)$ summands of W_λ transitively. It follows that W_λ is the S_n-module induced from the representation V_λ of the subgroup G_λ on the homology of $(x, \hat{1})$, for x of type λ. It remains to determine this representation. The intuitive idea is that the outer factors $S_{m_1} \times \ldots \times S_{m_i} \times \ldots$ act on the $(r+1)$ blocks as a subgroup of S_{r+1}, and hence like the restriction of the representation π_{r+1} to the Young subgroup $S_{m_1} \times \ldots \times S_{m_i} \times \ldots$, while the inner factors $S_1, S_2, \ldots, S_i, \ldots$ act trivially, since they fix the blocks of the partition x.

We now claim that the representation V_λ of G_λ on the homology of the interval $(x, \hat{1})$, for x of type λ, is precisely

$$\pi_{r+1}\downarrow_{S_{m_1}\times\ldots\times S_{m_i}\times\ldots}[h_1 \otimes \ldots \otimes h_i \otimes \ldots].$$

From the observation that the inner factors S_i act trivially on the blocks, we conclude that the decomposition into irreducibles of the G_λ-homology module must be of the form

$$\sum_{\underline{\mu}} c_{\underline{\mu}}\ s_{\mu^1}[h_1] \otimes s_{\mu^2}[h_2] \otimes \ldots \otimes s_{\mu^i}[h_i] \otimes \ldots,$$

for some positive integral multiplicities $c_{\underline{\mu}}$. (Notation as in Remark 1.4.0).

Consider the restriction of this representation to the subgroup $\times_i S_{m_i}$ of G_λ. On the one hand, from the preceding expression this restriction is clearly

$$\sum_{\underline{\mu}} c_{\underline{\mu}}\ s_{\mu^1} \otimes s_{\mu^2} \otimes \ldots \otimes s_{\mu^i} \otimes \ldots.$$

On the other hand, viewing $\times_i S_{m_i}$ as a subgroup of S_{r+1}, and repeating the observations that

(i) permutations σ in $S_{m_1} \times \ldots \times S_{m_i} \times \ldots$ permute the blocks of a partition ρ in $(x, \hat{1})$, and

(ii) $(x, \hat{1})$ is isomorphic to Π_{r+1} as a poset,

it is clear that the restriction $\tilde{H}(x, \hat{1})\downarrow^{G_\lambda}_{\times_i S_{m_i}}$ is isomorphic to the restriction $\pi_{r+1}\downarrow^{S_{r+1}}_{\times_i S_{m_i}}$. The claim now follows immediately.

Finally the statement of the theorem follows from the following technical fact about plethysm of symmetric functions:

Lemma 1.5 *Let Q be a symmetric function of homogeneous degree r, let f and g be homogeneous symmetric functions of degrees a, b, respectively. Then, in the notation of Remark 1.4.0, the degree n term in the plethysm $Q[f+g]$ is given by the symmetric function*

$$\sum_{\substack{(a^{m_a} b^{m_b}) \vdash n \\ m_a + m_b = r}} \left(Q\downarrow_{S_{m_a}\times S_{m_b}} [f \otimes g]\right) \uparrow^{S_n}_{S_{m_a}[S_a]\times S_{m_b}[S_b]}.$$

Note that here $Q\downarrow_{S_{m_a}\times S_{m_b}}$ may be a virtual linear combination of irreducibles of $S_{m_a} \times S_{m_b}$.

Proof

It suffices to consider the case of a Schur function $Q = s_\lambda$. Then from ([M], Chapter 1, (8.8)) we have

$$s_\lambda[f+g] = \sum_{\mu \subseteq \lambda} s_\mu[f] s_{\lambda/\mu}[g].$$

The degree n term is obtained by taking the sum over all partitions μ such that $a|\mu| + b(|\lambda| - |\mu|) = n$, that is, first over all partitions of n into part sizes a and b. We may rewrite the sum as

$$\sum_{\substack{\nu=(a^{m_a} b^{m_b}) \vdash n \\ m_a + m_b = r}} \sum_{\mu \vdash m_a} s_\mu[f]\ D_\mu s_\lambda[g],$$

where D_μ is the operator on the ring of symmetric functions that is adjoint to multiplication by s_μ. But it follows from properties of the Frobenius characteristic (see [M]) that the inner sum is precisely what we called $s_\lambda \downarrow_{S_{m_a}\times S_{m_b}} [f \otimes g]$. This is because the restriction of s_λ to the Young subgroup $S_{m_a} \times S_{m_b}$ equals the linear combination of outer tensor products $\sum_{\mu \vdash m_a} s_\mu \times D_\mu s_\lambda$. This finishes the proof of the lemma, and hence of the theorem. □

This result can be used to derive the Hanlon-Stanley formula for the representation π_n of S_n on $\tilde{H}_{n-3}(\Pi_n)$. Note that in contrast to [H1], the derivation given below of the formula for π_n does not make use of lattice-theoretic properties.

Example 1.6

(i) Take $P = \Pi_n^*$, the dual of the partition lattice. Using Theorems 1.2 and 1.4, we have that the degree n term in

$$\sum_{r=1}^{n} (-1)^{n-r} \pi_r [\sum_{i\geq 1} h_i]$$

vanishes for all $n \geq 2$, while $\pi_1 = h_1$. This yields

$$\sum_{r\geq 1} (-1)^{r-1} \pi_r [\sum_{i\geq 1} h_i] = h_1,$$

an identity in the plethysm ring of symmetric functions. Since h_1 is the unit in the plethysm ring, this describes the alternating sum of the π_n's as the plethystic inverse of $H = \sum_{i\geq 1} h_i$, (an observation made by Stanley). But H is the cycle indicator of the sum of the trivial representations of the symmetric groups, and an old formula of Cadogan ([C]) gives the plethystic inverse of H to be

$$I = \sum_{d\geq 1} \frac{1}{d}\, \mu(d)\, \log(1 + p_d).$$

Extracting the term of degree n, we recover the formula for the Frobenius characteristic of π_n which, as shown by Stanley ([St3]), follows from a computation of Hanlon ([H1]) on the fixed-point partition lattices:

$$\pi_n = \frac{1}{n} \sum_{d|n} \mu(d)\, p_d^{\frac{n}{d}}\, (-1)^{n-\frac{n}{d}},$$

where p_d denotes the dth power-sum. As shown in [St3], it follows that π_n coincides with the sign tensored with $e^{\frac{2\pi i}{n}} \uparrow_{C_n}^{S_n}$, where C_n is a cyclic subgroup of order n.

(ii) The plethystic inverse of H may also be calculated directly. The author is grateful to Ira Gessel for showing her the following argument, included for the sake of completeness.
Let f denote the inverse to be calculated; then $(1+H)[f] = h_1+1$, or equivalently $\left(\sum_{i\geq 0} h_i\right)[f] = 1+p_1$. Using the identity $\sum_{i\geq 0} h_i = \exp \sum_{i\geq 1} p_i/i$ and the fact that the map $g \mapsto g[f]$ is a homomorphism in the ring of symmetric functions, one has

$$\log(1 + p_1) = \sum_{i\geq 1} \frac{1}{i}\, p_i[f] = \sum_{i\geq 1} \frac{1}{i}\, f[p_i],$$

also using the fact that $p_n[g] = g[p_n]$ for all g. Now expand the logarithm and, for fixed m, take the plethysm with p_m, obtaining:

$$\sum_{i\geq 1} \frac{1}{i}\, (-1)^{i-1} p_m^i = \sum_{i\geq 1} \frac{1}{i}\, f[p_{mi}].$$

Let $\{a_m\}_{m\geq 1}$ be arbitrary complex numbers. Multiply both sides of the above equation by a_m and sum over m :

$$\sum_{m\geq 1}\sum_{i\geq 1}\frac{1}{i}(-1)^{i-1}\ a_m\ p_m^i = \sum_{n\geq 1}\frac{1}{n}\ f[p_n]\left(\sum_{m|n} m a_m\right).$$

Now choose the a_m so that the coefficient of $\frac{1}{n}f[p_n]$ is 1 if $n=1$ and 0 otherwise. This trick has the effect of reducing the right side of the preceding equation to $f[p_1]=f$. Since the unique solution for $m\,a_m$ is given by the number-theoretic Möbius function, the left side is

$$\sum_{m\geq 1}\sum_{i\geq 1}\frac{1}{i}\ \frac{\mu(m)}{m}\ (-1)^{i-1}p_m^i,$$

which gives the Hanlon-Stanley formula.

(iii) Now let Π_n^d denote the subposet of Π_{nd} consisting of those partitions all of whose block sizes are divisible by d. Note that this poset lacks a $\hat{0}$-element, which must be artificially adjoined. The rank one elements in the poset are the $\frac{(nd)!}{n!d!^n}$ partitions all of whose blocks have size d. In general, the rank of a partition x is given by $n+1$ minus the number of blocks of x. The rank of Π_n^d is thus n.
The homology representation of S_{nd} on Π_n^d was first studied in [CHR]. We denote the Frobenius characteristic for this representation by π_n^d. Corollary 4.7 of [CHR] is essentially a generating function for the π_n^d's. We show how to derive this using our methods.
As in (i), we compute the Whitney homology of the dual of Π_n^d. Again the salient factor is that if x is a partition with k blocks in Π_n^d, then as posets, the interval $(x,\hat{1})$ and the ordinary partition lattice Π_k are isomorphic. The ith Whitney homology $WH_i((\Pi_n^d)^*)$ decomposes into the direct sum of homologies $\bigoplus_{\substack{x:x \text{ has} \\ (i+1) \text{ blocks}}} \tilde{H}(x,\hat{1})_{\Pi_n^d}$. (The partition with only one block has rank zero in the dual poset). If x has type λ, then λ is now a partition of nd with $i+1$ parts, all of which are multiples of d, say $\lambda=\prod_j (jd)^{m_j}$, and the stabiliser subgroup in S_{nd} of x is $H_\lambda = \times_j S_{m_j}[S_{jd}]$. The arguments of the proof of Theorem 1.4 show that the H_λ-module structure of the interval $(x,\hat{1})$ is given by the symmetric function

$$\pi_{i+1}\big\downarrow_{S_{m_1}\times S_{m_2}\times\ldots}[h_d\otimes h_{2d}\otimes\ldots].$$

Hence as an S_{nd}-module, $ch(WH_i((\Pi_n^d)^*))$ (for $0\leq i\leq n-1$) is the degree nd term in the plethysm

$$\pi_{i+1}[\sum_{j\geq 1} h_{jd}].$$

Since $\pi_n^d = ch(WH_n((\Pi_n^d)^*))$, a final application of Lemma 1.1 now yields that π_n^d is the degree nd term in

$$\sum_{i\geq 0}^{n-1}(-1)^{n+1-i}\pi_{i+1}[\sum_{j\geq 1} h_{jd}].$$

It follows that

$$\sum_{n\geq 0}(-1)^n\pi_n^d = \sum_{i\geq 1}(-1)^i\pi_i[\sum_{j\geq 1}h_{jd}].$$

Substituting for π_i from (i), and making suitable translations, we see that this is the same as Corollary 4.7 on page 296 of [CHR].

(iv) Now let $\Pi_n^{(1,2)}$ be the subposet of Π_n consisting of partitions all of whose blocks are odd. Again, these posets and their homology representations were studied in [CHR]. Denote the characteristic of the homology representation by β_n. By considering the Whitney homology of the dual poset, and treating the cases for n odd and n even separately, we obtain the generating functions

$$h_1 = \sum_{m\geq 0}(-1)^m\beta_{2m+1}[\sum_{i\geq 0}h_{2i+1}],$$

and

$$-\left(\sum_{i\geq 1} h_{2i}\right) = \sum_{m\geq 1}(-1)^{m-1}\beta_{2m}[\sum_{i\geq 0} h_{2i+1}].$$

A similar argument can be applied to the case of $\Pi_n^{(1,k)}$, and we obtain formulas equivalent to Theorem 4.7 on page 297 of [CHR].

The plethystic inverse identity for $H = \sum_{i\geq 1} h_i$ can also be derived using a knowledge of the S_n-module structure of $WH(\Pi_n)$. This was originally determined by Lehrer and Solomon, by different methods.

For a set partition x of type λ, we shall determine the G_λ-module structure of the homology of an interval of the form $(\hat{0}, x)$ in Π_n, by examining the action of G_λ on a basis of the homology module. The arguments used to determine the module structure of the homology of an interval $(\hat{0}, x)$ in the partition lattice are different in flavour from those used for the case of intervals $(x, \hat{1})$. We remark here that our proof does not depend on the choice of basis, (as long as we work with bases of fundamental cycles), but to make the discussion more concrete, we use a basis constructed by Björner ([Bj2]) for geometric lattices.

First, we briefly review the theory of NBC bases in the context of the partition lattice. The atoms of Π_n are the partitions with $(n-1)$ blocks, and are thus completely determined by the unique non-singleton block $/i < j/$. A set S of atoms is *independent*, or a *base*, if $\vee S$ has rank $|S|$, and *dependent* otherwise. A *circuit* is a minimal dependent set. One checks easily that a set of atoms C in a geometric lattice is a circuit iff $\vee C$ has rank $|C|-1$. We shall use bases for homology constructed from maximal independent sets obtained from the atom-ordering $/12/ < /13/ < /23/ < /14/ < /24/ < /34/ < \ldots$ (Note that this atom-ordering gives rise to the following admissible labelling (in the sense of Stanley, see [St1], [Bj1]) of the edges of Π_n : if π is covered by τ, and τ joins blocks B_1 and B_2 of π, then the edge joining π to τ has label $max(minB_1, minB_2)$. This edge labelling, suggested by Ira Gessel, shows that Π_n is lexicographically shellable (see [Bj1])). A *broken* circuit is a subset S of atoms for which there is an atom a, preceding (in the specified atom-ordering) all the atoms of S, such that $\{a\} \cup S$ is a circuit. Note that a broken circuit is an independent set.

Every independent set of t atoms in a geometric lattice generates a Boolean sublattice, associated to which there is, up to sign, a unique fundamental cycle for a $(t-1)$-sphere in homology. Björner's theorem states that the fundamental cycles associated to the maximal independent sets of atoms which are *not* broken circuits, the $\widehat{\text{N}}$BC sets of atoms, form a basis for the homology of the geometric lattice. (See [Bj2], [Bj3].)

For the partition lattice Π_n with the atom-ordering chosen above, it is not hard to see that the NBC sets are precisely the subsets of atoms with the property that no two atoms have the same largest element in their respective non-singleton blocks. That is, if the set of atoms is $/i_1 < j_1/, \ldots /i_k < j_k/$, then $s \neq t$ implies $j_s \neq j_t$. (Note that this gives the correct number of maximal NBC sets in Π_n, viz., $(n-1)!$).

Theorem 1.7 *Let x be a partition in Π_n, of type $\lambda = 1^{m_1}2^{m_2}\ldots$ Then the top homology of the interval $(\hat{0}, x)$ is a $G_\lambda = \times_i S_{m_i}[S_i]$-module isomorphic to*

$$\iota_{m_1}[\pi_1] \otimes \epsilon_{m_2}[\pi_2] \otimes \ldots \otimes \iota_{m_{2i-1}}[\pi_{2i-1}] \otimes \epsilon_{m_{2i}}[\pi_{2i}] \otimes \ldots,$$

where ι_m and ϵ_m denote the trivial and the sign representations respectively.

Proof

We shall need to recall some facts about wreath product modules. If V is an S_m-module and W is an S_c-module, we write $V[W]$ for the wreath product module denoted by $(\otimes^m W)\tilde{} \otimes V$ in [JK]. (The Frobenius characteristic of this module induced up to S_{mc} is the plethysm $ch(V)[ch(W)]$). Denote by I_k the interval $\{c(k-1)+1, \ldots ck\}$, for $k = 1, \ldots, m$. Let σ_1 be the identity in S_{mc} and for $k \geq 2$ let σ_k denote the permutation in S_{mc} which interchanges the intervals I_1 and I_k, i.e., $\sigma_k(i) = c(k-1)+i$, $\sigma_k(c(k-1)+i) = i$, $i = 1, \ldots, k$, and $\sigma_k(j) = j$ otherwise.

Clearly the σ_k are all involutions, for $k > 1$, and they generate a subgroup isomorphic to S_m. Recall that, as a subgroup of S_{mc}, $S_m[S_c]$ can be identified with the normaliser $N_{S_{mc}}(\times^m S_c)$ of $\times^m S_c$ in S_{mc}.

Following the notation in [JK], let $(x_1, \ldots, x_m; \alpha)$ denote a typical element of $S_m[S_c]$, for $x_i \in S_c$ and α in the copy of S_m generated by the σ_k above. By definition, $S_m[S_c]$ acts on $(\otimes^m W) \otimes V$ by

$$(x_1, \ldots, x_m; \alpha).((\otimes_{j=1}^m w_j) \otimes v) = (\otimes_{j=1}^m x_j.w_{\alpha^{-1}(j)}) \otimes \alpha.v. \qquad (A)$$

(Here the w_j are in W, and v is in V.)

Since $(\hat{0}, x)$ is a product of lattices $\underbrace{\Pi_i \times \Pi_i \times \ldots \times \Pi_i}_{m_i \text{ times}}$, it suffices to analyse the representation in the case when $n = mc$ and x has type c^m, and thus $(\hat{0}, x)$ is poset-isomorphic to the m-fold product of partition lattices Π_c. The stabiliser group in this case is the wreath product $S_m[S_c]$. Without loss of generality, we may assume that x is the partition whose kth block consists of the interval I_k described above.

It is not hard to see that the action of each component of the m-fold product $S_c \times S_c \times \ldots S_c$ acting on $\tilde{H}(\hat{0}, x)$ coincides with π_c, i.e., the S_c-action on $\tilde{H}(\Pi_c)$. We shall show that, in the notations of the first paragraph, $\tilde{H}(\hat{0}, x) \simeq V_c[W]$, where W is the S_c-module $\tilde{H}(\Pi_c)$, and V_c is a one-dimensional S_m-module equal to the trivial representation if c is odd, and the sign representation if c is even.

We use the identification of $S_m[S_c]$ with the normaliser $N_{S_{mc}}(\times^m S_c)$, as described in the first paragraph. From the preceding remarks, it is enough to examine the action of the involutions σ_k, in the isomorphic copy of S_m inside $N_{S_{mc}}(\times^m S_c)$, on a basis of $\tilde{H}(\hat{0}, x)$. It turns out that any basis of spherical cycles, and in particular the NBC basis, makes the sign twists in the representation almost transparent.

Note that, if we write Π_{I_k} for the lattice of partitions of the set $\{c(k-1)+1, \ldots ck\}$, we can think of Π_{I_k} as the image under the permutation σ_k of the partition lattice $\Pi_c = \Pi_{I_1}$. Now $(\hat{0}, x)$ is a geometric lattice, so that Björner's theory applies. Clearly the maximal sets of atoms which are not broken circuits are of the form $B_1 \cup \sigma_2(B_2) \cup \ldots \cup \sigma_m(B_m)$, where B_k is a maximal independent set of atoms in $\Pi_c = \Pi_{I_1}$, which is not a broken circuit. (Equivalently, the key fact here is that the union of the B_k's also generates a Boolean lattice, or that the join of two spherical complexes is again spherical. Hence there is no particular reason to use the NBC basis; any other basis of spherical cycles, such as the recent one constructed in [Wa], will work just as well.) Each B_k is a set of size $c-1$, $B_k = \{/b_{k,i} < i/, i = 2, \ldots, c\}$. The atom-ordering is the one we used before: $/a < i/ < /b < j/$ if $i < j$ or if $i = j$ and $a_i < b_j$. The corresponding basis element in the homology of $(\hat{0}, x)$ is the fundamental cycle

$$\rho[B_1, \sigma_2(B_2) \ldots, \sigma_m(B_m)], \qquad (B)$$

where $[B_1, \sigma_2(B_2) \ldots, \sigma_m(B_m)]$ is the ordered $m(c-1)$-tuple of atoms obtained by taking the elements of the union $B = \cup_{k=1}^m \sigma_k(B_k)$, in the specified atom-ordering. That is, we identify the vector space $\otimes^m \tilde{H}(\Pi_c)$ with $\tilde{H}(\hat{0}, x)$ by means of the isomorphism defined on basis elements by

$$\otimes_{i=1}^m \rho[B_i] \mapsto \rho[B_1, \sigma_2(B_2) \ldots, \sigma_m(B_m)].$$

(More generally, for x of arbitrary type $\prod_i i^{m_i}$, this defines an isomorphism from $\tilde{H}(\hat{0}, x)$ to $\otimes_i \left(\otimes^{m_i} \tilde{H}(\Pi_i)\right)$.)

For clarity we write out explicitly what the image of this map is: (see [Bj3]) if $[A]$ is the ordered $m(c-1)$-tuple of atoms $[a_{1,2}, a_{1,3}, \ldots, a_{1,c-1}, a_{2,c+1}, \ldots, a_{k,(c-1)(k-1)+i}, \ldots]$, and τ is a permutation of the underlying set of atoms, i.e., $\tau \in S_{m(c-1)}$, let $[A^\tau]$ be the ordered tuple obtained by permuting the atoms of A according to τ. Finally let $e[A]$ denote the maximal chain uniquely determined by $[A]$, that is, the chain whose ith vertex is the join of the first i atoms of $[A]$. Then

$$\rho[A] = \sum_{\tau \in S_{m(c-1)}} \mathrm{sgn}(\tau)\ e[A^\tau]. \qquad (C)$$

It is clear that the action of S_{mc} on the homology is defined by $\sigma\,\rho[A] = \rho[\sigma(A)]$, where $[\sigma(A)]$ denotes the $m(c-1)$-tuple of atoms obtained by replacing each component of $[A]$, by its image under σ.

Hence the action of the involution σ_i on a basis cycle as in (B) satisfies

$$\begin{aligned}
&\sigma_i \,.\, \rho[B_1, \ldots, \sigma_k(B_k), \ldots, \sigma_m(B_m)] \\
&= \rho[\sigma_i(B_1), \ldots, \sigma_{i-1}(B_{i-1}), B_i, \sigma_{i+1}(B_{i+1}), \ldots, \sigma_m(B_m)] \\
&\text{since } \sigma_i \text{ permutes the intervals } I_1 \text{ and } I_i, \text{ and fixes everything else,} \\
&= (-1)^{(c-1)(c-1)(i-2+i-1)}\ \rho[B_i, \sigma_2(B_2), \ldots, \sigma_{i-1}(B_{i-1}), \sigma_i(B_1), \sigma_{i+1}(B_{i+1}), \ldots, \sigma_m(B_m)] \\
&\text{from the definition in } (C); \\
&= (-1)^{c-1}\ \rho[B_i, \sigma_2(B_2), \ldots, \sigma_{i-1}(B_{i-1}), \sigma_i(B_1), \sigma_{i+1}(B_{i+1}), \ldots, \sigma_m(B_m)].
\end{aligned}$$

In the particular case when $B_k = B$ for all $k = 1, \ldots, m$, we have

$$\sigma_i\, \rho[B, \sigma_2(B), \ldots, \sigma_m(B)] = (-1)^{c-1}\ \rho[B, \sigma_2(B), \ldots, \sigma_m(B)].$$

We have already noted that as vector spaces, $\tilde{H}(\hat{0}, x)$ is isomorphic to $W^{\otimes m}$, where W is the space $\tilde{H}(\Pi_c)$. By restricting to the subgroup $\times^m S_c$ of $S_m[S_c]$, it is clear that W is precisely the S_c-representation π_c. If V_c is the one-dimensional representation of S_c acting by multiplication by $(-1)^{c-1}$, then the above computation shows that as an $S_m[S_c]$-module, $\tilde{H}(\hat{0}, x)$ is isomorphic to the wreath product module $V_c[W]$. The result now follows. □

We can now give the structure of the rth Whitney homology of Π_n, that is, the S_n-module structure of the rth graded component of the Orlik-Solomon algebra for the root system A_{n-1}.

Theorem 1.8 *([LS], Theorem 4.5) For $r = 0, 1, \ldots n-1$, the characteristic of the rth Whitney homology of Π_n, is given by*

$$\sum_{\substack{\lambda \vdash n,\ \ell(\lambda)=n-r \\ \lambda=(1^{m_1}, 2^{m_2}, \ldots)}} h_{m_1}[\pi_1] e_{m_2}[\pi_2] \ldots e_{m_{2i}}[\pi_{2i}] h_{m_{2i+1}}[\pi_{2i+1}] \ldots,$$

the sum ranging over all (integer) partitions λ of n with exactly $n-r$ parts; m_i denotes the multiplicity of the part i.

This is also the term of degree n in the plethysm

$$(-1)^r h_{n-r}[\pi_1 - \pi_2 + \ldots].$$

Proof

The first statement follows from Theorem 1.2 and the preceding result; we need only add up all the modules $\tilde{H}(\hat{0}, x)$ for x at rank r, being careful to group them according to the orbit of x. Now convert to Frobenius characteristics.

The second statement follows from the first by using Lemma 1.5 and the rule of signs for plethysm: for symmetric functions f and g, (f homogeneous), $f[-g] = (-1)^{\deg f} \omega(f)[g]$; (see [M]). □

Note that, if we apply Lemma 1.1 to $WH(\Pi_n)$, we have, with the notations of Example 1.6, the plethystic identity

$$H[\sum_{r \geq 1} (-1)^{r-1} \pi_r] = h_1.$$

Remark 1.8.1 The theorem of Lehrer and Solomon may be restated in the following equivalent form: the S_n-equivariant structure of the Orlik-Solomon algebra, up to tensoring with the sign, coincides with the (1^n)-component of the exterior algebra of the free Lie algebra.

For, applying the involution ω to the expression for the Frobenius characteristic of $WH_r(\Pi_n)$ given by Theorem 1.8, we find that as S_n-modules, the characteristic of the sign representation tensored with the rth graded piece of the Orlik-Solomon algebra for A_{n-1} is given by the degree n term in

$$e_{n-r}[\textstyle\sum_{i\geq 1} \ell_i],$$

where $\ell(k) = \omega(\pi_k)$ is the characteristic of the action of S_k on the (1^k)-graded component of the free Lie algebra, and we have used the rule $\omega(f[g]) = (\omega^{\deg g}(f))[\omega(g)]$; (see [M]). It is not hard to see from our proof of Theorem 1.7 that any S_n-equivariant isomorphism (up to sign) from a basis of spherical cycles of the top homology of the partition lattice to a basis of the free Lie algebra, will automatically extend to an equivariant isomorphism from the whole Whitney homology to the exterior algebra over the free Lie algebra. This observation explains the work of [BB], where the authors extend the canonical bijection of [Bc] between NBC bases and Lyndon words, to give a combinatorial proof of the Lehrer-Solomon theorem.

The paper by Schechtman and Varchenko ([SV]) places the Lehrer-Solomon result in a more general homological setting, by constructing a suitable Koszul complex.

Proposition 1.9 *Assume P is a Cohen-Macaulay poset of rank n; denote by $P(\underline{r})$ the subposet of P obtained by selecting ranks $\{1,\ldots r\}$ in P. Let G be the automorphism group of P. Then the Whitney homology of $P(\underline{r})$ coincides with that of P in degrees $0, 1, \ldots, r$. In particular, the order homology of $P(\underline{r})$ is completely determined by the Whitney homology of P, by means of the G-equivariant recurrence*

$$\tilde{H}(P(\underline{r})) \oplus \tilde{H}(P(\underline{r-1})) = WH_r(P).$$

Here $1 \leq r \leq n-1$, and by definition $\tilde{H}(P(\underline{0})) = WH_0(P)$, while $\tilde{H}(P(\underline{r})) = 0$ if $r < 0$ or $r > n-1$.

Proof

The first statement follows immediately from Björner's theorem (Theorem 1.2). Now using Lemma 1.1, we have $\tilde{H}(P(\underline{r})) = WH_r(P) - WH_{r-1}(P) + \ldots + (-1)^r WH_0(P)$; and similarly for $\tilde{H}(P(\underline{r-1}))$. ☐

Using Björner's characterisation of $WH_r(P)$ in terms of the order homology of P ([Bj2]), we can construct a G-equivariant surjection from $WH_r(P)$ onto $\tilde{H}(P(\underline{r-1}))$, whose kernel is $\tilde{H}(P(\underline{r}))$.

Our second basic technique, which is most useful for describing the homology of the rank-selected subposet when the rank set is large, is the following. Note that Proposition 1.9 may be obtained as a special case of Theorem 1.10, by taking $P = P(\underline{r})$ and $Q = P(\underline{r-1})$.

Theorem 1.10 *Let P be a Cohen-Macaulay poset of rank r, G a finite group of automorphisms of P, and Q a G-invariant Cohen-Macaulay subposet of P. (Assume that Q contains $\hat{0}$and $\hat{1}$.) Then as G-modules:*

$$\begin{aligned}&(-1)^{r-\mathrm{rank}(Q)}\tilde{H}(Q) - \tilde{H}(P)\\ &= \bigoplus_{\substack{\hat{0}<x_1<\ldots<x_k<\hat{1}\\ x_i\notin Q}} (-1)^k \left(\tilde{H}(\hat{0},x_1)_P \otimes \tilde{H}(x_1,x_2)_P \otimes \ldots \otimes \tilde{H}(x_k,\hat{1})_P\right)\Big\uparrow_{\mathrm{stab}(x_1,\ldots,x_k)}^{G};\end{aligned}$$

where

1. *the sum runs over all chains of elements not in Q, where we take one such chain for each orbit under the action of G;*
2. *the homologies involved are all top homologies of open intervals $(x_i, x_{i+1})_P$ in the poset P;*
3. *$\mathrm{stab}(x_1,\ldots,x_k)$ is the intersection of the stabilisers of all $(k+1)$ intervals $(\hat{0},x_1)_P$, $(x_i,x_{i+1})_P$, $(x_k,\hat{1})_P$, in P.*

Proof

The starting point for the proof is a lemma of Baclawski ([Ba3]), which is precisely the Möbius function version of the formula above: For an arbitrary poset P, and subposet Q,

$$\mu(Q)-\mu(P)=\sum_{\substack{0<x_1<\ldots<x_k<1\\ x_i\notin Q}}(-1)^k\mu(\hat{0},x_1)_P\ \mu(x_1,x_2)_P\ \ldots\ \mu(x_k,\hat{1})_P.$$

Now let g be an element of the automorphism group of P that is also an automorphism of Q. Let P^g, Q^g, denote respectively the subposets of P and Q which are point-wise fixed by g. Applying Baclawski's formula to the poset P^g and its subposet Q^g, we find

$$\mu(Q^g)-\mu(P^g)=\sum_{\substack{0<x_1<\ldots<x_k<1\\ x_i\in P^g\setminus Q^g}}(-1)^k\mu(\hat{0},x_1)_{P^g}\ \mu(x_1,x_2)_{P^g}\ \ldots\ \mu(x_k,\hat{1})_{P^g}. \qquad (A)$$

Now suppose that P and Q are Cohen-Macaulay. The key observation here is that in this case, by the Hopf trace formula, the trace of the automorphism g acting on the reduced homology $\tilde{H}(P)$ is given by $(-1)^{\mathrm{rank}(P)}$ times the Möbius number $\mu(P^g)$ of the fixed-point subposet P^g. Note also that if P is Cohen-Macaulay, so is every interval (x,y) in P. Hence the formula becomes

$$\begin{aligned}&(-1)^{\mathrm{rank}(Q)-\mathrm{rank}(P)}\mathrm{tr}(g,\tilde{H}(Q))-\mathrm{tr}(g,\tilde{H}(P))\\ &=\sum_{\substack{0<x_1<\ldots<x_k<1\\ x_i\in P\setminus Q}}(-1)^k\mathrm{tr}(g,\tilde{H}(\hat{0},x_1))\ \mathrm{tr}(g,\tilde{H}(x_1,x_2))\ \ldots\ \mathrm{tr}(g,\tilde{H}(x_k,\hat{1})). \qquad (B)\end{aligned}$$

Note that we can replace $P^g\setminus Q^g$ with $P\setminus Q$ because, by the preceding remarks, the product of traces vanishes if the chain is not in P^g. (The Möbius number $\mu(P^g)$ can be computed as an alternating sum of chains in P that are fixed by g). Assume that the automorphism group G of P is finite. For a chain $c=x_1<\ldots<x_k$ in $P\setminus Q$, let $\mathrm{stab}(c)$ denote the stabiliser subgroup of the chain c. Write $W(c)$ for the tensor product of $\mathrm{stab}(c)$-modules

$$\tilde{H}(\hat{0},x_1)\otimes\tilde{H}(x_1,x_2)\otimes\ldots\otimes\tilde{H}(x_k,\hat{1}).$$

We claim that the right side of (B) is in fact equal to

$$\sum_{c=x_1<\ldots<x_k,\ x_i\notin Q}(-1)^k\mathrm{tr}(g,W(c)\big\uparrow_{\mathrm{stab}(c)}^G), \qquad (C)$$

where the sum runs over all chains c such that $g\in\mathrm{stab}(c)$, and no two chains are in the same orbit. We compute the trace of $g\in G$ on the induced module $W(c)\big\uparrow_{\mathrm{stab}(c)}^G$. If $\{g_i\}_i$ are a set of coset representatives of $\mathrm{stab}(c)$ in G, then by the well-known formula for the character of an induced representation,

$$\begin{aligned}\mathrm{tr}(g,W(c)\big\uparrow_{\mathrm{stab}(c)}^G)&=\sum_{\{i:g_i^{-1}gg_i\in\mathrm{stab}(c)\}}\mathrm{tr}(g_i^{-1}gg_i,W(c))\\ &=\sum_{\{i:g\in g_i.\mathrm{stab}(c)\}}\mathrm{tr}(g,g_i.W(c))\\ &=\sum_{\{i:g\in\mathrm{stab}(g_i.c)\}}\mathrm{tr}(g,W(g_i.c)).\end{aligned}$$

Substituting in (C) and comparing with (B), we see that the claim is justified. $\square$

In the particular case when S is a subset of the ranks $\{1,\ldots,r-1\}$ of P, and $Q=P_S$ is the rank-selected subposet of P consisting of the ranks in S, together with the minimal and maximal elements $\hat{0}$ and $\hat{1}$, this gives a formula for the homology of P_S in terms of the homology of P and intervals of P. The formula now reads: As G-modules,

$$
\begin{aligned}
&(-1)^{r-|S|}\tilde{H}(P_S) - \tilde{H}(P)\\
&= \bigoplus_{\substack{\hat{0}<x_1<\ldots<x_k<\hat{1}\\ \mathrm{rank}(x_i)\notin S}} (-1)^k \left(\tilde{H}(\hat{0},x_1)_P \otimes \tilde{H}(x_1,x_2)_P \otimes \ldots \otimes \tilde{H}(x_k,\hat{1})_P\right) \uparrow^G_{\mathrm{stab}(x_1,\ldots,x_k)};
\end{aligned}
$$

where the sum runs over all chains of elements whose ranks are *not* in S, and we take one such chain for each orbit under the action of G.

Remark 1.10.1 Theorem 1.10 and Proposition 1.9 both generalise to the non-Cohen-Macaulay case, although the resulting formulas are less simple. In the general case, the Möbius-theoretic equation (A) is interpreted by using the fact that, for an element $g \in G$, $\mu(P^g)$ is the trace of g on $\tilde{H}_{\pm}(P)$, where, for any poset Q, $\tilde{H}_{\pm}(Q)$ denotes the virtual sum of G-modules $\oplus_i(-1)^i\tilde{H}_i(Q)$. Again, this is a consequence of the Hopf trace formula applied to the order complex of Q. Let r be the length of the longest chain in P, from the minimal element to the maximal element. The formula in the statement of Theorem 1.10 for the homology G-modules then becomes

$$
\begin{aligned}
&\tilde{H}_{\pm}(Q) - \tilde{H}_{\pm}(P)\\
&= \bigoplus_{\substack{0<x_1<\ldots<x_k<1\\ x_i\notin Q}} (-1)^k \left(\tilde{H}_{\pm}((\hat{0},x_1)_P) \otimes \tilde{H}_{\pm}((x_1,x_2)_P) \otimes \ldots \otimes \tilde{H}_{\pm}((x_k,\hat{1})_P)\right) \uparrow^G_{\mathrm{stab}(x_1,\ldots,x_k)}.
\end{aligned}
$$

In particular, if P is ranked (but not necessarily Cohen-Macaulay), Proposition 1.9 becomes

$$\tilde{H}_{\pm}(P(\underline{r})) - \tilde{H}_{\pm}(P(\underline{r-1})) = H_{r,\pm}(P),$$

where $H_{r,\pm}(P)$ denotes the virtual G-module $\oplus_{\{x\in P:\mathrm{rank}(x)=r\}}\tilde{H}_{\pm}(\hat{0},x)$.

The next computation gives an idea of how we can use this theorem to determine homology representations for the partition lattice.

Example 1.11 We use Theorem 1.10 to derive a general recurrence for the rank-selected homology representations of S_n acting on the Boolean lattice B_n. Let S be a subset of the nontrivial ranks $[1, n-1]$. Denote by $(B_n)_S$ the subposet of B_n consisting of the ranks in S together with $\hat{0}$ and $\hat{1}$, and by $\gamma_n(S)$ the corresponding homology representation of S_n. Let $S = \{s_1 < s_2 < \ldots < s_k\}$, and let $S - s_1$ denote the subset $\{s_2 - s_1, \ldots, s_k - s_1\}$ of $[1, n-s_1-1]$. (We agree that $S - s_1$ is empty if $S = \{s_1\}$.) Apply Theorem 1.10 with $P = (B_n)_S$ and $Q = (B_n)_{S\setminus\{s_1\}}$. The left-hand side of the formula becomes $-\gamma_n(S) - \gamma_n(S\setminus\{s_1\})$, while the right-hand side is

$$(-1)\left(\tilde{H}(\hat{0},x)_P \otimes \tilde{H}(x,\hat{1})_P\right) \uparrow^{S_n}_{\mathrm{stab}(x)},$$

for any representative x in the orbit of subsets of size s_1. Thus $\mathrm{stab}(x) = S_{s_1} \times S_{n-s_1}$. Clearly $\tilde{H}(\hat{0},x)_P$ is the trivial module for $\mathrm{stab}(x)$. On the other hand, the interval $(x,\hat{1})_P$ is poset-isomorphic to the rank-selected Boolean subposet $(B_{n-s_1})_{S-s_1}$, and its homology representation coincides with $1 \otimes \gamma_{n-s_1}(S-s_1)$. The inner tensor product therefore equals $1_{S_{s_1}} \otimes \gamma_{n-s_1}(S-s_1)$. Hence we obtain the recurrence

$$\gamma_n(S) + \gamma_n(S\setminus\{s_1\}) = \left(1_{S_{s_1}} \otimes \gamma_{n-s_1}(S-s_1)\right) \uparrow^{S_n}_{S_{s_1}\times S_{n-s_1}}.$$

The question of decomposing $\gamma_n(S)$ into irreducibles is answered by the following elegant rule, determined by Solomon ([So]; see also [St3]): The multiplicity of the irreducible indexed by $\lambda \vdash n$ in $\gamma_n(S)$ equals the number of standard Young tableaux of shape λ with descent set S, i.e., with the property that $i+1$ appears in a lower row than i in the tableau if and only if $i \in S$.

2. Rank-selection in the partition lattice

We are now ready to apply the methods developed above to specific cases of rank-selection in the partition lattice Π_n.

As in Section 1, write $\Pi_n(\bar{r})$ for the subposet of Π_n obtained by selecting the top r ranks of Π_n, excluding the top element of rank $(n-1)$. The ranks chosen are thus $n-1-r, n-r, \ldots, n-3, n-2$. Also, for any Cohen-Macaulay poset P, we generally suppress the homology degree and write $\tilde{H}(P)$ for the (reduced) top homology of P. Denote by $V_n(\bar{r})$ the S_n-representation on the top homology of $\Pi_n(\bar{r})$. (Define $V_n(\bar{0}) = h_n$; if $r > n-2$ or $r < 0$, set $V_n(\bar{r}) = 0$.) We have

Theorem 2.1

(1) The Frobenius characteristic of the representation $V_n(\bar{r})$ is given by the degree n term in the plethysm

$$(\pi_{r+1} - \pi_r + \ldots + (-1)^r \pi_1)[h_1 + h_2 + \ldots + h_n],$$

or equivalently in

$$(\pi_{r+2} - \pi_{r+3} + \ldots + (-1)^{n-r} \pi_n)[h_1 + h_2 + \ldots + h_n].$$

(2) The restriction of $V_n(\bar{r})$ to S_{n-1} is a permutation module, whose orbit decomposition is given by

$$\bigoplus_{\substack{\lambda \vdash (n-1), \ell(\lambda)=r+1 \\ \lambda = 1^{m_1} 2^{m_2} \ldots}} \binom{r+1}{m_1, m_2, \ldots} 1 \uparrow_{S_\lambda}^{S_{n-1}},$$

where the sum runs over all partitions λ of $n-1$ of length $\ell(\lambda) = r+1$, S_λ is the Young subgroup corresponding to λ, and m_i denotes the multiplicity of the part i in λ.

Note that (2) generalises the well-known result of [St3] that the restriction of $\tilde{H}(\Pi_n)$ to S_{n-1} is the regular representation.

Proof

Part (1) follows by first considering the implications of Proposition 1.9. This shows that, for $0 \le i \le r$, the ith Whitney homology of the dual of $\Pi_n(\bar{r})$ coincides with the ith Whitney homology of the dual of Π_n. Hence by the basic Lemma 1.1,

$$V_n(\bar{r}) = WH_{r+1}(\Pi_n^*) - WH_r(\Pi_n^*) + \ldots + (-1)^{r+1} WH_0(\Pi_n^*).$$

Using Theorem 1.4 now gives the desired plethystic formula. (Note that the representation on $WH_0(\Pi_n^*)$ is trivial.) The equivalence of the two statements is a consequence of the acyclicity of Whitney homology.

For Part (2), we use a standard symmetric functions technique: the characteristic of the restriction of $V_n(\bar{r})$ to S_{n-1} is easily obtained by taking the derivative of the expression in Part (1) with respect to the power-sum symmetric function p_1. The reader is referred to [M] for more details on manipulating plethysms. We use down and up arrows to indicate respectively restriction and induction of modules. Also we write $\mathrm{ch}(V)$ for the Frobenius characteristic of the module V. For $r = n-2$, that is, for the homology of the whole partition lattice, the restriction to S_{n-1} is the regular representation ([St3]). This is easily established by an inductive application of the above technique, using the fact that π_n is equal to the degree n term in the alternating sum $\sum_{i=1}^{n} (-1)^{i-1} \pi_{n-i}[\sum_{j\ge 1} h_j]$. (See Example 1.6 (i)). The details are explained below for the case of general r.

It is enough to compute the restriction of $WH_i(\Pi_n^*)$ to S_{n-1}. Since $\mathrm{ch}(WH_i(\Pi_n^*))$ is given by the degree n term in $\pi_{i+1}[\sum_{j\ge 1} h_j]$, the characteristic of $WH_i(\Pi_n^*)\downarrow_{S_{n-1}}$ is given by the degree $(n-1)$ term in

$$\frac{\partial}{\partial p_1} \pi_{i+1}[\sum_{j\ge 1} h_j] = p_1^i[\sum_{j\ge 1} h_j] \left(1 + \sum_{j\ge 1} h_j\right) = \left(\sum_{j\ge 1} h_j\right)^{i+1} + \left(\sum_{j\ge 1} h_j\right)^{i},$$

for $i \geq 1$, while $WH_0(\Pi_n^*)\downarrow_{S_{n-1}}$ has characteristic h_{n-1}.

Now substitute into the alternating sum formula for $V_n(\bar{r})$ in the first paragraph of the proof: the characteristic of $V_n(\bar{r})\downarrow_{S_{n-1}}$ is given by the degree $(n-1)$ term in $\left(\sum_{j\geq 1} h_j\right)^{r+1}$. This in turn simplifies into

$$\sum_{\substack{\lambda\vdash(n-1),\ell(\lambda)=r+1\\ \lambda=1^{m_1}2^{m_2}\ldots}} \binom{r+1}{m_1, m_2, \ldots} h_\lambda,$$

as claimed. $\square$

We note that computations of character values from our plethystic formula show that the representations $V_n(\bar{r})$ are not, in general, permutation modules for S_n.

Next denote by $V_n(\underline{r})$ the S_n-representation on the top homology of the subposet $\Pi_n(\underline{r})$, consisting of the first r nonzero ranks of Π_n. (Define $V_n(\underline{0}) = h_n$; if $r > n-2$ or $r < 0$, set $V_n(\underline{r}) = 0$.)

Theorem 2.2

(1) The characteristic of the representation $V_n(\underline{r})$ is given by the degree n term in the plethysm

$$(-1)^r(h_{n-r} + h_{n-r+1} + \ldots + h_n)[\pi_1 - \pi_2 + \ldots + (-1)^{n-1}\pi_n],$$

or equivalently in

$$(-1)^{r-1}(h_1 + \ldots + h_{n-r-1})[\pi_1 - \pi_2 + \ldots + (-1)^{n-1}\pi_n].$$

(2) Using down and up arrows to indicate respectively restriction and induction of modules, we have the following recursive property relating the S_n-module $V_n(\underline{r})$ to its structure as an S_{n-1}-module:

$$V_n(\underline{r})\downarrow_{S_{n-1}} \simeq V_{n-1}(\underline{r}) \oplus V_{n-1}(\underline{r-1})\downarrow_{S_{n-2}}\uparrow^{S_{n-1}},$$

for all $r = 1, \ldots, n-2$.

(3) Let σ be a permutation in S_n of type $\prod_i i^{m_i}$, that is, with m_i cycles of length i. The generating function for the character values $\mathrm{tr}(\sigma, V_n(\underline{r}))$ of σ on $V_n(\underline{r})$, $0 \leq r \leq n-2$, is

$$\sum_{r=0}^{n-2} \mathrm{tr}(\sigma, V_n(\underline{r}))\ t^r = \frac{(-1)^n}{1+t} \prod_i \prod_{k=0}^{m_i-1} \left(\sum_{d|i} \mu(d)(-1)^{\frac{i}{d}} t^{i-i/d} - ki\ t^i \right).$$

Remark 2.2.1 The expression $(1+t)\ \sum_{r=0}^{n-2} \mathrm{tr}(\sigma, V_n(\underline{r}))\ t^r$ is computed in [OS] (p. 183, Example 4.10) for small values of n. Note that this is essentially a kind of "characteristic polynomial "of the subposet of Π_n fixed by σ. Also note that in general the factors in the numerator of the right-hand side are not linear.

Remark 2.2.2 Part (3) also follows from Corollary 4.4 of [CHR].

Proof of Theorem 2.2:

Part (1) follows exactly as before, except that we now use the Lehrer-Solomon result, Theorem 1.7, in place of Theorem 1.4.

Part (2) can again be proved using symmetric functions: we compute as before that the characteristic of $V_n(\underline{r})\downarrow_{S_{n-1}}$ is the degree $(n-1)$ term in

$$(-1)^r(h_{n-1-r} + \ldots + h_{n-1})[\pi_1 - \pi_2 + \ldots + (-1)^{n-1}\pi_n]\ \left(1 - p_1 + p_1^2 - \ldots + (-1)^{n-1}p_1^{n-1}\right),$$

which is the degree $(n-1)$ term in

$$(-1)^r(h_{n-1-r}+\ldots+h_{n-1})[\pi_1-\pi_2+\ldots+(-1)^{n-2}\pi_{n-1}]$$
$$+p_1\ \ (-1)^{r-1}(h_{n-2-(r-1)}+\ldots+h_{n-2})[\pi_1-\pi_2+\ldots+(-1)^{n-2}\pi_{n-1}]\times\left(\sum_{i=0}^{n-2}(-1)^i p_1^i\right).$$

The first summand is the characteristic of $V_{n-1}(\underline{r})$. Replacing n and r by $n-1$ and $r-1$ respectively in the first line of the differentiation, one sees that the second summand is precisely p_1 times the characteristic of $V_{n-1}(\underline{r-1})\downarrow_{S_{n-2}}$, as required.

To show part (3), we use Theorem 1.7 of Lehrer and Solomon. For technical reasons which simplify the manipulations, we start with the formula

$$\sum_{i=0}^{n-1}\omega(\mathrm{ch}(WH_i(\Pi_n)))\,(-t)^{n-i}=\prod_{j\geq 1}(1-p_j)^{\ell_j(t)},\quad\text{where we have set } \ell_j(k)=\frac{1}{j}\sum_{d|j}\mu(d)k^{\frac{j}{d}}.$$

Here ω is the involution in the ring of symmetric functions which sends h_n to e_n. Note that this is simply the generating function for the Frobenius characteristics of the graded components of the Orlik-Solomon algebra of type A_{n-1}. This is equivalent to the expression derived in Corollary 4.4 of [CHR], by direct computation on the partition lattice.

Note that for positive integer values of t, $\ell_j(t)$ is the dimension of the degree j component of the free Lie algebra over t-fold tensor space, and is thus a positive integer. Hence for such t, the generating function $\sum_{i=0}^{n-1}\omega(\mathrm{ch}(WH_i(\Pi_n)))\,(-t)^{n-i}$ is given by the degree n term in

$$\prod_{j=1}^{n}\left(\sum_{m_j=0}^{\ell_j(t)}(-1)^{m_j}p_j^{m_j}\binom{\ell_j(t)}{m_j}\right),$$

and is therefore equal to

$$\sum_{\substack{\lambda\vdash n\\ \lambda=1^{m_1}2^{m_2}\ldots}}\frac{p_\lambda}{z_\lambda}\prod_i\left((-i)^{m_i}\ell_i(t)(\ell_i(t)-1)\ldots(\ell_i(t)-m_i+1)\right),\quad\text{where } z_\lambda=\prod_i i^{m_i}.$$

Since this polynomial identity in t holds for all positive integer values of t, it must hold for all t. Applying the involution ω once more, and extracting the coefficient of $\frac{p_\lambda}{z_\lambda}$, we obtain the following generating function for the trace of an element g with cycle-type λ, acting on the Whitney homology of Π_n :

$$\sum_{i=0}^{n-1}\mathrm{tr}(g,WH_i(\Pi_n))\,(-t)^{n-i}=(-1)^n\prod_i i^{m_i}\left(\ell_i(t)(\ell_i(t)-1)\ldots(\ell_i(t)-m_i+1)\right).$$

Let $P_g(t)=\sum_{i=0}^{n-1}\mathrm{tr}(g,WH_i(\Pi_n))\,t^i$. Then from the alternating sum formula of Lemma 1.1, the required generating function is given by

$$\frac{P_g(t)}{(1+t)}=\frac{1}{(1+t)}t^n(-1)^n\prod_i i^{m_i}\left(\ell_i(-t^{-1})(\ell_i(-t^{-1})-1)\ldots(\ell_i(-t^{-1})-m_i+1)\right),$$

which now simplifies to the desired result. □

These results can be used to deduce some more specific information about the representation $V_n(\underline{r})$.

Corollary 2.3 *Let $n\geq 3$. Assume as usual that $0\leq r\leq n-2$. In $V_n(\underline{r})$:*

(i) The multiplicity of the trivial representation is zero for all $r \geq 1$, while the irreducible indexed by $(n-1,1)$ appears exactly once. The first part of this statement was originally proved by Hanlon ([Ha2]), in response to a conjecture of Stanley ([St3]).
(ii) The multiplicity of the sign representation is zero for all r, while the irreducible indexed by $(2,1^{n-2})$ appears exactly once for $r = n-2$, and does not appear for $1 \leq r \leq n-3$.
(iii) $V_n(\underline{r})$ is not a permutation module for S_n, if $r \geq 2$.

Proof

Note that $\mathrm{ch}(V_n(\underline{0}) = h_n$, while (for $n \geq 3$) $\mathrm{ch}(V_n(\underline{1}) = h_2\, h_{n-2} - h_n$, by direct computation on the rank 2 poset $\Pi_n(\underline{1})$. The statements are therefore true for $r = 1$, for all $n \geq 3$.

First, it is well-known and easy to see directly from the formula of Example 1.6 for the characteristic of π_n, i.e., of $V_n(\underline{n-2})$, that the trivial (respectively sign) representation does not appear in π_n for $n > 2$ (respectively for $n \geq 2$).

We now examine the module $WH_r(\Pi_n)$. Its characteristic, from Theorem 1.7, is

$$\sum_{\substack{\lambda \vdash n,\ \ell(\lambda)=n-r \\ \lambda=(1^{m_1},2^{m_2},\ldots)}} h_{m_1}[\pi_1] e_{m_2}[\pi_2] \ldots e_{m_{2i}}[\pi_{2i}] h_{m_{2i+1}}[\pi_{2i+1}] \ldots,$$

where the sum ranges over all (integer) partitions λ of n with exactly $n-r$ parts, and m_i denotes the multiplicity of the part i.

From the preceding remarks, the contribution to the trivial representation can come only from the unique term of the form $h_{m_1}[\pi_1] e_{m_2}[\pi_2]$, where $m_1 + 2m_2 = n$, and $m_1 + m_2 = n - r$. (Note that this implies $m_2 = r$.) For a nonzero contribution to the trivial representation, one must clearly have m_2, and hence r, equal to 0 or 1. Hence the trivial representation appears exactly once in $WH_r(\Pi_n)$ for $r = 0$ or $r = 1$, and never otherwise. By Lemma 1.1, it does not appear in $V_n(\underline{r})$ for $r \geq 1$.

An even simpler analysis shows that the sign representation never appears in the Whitney homology of Π_n, hence never in $V_n(\underline{r})$, if $n \geq 2$.

Now using part (2) of Theorem 2.2, we compute, by Frobenius reciprocity,

$$\begin{aligned} < 1\uparrow_{S_{n-1}}^{S_n}, V_n(\underline{r}) > &=< 1_{S_{n-1}}, V_n(\underline{r})\downarrow_{S_{n-1}} > \\ &=< 1_{S_{n-1}}, V_{n-1}(\underline{r}) > + < 1_{S_{n-1}}, V_{n-1}(\underline{r-1})\downarrow_{S_{n-2}}\uparrow^{S_{n-1}} > \\ &=< 1\uparrow_{S_{n-2}}^{S_{n-1}}, V_{n-1}(\underline{r-1}) >, \end{aligned}$$

using the fact that the trivial representation does not appear for $r \geq 1$. Continuing the reduction down to $V_{n-r+1}(\underline{1})$, we find that $< 1\uparrow_{S_{n-1}}^{S_n}, V_n(\underline{r}) >= 1$ for all $1 \leq r \leq n-2$. But $1\uparrow_{S_{n-1}}^{S_n} = 1_{S_n} + (n-1,1)$, and the result follows.

Statement (ii) is deduced similarly, by computing the inner product with the sign representation induced from S_{n-1} up to S_n.

Part (iii) is clear from part (i). ◻

In contrast to the restricted modules $V_n(\bar{r})\downarrow_{S_{n-1}}$, which were shown to be permutation modules, we can show that the restriction of $V_n(\underline{r})$ to S_{n-1} is a permutation module iff $r = 0$ or $r = n-2$. (This is also suggested by inspection of the character values using Theorem 2.2 (iii)). Although part (iii) in the corollary below logically follows from results of Section 4, we record it here since it is in the spirit of the accompanying results.

Corollary 2.4 *Let $n \geq 3$. Assume that $1 \leq r \leq n-3$.*
(i) The restricted module $V_n(\underline{r})\downarrow_{S_{n-1}}$ is a permutation module iff $r = 0$ or $r = n-2$.
(ii) In the restriction $V_n(\underline{r})\downarrow_{S_{n-1}}$, the trivial representation appears exactly once, while the irreducible $(n-2,1)$ appears $2r$ times.

(iii) The irreducible $(n-3,2)$ (for $n \geq 5$) appears in $V_n(\underline{r})\downarrow_{S_{n-1}}$ with nonzero multiplicity $d_n(r)$ equal to

$$\frac{r}{2}(3r-2)+\frac{1}{2}\phi(r)$$

where $\phi(r)$ is equal to 1 if r is odd, 0 if $r \equiv 0 \bmod 4$, and 2 if $r \equiv 2 \bmod 4$.

Proof

(i) We proceed by contradiction. If $V_n(\underline{r})\downarrow_{S_{n-1}}$ is a permutation module, by (i) there can be only one orbit. Let G_r be a subgroup of S_{n-1} such that $V_n(\underline{r})\downarrow_{S_{n-1}} = 1\uparrow_{G_r}^{S_{n-1}}$. Let N_1 be the inner product

$$< V_n(\underline{r})\downarrow_{S_{n-1}}, 1\uparrow_{S_{n-2}}^{S_{n-1}} >,$$

and let N_2 be the inner product

$$< V_n(\underline{r})\downarrow_{S_{n-1}}, 1\uparrow_{S_{n-3}\times S_2}^{S_{n-1}} > .$$

Denote by $d_n(r)$ the multiplicity of $(n-3,2)$ in $V_n(\underline{r})\downarrow_{S_{n-1}}$. Using the recurrence of Theorem 2.2 (ii), we find that

$$N_2 = d_n(r) + N_1. \qquad (A)$$

On the other hand, a standard result from representation theory tells us that if G_r and H are subgroups of a finite group G, then the inner product $< 1\uparrow_{G_r}^{G}, 1\uparrow_{H}^{G} >$ is the number of distinct double cosets $G_r\, x\, H$ in G. The double coset $G_r\, x\, H$ is just the orbit of $x\, H$ under the action of G_r on the left cosets of H by translation. Hence we can use Burnside's formula to compute the number of orbits of this action, that is, the number of these double cosets. Write $G\backslash H$ for a distinct set of coset representatives of H in G, and $N(H)$ for the normaliser of H in G. Recall that two conjugates xHx^{-1} and yHy^{-1} either have trivial intersection or coincide, the latter iff x and y give the same left coset of $N(H)$ in G. Then the required number of double cosets is

$$\begin{aligned}
&\frac{1}{|G_r|}\sum_{g\in G}|\{x\,H : gxH = xH\}| = \frac{1}{|G_r|}|\{(g,x) : g\in G, x\in G\backslash H, g\in xHx^{-1}\}| \\
&= \frac{1}{|G_r|}|\cup_{x\in G\backslash H}\, xHx^{-1}| = \frac{1}{|G_r|}|\cup_{x\in G\backslash N(H)}\, xHx^{-1}| \\
&= \frac{1}{|G_r|}(\frac{|G|}{|N(H)|}(|H|-1)+1).
\end{aligned}$$

Now use this formula for the subgroups $H = S_{n-2}\times S_1$ and $H = S_{n-3}\times S_2$ of the group $G = S_{n-1}$. It is easy to see that in each case $N(H) = H$. Hence we have

$$N_1 = \frac{1}{|G_r|}((n-1)! - n + 2) \text{ and } N_2 = \frac{1}{|G_r|}((n-1)! - \frac{n^2-3n}{2}). \qquad (B)$$

Finally, compute $(N_2 - N_1)|G_r|$ using (A) and (B). We are forced to conclude that

$$|G_r|d_n(r) = -\frac{n^2-5n+4}{2} = -\frac{1}{2}(n-1)(n-4) \leq 0.$$

Note that $n \geq r+3 \geq 4$. If $n \geq 5$, this is negative, which is clearly impossible. For $n = 4$, the only case that needs checking is $r = 1$, for which it follows easily that $\mathrm{ch}(V_4(\underline{1})) = h_2^2 - h_4$, and $\mathrm{ch}(V_4(\underline{1})\downarrow) = 2h_2h_1 - h_3$. The character value on a 3-cycle is clearly negative.

(ii) The first statement follows from (i) of Corollary 2.3 above. Using the recurrence of part (ii), Theorem 2.2, we obtain the recurrence

$$< V_n(\underline{r})\downarrow_{S_{n-1}}, (n-2,1) >= 2+ < V_{n-1}(\underline{r-1})\downarrow_{S_{n-2}}, (n-3,1) >;$$

iterate $(r-1)$ more times, noting that for $r = 0$, the multiplicity on the left is zero.

(iii) This time we obtain the recurrence (valid only if $r \leq n-3$)

$$d_n(r) =< V_n(\underline{r})\downarrow_{S_{n-1}}, (n-3,2) >=< V_{n-1}(\underline{r}), (n-3,2) > + d_{n-1}(r-1) + 2(r-1),$$

where we have made use of (ii). The calculation of the first multiplicity on the right is carried out in Lemma 4.6 (3). We obtain $d_n(r) - d_{n-1}(r-1) = 3r - \frac{5}{2} + \frac{1}{2}(-1)^{\lfloor \frac{r}{2} \rfloor}$; now take this recurrence all the way down to $d_{n-r+1}(1) - d_{n-r}(0)$. $\square$

Remark 2.4.1 It follows from the corresponding result for the S_n-representation ℓ_n on the (1^n)-graded component of the free Lie algebra ([K]), that for $n \geq 7$, every irreducible except for (1^n) appears at least once in π_n. The exceptions are for $n = 2m$, $1 \leq m \leq 3$, in which case π_{2m} does not contain the irreducible (m, m).

Next we investigate the structure of $V_n(\underline{r})$ as an $(S_{n-2} \times S_2)$-module. This will be useful for our computations of the multiplicity of the trivial representation in Section 4. That $S_{n-2} \times S_2$ is a natural choice of subgroup can be seen by investigating the basis for homology constructed by Björner for geometric lattices. It is not hard to see that the truncated posets $\Pi_n(\underline{r})$ are in fact geometric lattices for all r. Therefore Björner's theory of non-broken circuits applies (see [Bj2], [Bj3]).

Proposition 2.5 *The following vector space decomposition holds:*

$$\tilde{H}(\Pi_n(\underline{r})) = \bigoplus_{\substack{x \in \Pi_n, x > /12/, \\ \mathrm{rank}(x) = (r+1)}} \tilde{H}(\hat{0}, x)_{\Pi_n}.$$

In particular, the characteristic of $V_n(\underline{r})\downarrow_{(S_{n-2} \times S_2)}$ *is*

$$\sum_{\substack{\lambda \vdash n, \lambda = 1^{m_1} 2^{m_2} \ldots, \\ \sum m_i = n-r-1}} \sum_{\substack{j \geq 2 \\ m_j \geq 1}} \left(\prod_{i \neq j} \prod_{i \text{ odd}} h_{m_i}[\pi_i] \prod_{i \text{ even}} e_{m_i}[\pi_i] \right) \delta_{m_j - 1}[\pi_j] \quad \left(D_{(2)}\pi_j \cdot h_2 + D_{(1^2)}\pi_j \cdot e_2 \right),$$

where δ_{m_j-1} *is* h_{m_j-1} *if* j *is odd, and equals* e_{m_j-1} *if* j *is even;* D_λ *denotes the operation of skewing by the irreducible indexed by* λ, *(as in the proof of Lemma 1.5).*

Proof

We examine the rank $(r+1)$ subposet $\Pi_n(\underline{r})$. The meet and join operations are inherited, with the caveat that for x, y in $\Pi_n(\underline{r})$, we set $x \vee y = 1$ if $x \vee y$ is not in $\Pi_n(\underline{r})$. It is easy to check that upper semi-modularity is preserved. $\Pi_n(\underline{r})$ is thus a geometric lattice of rank $(r+1)$. Clearly the independent sets of atoms of size less than $(r+2)$ are the same in both $\Pi_n(\underline{r})$ and in Π_n. We use the same atom-ordering as in the discussion preceding Theorem 1.8. To construct Björner's homology basis for $\Pi_n(\underline{r})$ we need to identify the independent sets of size $(r+1)$ which are not broken circuits in $\Pi_n(\underline{r})$. It is enough to consider whether an independent set S of size $(r+1)$, which is not a broken circuit in Π_n, can be a broken circuit in $\Pi_n(\underline{r})$. This would mean that

(i) there is an atom a preceding all the atoms in S, such that

(ii) the rank of $\vee S \vee a$ is $|S| = (r+1)$ in $\Pi_n(\underline{r})$, but equals $|\{a\} \cup S| = r+2$ in Π_n.

If S contains the atom $/12/$, clearly (i) cannot hold, and S cannot be a broken circuit in $\Pi_n(\underline{r})$. If the least atom $/12/$ is not in S, then the fact that S is not a broken circuit in Π_n automatically forces condition (ii), and thus S must be a broken circuit in $\Pi_n(\underline{r})$. We conclude that the maximal NBC sets in $\Pi_n(\underline{r})$ are those NBC subsets of Π_n, of size $(r+1)$, which contain the least atom $/12/$.

Denote by $\rho(S)$ the fundamental cycle associated to a maximal NBC set of atoms S. All NBC subsets contributing a basis cycle contain the atom $/12/$, whose stabiliser is the subgroup $(S_{n-2} \times S_2)$. The significance of this subgroup in the homology representation of $\Pi_n(\underline{r})$ is now clear.

The first statement of the proposition follows from the NBC basis B for homology constructed above.

As before, we identify $S_{n-2} \times S_2$ with the stabiliser of the atom $/12/$. If x, y are partitions lying above $/12/$, clearly they are in the same $(S_{n-2} \times S_2)$-orbit iff they have the same type λ, and the blocks containing $\{1, 2\}$ have the same size, say $i*$. When this happens we shall say that x and y have the same *refined* type. Thus a refined type consists of a pair $(i*, \lambda)$ where $i* \geq 2$, and $\lambda \vdash n$. If $\lambda = 1^{m_1} 2^{m_2} \ldots$, the stabiliser of x (and y) in this case is $G_{(i*,\lambda)} = \times_{j \neq i*} S_{m_j}[S_j] \times S_{m_{i*}-1}[S_{i*}] \times S_{i*-2} \times S_2$.

It is clear that the $(S_{n-2} \times S_2)$-action breaks up the homology module into invariant subspaces indexed by the possible refined types of x. The representation afforded by the homology of one such interval $(\hat{0}, x)$

is clearly the restriction from G_λ to the subgroup $G_{(i*,\lambda)}$ of the representation computed in Theorem 1.8. Let μ denote the integer partition of $n - m_{i*}i*$ consisting of all the parts of λ except those equal to $i*$. Then $G_\lambda = G_\mu \times S_{m_{i*}}[S_{i*}]$, while $G_{(i*,\lambda)} = G_\mu \times (S_{m_{i*}-1}[S_{i*}]) \times (S_{i*-1} \times S_2)$. Clearly the action of G_μ is unchanged, and is given by the formula of Theorem 1.8, that is, it is the outer tensor product $\otimes_{j\neq i*}\gamma_{m_j}[\pi_j]$, where γ_{m_j} is the trivial representation for j odd, and the sign representation for j even. By transitivity, it is enough to compute the action of the $S_{m_{i*}}[S_{i*}]$-module $\gamma_{m_{i*}}[\pi_{i*}]$, restricted first to $S_{m_{i*}-1}[S_{i*}] \times S_{i*}$, and then to $S_{m_{i*}-1}[S_{i*}] \times S_{i*-1} \times S_2$. We claim that the first restriction is given by

$$\delta_{m_{i*-1}}[\pi_{i*-1}] \otimes \pi_{i*}, \qquad (A)$$

where δ_j is trivial if $(j+1)$ is odd, and equals the sign otherwise. This follows from the following technical fact about wreath product modules:

Lemma 2.6 *Suppose V and W are respectively S_{a+b}- and S_n-modules. Suppose V decomposes into irreducible $(S_a \times S_b)$-modules as $\oplus_{\mu,\nu} c_{\mu,\nu}(V^\mu \otimes V^\nu)$; here the $c_{\mu,\nu}$ are integer multiplicities, and V^μ, V^ν are respectively S_a- and S_b-irreducibles. Then the restriction of the wreath product module $V[W]$ from $S_{a+b}[S_n]$ to $S_a[S_n] \times S_b[S_n]$ is given by*

$$\oplus_{\mu,\nu} c_{\mu,\nu}(V^\mu[W] \otimes V^\nu[W]).$$

Proof

The notation is as in the proof of Theorem 1.8. For convenience we write $m = a + b$. We use an $(S_a \times S_b)$-invariant decomposition

$$V = \oplus_{\mu,\nu} c_{\mu,\nu}(V^\mu \otimes V^\nu).$$

Choose a basis for V consisting of tensors of the form $v = v_i^\mu \otimes v_i^\nu$, where i ranges from 1 to $c_{\mu,\nu}$. Let (α, β) be a permutation in $(S_a \times S_b)$. Because of its canonical embedding in $S_m[S_n]$, (inherited from the embedding of $S_a \times S_b$ in S_m), the action of the element $(x_1, \ldots, x_m; (\alpha, \beta))$ of $(S_a[S_n] \times S_b[S_n])$ on a basis element $(\otimes^m w_j) \otimes (v_i^\mu \otimes v_i^\nu)$ is given by

$$\begin{aligned}
&(x_1, \ldots, x_m; (\alpha,\beta)).(\otimes^m w_j) \otimes (v_i^\mu \otimes v_i^\nu) \\
&= (\otimes_{j=1}^a x_j.v_{\alpha^{-1}(j)}) \otimes (\otimes_{j=a+1}^{a+b} x_j).v_{\beta^{-1}(j)} \otimes (\alpha.v_i^\mu) \otimes (\beta.v_i^\nu) \\
&= \left((\otimes_{j=1}^a x_j.v_{\alpha^{-1}(j)}) \otimes (\alpha.v_i^\mu)\right) \otimes \left((\otimes_{j=a+1}^{a+b} x_j.v_{\beta^{-1}(j)}) \otimes (\beta.v_i^\nu)\right).
\end{aligned}$$

The first and second factors coincide respectively with the actions of $S_a[S_n]$ on $V^\mu[W]$ and of $S_b[S_n]$ on $V^\nu[W]$. This proves the lemma. □

Returning to the proof of Proposition 2.5, claim (A) is established by applying the preceding lemma in the simplest case, when $a = n-1$ and $b = 1$.

The restriction of π_{i*} to $S_2 \times S_{i*-2}$ is given by

$$(\iota_2 \otimes D_{(2)}\pi_{i*}) + (\epsilon_2 \otimes D_{1^2}\pi_{i*}),$$

where ι_2 (respectively ϵ_2) is the trivial (respectively sign) representation.

Finally, the $(S_{n-2} \times S_2)$-action is obtained by inducing up. □

Remark 2.5.1 Note that the Betti number of $\tilde{H}(\Pi_n(\underline{r})$ is

$$\sum_{2\leq a_1<a_2<\ldots<a_r\leq n-1} a_1 a_2 \ldots a_r.$$

This follows by counting the number of maximal NBC sets in $\Pi_n(\underline{r})$ which are NBC subsets of Π_n, of size $(r+1)$, and contain the least atom /12/.

Remark 2.5.2 The preceding discussion clearly applies to any geometric lattice, the role of the atom $/12/$ being replaced by the least atom in the specified atom-ordering.

Note also that the action of S_{n-1} on the basis

$$B = \{\rho(S) : S \text{ is an NBC set of } (r+1) \text{ atoms containing the atom } /12/\},$$

reflects statement (ii) of Theorem 2.2: First, it is clear that the subset

$$B_1 = \{\rho(S) \in B : n \text{ is a singleton block in all atoms of } S\},$$

spans an invariant subspace for S_{n-1}, isomorphic to the homology of $\Pi_{n-1}(\underline{r})$. The complementary subset is

$$B_2 = \{\rho(S) \in B : \text{ there is an atom of the form } /i < n/ \text{ in } S\},$$

which also spans an S_{n-1}-invariant subspace W_2. In turn, the subset

$$B_2' = \{\rho(S) \in B_2 : \text{the atom } /n-1 < n/ \text{ is in } S\}$$

spans an invariant subspace W_2' for the subgoup S_{n-2} fixing $n-1$, which is clearly isomorphic to the homology of $\Pi_{n-1}(\underline{r-1})$, as an S_{n-2}-module. The space W_2 spanned by the basis cycles in B_2 decomposes as $W_2 = \oplus_{i=1}^{n-1}(i, n-1)W_2'$, the transpositions $\{(i, n-1)\}_{i=1}^{n-1}$ are coset representatives for S_{n-2} in S_{n-1}, and the direct summands are permuted transitively by S_{n-1}. Thus W_2 is isomorphic to $V_{n-1}(\underline{r-1})\downarrow_{S_{n-2}}\uparrow^{S_{n-1}}$.

The restrictions $D_{(2)}\pi_n$ and $D_{(1^2)}\pi_n$ have particularly simple expressions in terms of the power-sum symmetric functions, which can be derived using the expression for the characteristic of π_n (Example 1.6). For future use, we record this in

Lemma 2.7 *Let $m \leq n$. Then $D_{(m)}\pi_n$ is given by*

$$\frac{1}{n} \sum_{d|(n,m)} \mu(d)(-1)^{n-\frac{n}{d}} \binom{\frac{n}{d}}{\frac{m}{d}} p_d^{\frac{n-m}{d}}.$$

Hence if $(m_1, \ldots, m_k)$ is such that $\sum_{i=1}^k m_i = n$, then $\prod_{i=1}^k D_{(m_i)}\pi_n =< \pi_n, h_{m_1} \ldots h_{m_k} >$ is given by

$$\frac{1}{n} \sum_{d|(n,m_1,\ldots,m_k)} \mu(d)(-1)^{n-\frac{n}{d}} \binom{\frac{n}{d}}{\frac{m_1}{d}, \ldots, \frac{m_k}{d}}.$$

In particular, let $\delta_{n,\text{even}}$ equal 1 if n is even, 0 otherwise. Then

(i) $D_{(2)}\pi_n = \frac{n-1}{2} p_1^{n-2} + (-1)^{n-\frac{n}{2}+1} \frac{1}{2} p_2^{\frac{n}{2}-1} \delta_{n,\text{even}}$;

(ii) $D_{(1^2)}\pi_n = \frac{n-1}{2} p_1^{n-2} + (-1)^{\frac{n}{2}} \frac{1}{2} p_2^{\frac{n}{2}-1} \delta_{n,\text{even}}$.

Proof

It is enough to use the fact that $D_{(m)}(f)$ is the function in the variables $x_1, x_2, \ldots$, which is equal to the coefficient of x_0^m in the symmetric function $f(x_0, x_1, \ldots)$. (See [M]).

Also note that $D_{(1^m)}(f) = \omega(D_{(m)}(\omega(f)))$. □

Next we consider selecting ranks in arithmetic progression. Let $d \geq 2$, and let R_k denote the characteristic of the S_k-representation on the top homology of the subposet of Π_k obtained by selecting all ranks $k-2 \geq i \geq 1$ such that $k-i$ is a multiple of d, together with $\hat{0}$ and $\hat{1}$ adjoined. (Recall that the rank of a partition x in Π_n is n minus the number of blocks in x; the subposet therefore consists of all partitions with *number* of blocks divisible by d). For $1 \leq i \leq d-1$, define $R_i = h_i$. Note that $R_d = h_d$, the poset in this case being empty once the least and greatest elements are removed.

The following plethystic identity completely determines the characteristics R_k :

Theorem 2.8 *Assume $d \geq 2$. Then*

$$\sum_{n \geq 0} (-1)^n \sum_{i=1}^{d-1} R_{nd+i} = (h_1 - R_d + R_{2d} - \ldots)[h_1 + h_2 + \ldots].$$

Proof

Denote by $P_{k,d}$ the subposet of Π_k consisting of those ranks corresponding to number of blocks divisible by d. For clarity we consider two separate cases:

(i) Assume $k = nd$. Then the rank-set chosen is $\{d, 2d, \ldots (n-1)d\}$. Note that the $\hat{0}$-element of Π_{nd} serves as the $\hat{0}$-element for the poset $P_{nd,d}$, but a $\hat{1}$-element must be artificially adjoined. Also, the rank of $P_{nd,d}$ is n.

We compute the Whitney homology of the dual poset $P^*_{nd,d}$. As vector spaces, we have the decomposition $WH_i(P^*_{nd,d}) = \bigoplus_{\substack{x\in P_{nd,d} \\ \mathrm{rank}(x)=(n-i)}} \tilde{H}(\hat{0}, x)$. Observe that the interval $(\hat{0}, x)$, for x in $P^*_{nd,d}$ of rank $(n-i)d$, is isomorphic as a poset to $P_{id,d}$. By arguments similar to those of Theorem 1.4, we find that $\mathrm{ch}(WH_i(P^*_{nd,d}))$ is given by the degree (nd) term in the plethysm $R_{id}[\sum_{j\geq 1} h_j]$, for $1 \leq i \leq n-1$. Since $\mathrm{ch}(WH_0(P^*_{nd,d})) = h_{nd}$, and $R_k = R_{nd} = \mathrm{ch}(WH_n(P^*_{nd,d}))$, we have, by Lemma 1.1, that R_{nd} is the degree (nd)-term in the alternating sum of plethysms

$$(R_{(n-1)d} - R_{(n-2)d} + \ldots + (-1)^{n-2}R_d + (-1)^{n-1}h_1)\left[\sum_{j\geq 1} h_j\right]. \quad (A)$$

(ii) Now let $k = nd + i$, where $1 \leq i \leq d-1$. Thus the rank-set chosen is $\{i, i+d, i+2d, \ldots, i+(n-1)d\}$. Now the poset $P_{nd+i,d}$, which has rank $(n+1)$, lacks both the $\hat{0}$- and the $\hat{1}$-element, which we adjoin. Again computing the Whitney homology of the dual poset, we find that $R_{nd+i} = \mathrm{ch}(WH_{n+1}(P^*_{nd+i,d}))$, $h_{nd+i} = \mathrm{ch}(WH_0(P^*_{nd+i,d}))$, and for $1 \leq j \leq n$, $\mathrm{ch}(WH_j(P^*_{nd+i,d}))$ is given by the degree $(nd+i)$ term in the plethysm $(R_{nd} - R_{(n-1)d} + \ldots + (-1)^n h_1)[\sum_{s\geq 1} h_s]$. Hence by Lemma 1.1, R_{nd+i} is the degree $(nd+i)$ term in

$$R_{nd} - R_{(n-1)d} + \ldots + (-1)^n h_1)\left[\sum_{s\geq 1} h_s\right]. \quad (B)$$

Combining (A) and (B), we see that in the infinite series

$$(h_1 - R_d + R_{2d} - \ldots)\left[\sum_{j\geq 1} h_j\right],$$

the terms of degree nd vanish identically, while the term of degree $(nd+i)$ is $(-1)^n R_{nd+i}$, for $1 \leq i \leq d-1$. Hence the result. $\square$

Next let $Q_k(i)$ be the characteristic of the S_k action on the top homology of the subposet of Π_k obtained by selecting those ranks corresponding to partitions such that the number of blocks is congruent to i mod d. Define $Q_i(i) = h_i$ and for $1 \leq j \leq d-1$, set $Q_j(i) = h_j$.

One has the following plethystic generating functions:

Theorem 2.9 *Let $d \geq 2$.*

1. *For $i = 2, \ldots, d-1$:*

$$(h_1 - Q_i(i) + Q_{i+d}(i) - Q_{i+2d}(i) + \ldots + (-1)^n Q_{i+d(n-1)}(i) + \ldots)\left[\sum_{i\geq 1} h_i\right]$$
$$= \sum_{n\geq 0}(-1)^n \sum_{0\leq j\leq d-1, j\neq i} Q_{nd+j}(i)$$

2. *For $i = 1$:*

$$(h_1 - Q_{1+d}(1) + Q_{1+2d}(1) - \ldots + (-1)^{n-1} Q_{1+(n-1)d} + \ldots) \left[\sum_{i\geq 1} h_i\right]$$
$$= \sum_{n\geq 1} (-1)^{n-1} \sum_{0\leq j\leq d-1, j\neq 1} Q_{nd+j}(1).$$

Proof

Essentially the same analysis as in the previous theorem. The difference in sign between the case $i \neq 1$ and $i = 1$ arises from the fact that in the second case, the subposet inherits the $\hat{1}$-element from the partition lattice. □

We turn now to our first application of Theorem 1.10. The reader is referred to the notation introduced in the proof of Theorem 1.4.

Theorem 2.10 *Let Q_k denote the poset obtained from Π_n ($n \geq 4$) by deleting all elements at rank k. The characteristic of S_n acting on the top homology of Q_k is given by the degree n term in the symmetric function*

$$(-1)^k \pi_{n-k}[\pi_1 - \pi_2 + \ldots + (-1)^{n-1}\pi_n] - \pi_n.$$

Proof

From the formula of Theorem 1.10, we have

$$\tilde{H}(Q_k) \oplus \tilde{H}(\Pi_n) = \bigoplus_{\substack{0<x<1 \\ \mathrm{rank}(x)=k}} \left(\tilde{H}(\hat{0},x)_{\Pi_n} \otimes \tilde{H}(x,\hat{1})_{\Pi_n}\right) \uparrow_{\mathrm{stab}(x)}^{S_n},$$

where the sum runs over one x for each distinct stabiliser subgroup, that is, we take one x in each S_n-orbit.

If x has type λ, then $\mathrm{stab}(x) = G_\lambda$, and the sum above is indexed by all the integer partitions λ of n with $(n-k)$ parts. The G_λ-module structure of the homologies of the intervals $(\hat{0}, x)$ and $(x, \hat{1})$ are given by the first formula of Theorem 1.7 and the proof of Theorem 1.4 respectively. More precisely, if m_i denotes the multiplicity of the part i in the type λ of x, (so that $\sum m_i = n-k$), then

$$\tilde{H}(\hat{0},x) = h_{m_1}[\pi_1] \otimes e_{m_2}[\pi_2] \otimes \ldots \otimes e_{m_{2i}}[\pi_{2i}] \otimes h_{m_{2i+1}}[\pi_{2i+1}] \otimes \ldots, \qquad (A)$$

where, following the notation of the proof of Theorem 1.4, the tensors indicate that this is an outer tensor product. Similarly, if $\pi_{n-k}\downarrow_{S_{m_1}\times\ldots\times S_{m_i}\times\ldots}$ decomposes so that the irreducible indexed by the vector of partitions $\underline{\mu} = (\mu^1, \ldots, \mu^i, \ldots)$ appears with multiplicity $c_{\underline{\mu}}$, then

$$\tilde{H}(x,\hat{1}) = \sum_{\underline{\mu}} c_{\underline{\mu}}\, s_{\mu^1}[h_1] \otimes s_{\mu^2}[h_2] \otimes \ldots \otimes s_{\mu^i}[h_i] \otimes \ldots. \qquad (B)$$

(Recall that this representation was denoted by $\pi_{n-k}\downarrow_{S_{m_1}\times\ldots\times S_{m_i}\times\ldots}[h_1 \otimes \ldots \otimes h_i \otimes \ldots]$ in the proof of Theorem 1.4). It remains to compute the *inner* tensor product of these two representations. For this it suffices to note the following facts: Let f_1, f_2 (respectively g_1, g_2) be Frobenius characteristics for representations of S_m (respectively S_n). As in [M], we use $*$ to denote the internal product of symmetric functions (i.e., the inner tensor product of the corresponding representations). Then

(i) the inner tensor product of the $(S_m \times S_n)$-representations $(f_1 \otimes g_1)$ and $(f_2 \otimes g_2)$ is given by

$$(f_1 \otimes g_1) \; * \; (f_2 \otimes g_2) = (f_1 * f_2) \otimes (g_1 * g_2);$$

(ii) by examining wreath product modules as in Lemma 2.6, one can show that the inner tensor product of the $S_m[S_n]$-representations $f_1[g_1]$ and $f_2[g_2]$ is given by the plethysm

$$(f_1[g_1]) \ * \ (f_2[g_2]) = (f_1 * f_2)[g_1 * g_2];$$

(iii) $h_n * f = f$ and $e_n * f = \omega(f)$ for f of degree n, and

(iv) $f[-g] = (-1)^{\deg(f)}\omega(f)[g]$.

Now use the linearity of the operation $*$ to compute, as a G_λ-module, the inner tensor product of (A) and (B), giving:

$$\begin{aligned}
&\sum_{\underline{\mu}} c_{\underline{\mu}} \ \left(\otimes_i s_{\mu^{2i-1}}[\pi_{2i-1}] \ \otimes \ \otimes_i \omega(s_{\mu^{2i}})[\pi_{2i}]\right) \qquad \text{by } (i)-(iii),\\
=&\sum_{\underline{\mu}} c_{\underline{\mu}} \ \left(\otimes_i s_{\mu^{2i-1}}[\pi_{2i-1}] \ \otimes \ \otimes_i (-1)^{m_{2i}} s_{\mu^{2i}}[-\pi_{2i}]\right) \qquad \text{by } (iv),\\
=&(-1)^k \ \pi_{n-k}\big\downarrow_{S_{m_1}\times\ldots\times S_{m_i}\times\ldots}[\pi_1 \otimes (-\pi_2) \otimes \ldots \otimes (-1)^{i-1}\pi_i \otimes \ldots].
\end{aligned}$$

Finally, we have to sum such expressions over all set partitions x at rank k, taking one x of each type. Collecting terms first by the type, and then summing over all types, i.e., over all integer partitions λ of n with $(n-k)$ parts, we obtain, by Lemma 1.5, that the result of the induction up to S_n is the degree n term in the plethysm $(-1)^k\pi_{n-k}[\sum_{i=1}^n(-1)^{i-1}\pi_i]$. The result follows. $\square$

We record a consequence of the preceding calculations for future use:

Remark 2.10.1 Let $\lambda = \prod_i i^{m_i}$ be an integer partition, and let f and g be symmetric functions of degree $\sum_i m_i$ (corresponding to (virtual) representations of S_{m_i}) . Let ρ_i and τ_i be two series of symmetric functions of degree i, $i \geq 1$, (corresponding to (virtual) representations of S_i). Then the inner tensor product of G_λ-modules corresponding to $f\big\downarrow_{\times_i S_{m_i}}[\rho_1 \otimes \rho_2 \otimes \ldots]$ and $g\big\downarrow_{\times_i S_{m_i}}[\tau_1 \otimes \tau_2 \otimes \ldots]$ evaluates according to the rule:

$$f\big\downarrow_{\times_i S_{m_i}}[\rho_1 \otimes \rho_2 \otimes \ldots] \ * \ g\big\downarrow_{\times_i S_{m_i}}[\tau_1 \otimes \tau_2 \otimes \ldots] = (f \ * \ g)\big\downarrow_{\times_i S_{m_i}}[(\rho_1 \ * \ \tau_1) \otimes (\rho_2 \ * \ \tau_2) \otimes \ldots].$$

Example 2.11 We illustrate by computing the homology representation of the subposet obtained by deleting all the atoms. Thus $k = 1$, and we need to compute the degree n term in $\pi_{n-1}[\sum_{i\geq 1}(-1)^{i-1}\pi_i]$.

Recalling the notation in Remark 1.4.0, this is given by the sum over all partitions λ of n into $(n-1)$ parts, of terms of the form

$$\left(\pi_{n-1}\big\downarrow_{S_{m_1}\times S_{m_2}\times\ldots}[\pi_1 \otimes (-\pi_2) \otimes \ldots]\right)\big\uparrow^{S_n}_{\times_i S_{m_i}[S_i]},$$

where m_i is the multiplicity of i in λ. Clearly the only partition of n into $(n-1)$ parts is $(1^{n-2}, 2)$, and thus (see Lemma 1.5) we need to compute

$$\left(\pi_{n-1}\big\downarrow_{S_{n-2}\times S_1}[\pi_1 \otimes (-\pi_2)]\right)\big\uparrow^{S_n}_{S_{n-2}[S_1]\times S_1[S_2]}.$$

But $\pi_{n-1}\big\downarrow_{S_{n-2}\times S_1} = p_1^{n-2} \otimes p_1$, and the preceding expression reduces to

$$(p_1^{n-2}[\pi_1] \otimes p_1[-\pi_2])\big\uparrow^{S_n} = p_1^{n-2} \ (-\pi_2) = -p_1^{n-2} \ h_2.$$

The Frobenius characteristic is therefore $p_1^{n-2} \ h_2 - \pi_n$.

For $1 \leq k \leq r \leq n-2$ define symmetric functions $\beta_{n-k}([1, r-k])$ by the formula

$$\beta_{n-k}([1,r-k]) = (-1)^{r-k}(h_{n-r} + h_{n-r+1} + \ldots + h_{n-k}) \left[\pi_1 - \pi_2 + \ldots + (-1)^{n-k-1}\pi_{n-k}\right] |_{\deg(n-k)}.$$

Comparison with Theorem 2.2 shows that $\beta_{n-k}([1, r-k])$ is in fact the characteristic of the S_{n-k}-module $V_{n-k}(\underline{r-k})$. Thus $g_n^r = \mathrm{ch}(V_n(\underline{r})) = \mathrm{ch}(V_n([1,r]))$, while $g_{n-r}^r = h_{n-r}$.

The proof of Theorem 2.10 generalises easily to give

Corollary 2.12 *The homology representation of the subposet of Π_n consisting of the interval $[1, r]$ with the rank k removed, for $r > k \geq 1$, is given by*

$$(-1)^k \beta_{n-k}([1, r-k]) \left[\pi_1 - \pi_2 + \ldots\right] |_{\deg n} - \beta_n([1, r]).$$

We come now to our most general formula for the homology representation.

Theorem 2.13 *Let $S = \{s_1 < s_2 < \ldots < s_r\}$ be a subset of the ranks $\{1, \ldots, n-2\}$ of Π_n, and denote by $\beta_n(S)$ the Frobenius characteristic of the homology of the rank-selected subposet of Π_n corresponding to the set S. We have the recurrence*

$$\beta_n(S) + \beta_n(S \backslash \{s_1\}) = \beta_{n-s_1}(S - s_1) \left[\sum_{i \geq 1} h_i\right] |_{\deg n},$$

where $S - s_1$ denotes the subset of ranks $\{s_2 - s_1 < s_3 - s_1 < \ldots < s_r - s_1\}$ in Π_{n-s_1}.

Proof

We remind the reader that the right side of this recurrence is evaluated as in Remark 1.4.0 and Lemma 1.5; the exact formula is

$$\sum_{\substack{\lambda \vdash n \\ \lambda = \prod i^{m_1}, \sum m_i = n - s_1}} \left(\beta_{n-s_1}(S - s_1) \downarrow_{\times_i S_{m_i}} [h_1 \otimes h_2 \otimes \ldots]\right) \uparrow_{\times_i S_{m_i}[S_i]}^{S_n}.$$

The proof is another application of Theorem 1.10, reduced to deleting one rank. The details of the calculation closely follow the proof of Theorem 2.10.

The fact that we delete the bottom-most rank simplifies the calculation, for then the contribution to the inner tensor product from the term $\tilde{H}(\hat{0}, x)_P$, where P is the poset $\Pi_n(S)$, and x has rank s_1 in Π_n, is simply the trivial representation of $\mathrm{stab}(x)$. (If x has type λ with m_i parts of size i, then this representation has characteristic

$$h_{n-s_1} \downarrow_{\times_i S_{m_i}} [h_1 \otimes h_2 \otimes \ldots]).$$

□

As an application of the preceding recurrence, we turn to the case of segments of consecutive ranks. For $0 \leq r < s \leq n-2$, denote by $V_n([r+1, s])$ the reduced top homology of the subposet of Π_n obtained by selecting all elements in ranks $r+1, r+2, \ldots, s$. By convention $V_n([0, s]) = 0$, while $V_n([s+1, s])$ is the trivial module. The general recurrence for rank-selected homology gives the following recursive formula for computing $V_n([r+1, s])$ as an S_n-module.

Corollary 2.14 *With the preceding definitions, writing $\beta_n([a, r])$ for the Frobenius characteristic of the S_n-module $V_n([a, r])$, we have, for $0 \leq a \leq r \leq n-2$:*

$$\beta_n([a+1, r]) + \beta_n([a, r]) = \beta_{n-a}([1, r-a]) \left[\sum_{i \geq 1} h_i\right] |_{\deg n}.$$

This, together with Lemma 1.1, gives the formulas

$$
\begin{aligned}
&\beta_n([a,r]) \\
&= (\beta_{n-a}([1,r-a]) - \beta_{n-a-1}([1,r-a-1]) + \ldots + (-1)^{r-a}\beta_{n-r}(\emptyset)\left[\sum_{i\geq 1} h_i\right]|_{\deg n} - (-1)^{r-a}h_n \\
&= (\beta_{n-a+1}([1,r-a+1]) - \beta_{n-a+2}([1,r-a+2]) + \ldots + (-1)^{k-1}\beta_{n-a+k}([1,r-a+k]) + \ldots \\
&+ (-1)^{a-1}\beta_n([1,r]))\left[\sum_{i\geq 1} h_i\right]|_{\deg n}.
\end{aligned}
$$

Example 2.15 We record two particular cases of rank-selection that are easily derived; the first by inspection, and the second from the preceding arguments.

0. The characteristic $\beta_n(\{a\})$ of the homology of $\Pi_n\{a\} = \{x \in \Pi_n : x \text{ is of rank } a\}$ is

$$h_{n-a}\left[\sum_{i\geq 1} h_i\right]|_{\deg n} - h_n.$$

1. Let $a < b$ be two ranks chosen among the $n-2$ nontrivial ranks $\{1, 2, \ldots, n-2\}$ of Π_n. Denote the corresponding rank-selected subposet by $\Pi_n\{a,b\}$. The characteristic $\beta_n(\{a,b\})$ of the homology representation of $\Pi_n\{a,b\}$ satisfies the recurrence

$$\beta_n(\{a,b\}) + \beta_n(\{b\}) = \beta_{n-a}(\{b-a\})\left[\sum_{i\geq 1} h_i\right]|_{\deg n}.$$

We end this section with a brief discussion of Betti numbers. Write $d_n(S)$ for the Betti number of the rank-selected subposet $\Pi_n(S)$. Note that $d_n(S)$ is the dimension of the top homology of $\Pi_n(S)$. Theorem 2.13 gives the following recurrence for these numbers:

Proposition 2.16 *Let $S(n,k)$ denote the Stirling number of the second kind, which counts the number of set partitions in Π_n with k blocks. Then*

$$d_n(S) + d_n(S\backslash\{s_1\}) = d_{n-s_1}(S-s_1)\ S(n, n-s_1).$$

The formulas we have derived allow us to compute more explicitly some of these Betti numbers.

Proposition 2.17 *Let $c(n,k)$ denote the number of permutations in S_n whose cycle decomposition consists of k cycles.*

(i) $d_n([n-1-r, n-2]) = (r+1)!\ S(n-1, r+1)$.

(ii) $d_n([1,r])$ *satisfies the following additional recurrences:*

$$d_n([1,r]) + d_n([1,r-1]) = c(n, n-r);$$

and

$$d_n([1,r]) = d_{n-1}([1,r]) + (n-1)d_{n-1}([1,r-1]);$$

(iii) $d_n([1,n-2]\backslash\{k\} = (n-k-1)!c(n,n-k) - (n-1)!$.

Proof

(i) follows from Theorem 2.1 (2).
(ii) follows from Proposition 1.9 (note that the dimension of the rth Whitney homology of Π_n is precisely $c(n, n-r)$), and from Theorem 2.2 (2).
(iii) follows from the formula of Theorem 2.10. ☐

Finally we note that using the theory of EL-labellings, and Gessel's labelling for Π_n, we also have the following general expression for $d_n(S)$:

Proposition 2.18 *([ES, Theorem 1.7]) Let $Des(\sigma)$ denote the descent set of a permutation σ, i.e., the set $\{i : \sigma(i) > \sigma(i-1)\}$. Let $\sigma^*(k)$ be one more than the number of indices i less than k for which $\sigma(i) < \sigma(k)$. Then*

$$d_n(S) = \sum_{\substack{\sigma \in S_{n-1} \\ Des(\sigma)=S}} \prod_{i=1}^{n-1} \sigma^*(i).$$

3. The action of S_n on chains in the partition lattice

Denote by $\alpha_n(S)$ the permutation representation of S_n acting on the maximal chains of the rank-selected subposet $\Pi_n(S)$ of Π_n corresponding to the subset S, (with $\hat{0}$ and $\hat{1}$ adjoined, as usual). A standard Hopf trace formula computation shows that

$$\beta_n(T) = \sum_{S \subseteq T} \alpha_n(S)\,(-1)^{|T|-|S|},$$

thereby expressing the homology module as an alternating sum of the permutation modules of chains. (This holds for any Cohen-Macaulay poset). In contrast, Theorem 2.13 gives an expression for the homology modules β_n as an alternating sum of *homology* modules of lower degree.

The permutation representation $\alpha_n(S)$ is consequently the sum of the homology modules $\beta_n(T)$, as T ranges over subsets of S. That is, the homology modules $\beta_n(T)$, $T \subseteq S$, refine the permutation module $\alpha_n(S)$ of maximal chains in $\Pi_n(S)$. This permutation representation is frequently interesting in its own right. In the case of the partition lattice, Theorem 2.10 has the following corollary:

Proposition 3.1 $\alpha_n(S) = \alpha_{n-s_1}(S - s_1) \left[\sum_{i\geq 1} h_i\right] \big|_{\deg n}$.

(By abuse of notation, we have used $\alpha_n(S)$ to denote the characteristic of the representation on the maximal chains of $\Pi_n(S)$.)

Hence the dimensions $A_n(S)$ of the representations $\alpha_n(S)$ satisfy the recurrence

$$A_n(S) = A_{n-s_1}(S - s_1)\ S(n, n - s_1).$$

When $S = [1, n-2]$, this confirms the fact that the total number of chains in Π_n is $\frac{n!(n-1)!}{2^{n-1}}$.

In particular, we obtain an elegant orbit decomposition for the action α_n of S_n on the maximal chains of Π_n :

Theorem 3.2

$$\alpha_n = \bigoplus_{i=1}^{\lfloor n/2 \rfloor} a_i(n)\ \ 1\Big\uparrow_{S_2^i \times S_1^{n-2i}}^{S_n},$$

where the $a_i(n)$ are determined by the recurrence

$$a_i(n+1) = i a_i(n) + (n - 2i + 2) a_{i-1}(n),$$

with initial conditions $a_0(1) = 1 = a_1(2)$, $a_0(n) = 0$, $n > 1$, and $a_i(n) = 0$ if $2i > n$. *(Observe that the character values are supported on the set of involutions).*

Proof

We shall show first that α_n admits the following recursive description:

$$\alpha_n = \left(1_{S_2} \otimes \alpha_{n-1}\downarrow_{S_{n-2}}^{S_{n-1}}\right) \uparrow^{S_n} \qquad (A).$$

When S is the full set of ranks, Proposition 3.1 says that α_n is the degree n term in the plethysm $\alpha_{n-1}[\sum_{i\geq 1} h_i]$. The computation is now almost identical to the one of Example 2.11. The only partition of n with $n-1$ parts is $(1^{n-2}, 2)$.

Since $\alpha_{n-1}\downarrow_{S_{n-2}\times S_1}^{S_{n-1}} = \alpha_{n-1}\downarrow_{S_{n-2}} \otimes h_1$,, we compute that

$$\alpha_n = \alpha_{n-1}\downarrow_{S_{n-2}}[h_1]\ h_1[h_2] = \alpha_{n-1}\downarrow_{S_{n-2}}\ h_2, \quad \text{as claimed.}$$

Now proceed by induction. Note first that clearly $\alpha_i = h_i$ for $i = 1, 2$. In terms of symmetric functions we need to show that

$$\alpha_n = \sum_{i=1}^{\lfloor n/2 \rfloor} a_i(n)\ \ h_2^i h_1^{n-2i}, \qquad (B)$$

where the $a_i(n)$ satisfy the stated recurrence.

Assume this holds for n. Recalling that the characteristic of the restriction of α_n is given by $\frac{\partial}{\partial p_1}\alpha_n$, we may translate (A) in terms of symmetric functions as follows:

$$\alpha_{n+1} = h_2\ \frac{\partial}{\partial p_1}\alpha_n.$$

Now use the inductive hypothesis (B) to obtain:

$$\begin{aligned}\alpha_{n+1} =& h_2 \left(\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} a_i(n)\, i h_2^{i-1} h_1 h_1^{n-2i} \ + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor -1} a_i(n)\,(n-2i) h_1^{n-2i-1} h_2^i\right)\\ =& \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} a_i(n+1) h_2^i h_1^{n-2i+1};\end{aligned}$$

comparing coefficients, we find that $a_i(n+1) = i a_i(n) + (n-2i+2) a_{i-1}(n)$. ☐

The recurrence for the $a_i(n)$ is remarkably similar to the one satisfied by the Eulerian numbers $A(n,i)$ (which count the number of permutations in S_n with i descents). As observed jointly with R. Simion, it can be verified that this recurrence is compatible with the following combinatorial description for the multiplicity $a_i(n)$: It is the number of permutations in S_{n-2} with exactly $i-1$ descents, none consecutive, and with the "hereditary"property that when the letters $n-2$, $n-3, \ldots 3, 2, 1$ are erased in succession, the property of non-consecutive descents is preserved after each erasure. A slightly different but equinumerous set of permutations appears in work of Foata and Schützenberger ([FSch]), who call them André permutations. In fact in two subsequent papers ([FS1], [FS2]), Foata and Strehl construct an action of S_2^{n-1} on the set of $n!$ permutations whose orbit decomposition is remarkably reminiscent of ours. The Foata-Strehl action explains the following

Remark 3.2.1 Let c_n denote the total number of maximal chains in Π_n, so that $c_n = \frac{n!(n-1)!}{2^{n-1}}$. Equating dimensions in the statement of Theorem 3.2, we obtain the identity

$$(n-1)! = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} a_i(n) 2^{n-1-i}.$$

Note also that if we count maximal chains recursively by first fixing the element of rank $(n-2)$ in the chain, (as in [St3, Theorem 7.7]), we obtain the recurrence

$$c_n = \frac{1}{2}\sum_{i=1}^{n-1}\binom{n}{i}\binom{n-2}{i-1}c_i c_{n-i}.$$

(Here the first binomial coefficient counts the number of choices for the element at rank $(n-2)$, while the second counts the number of ways to intertwine two chains in Π_i and Π_{n-i} to obtain a chain in Π_n.)

The precise connection between the Foata-Strehl representation $W_{FS}(n)$ of S_2^{n-1} and the action of S_n on the maximal chains of Π_n is as follows:

Proposition 3.3

$$\alpha_{n+1}\uparrow_{S_{n+1}}^{S_{2n}} = (W_{FS}(n)\otimes 1_{S_2})\uparrow_{S_2{}^{n-1}\times S_2}^{S_{2n}}.$$

In [St3, Theorem 7.7], Stanley shows that the multiplicity of the trivial representation in α_n is the Euler number E_{n-1}. It follows from Theorem 3.2 that the André permutations in S_{n-1} counted by $a_i(n)$ refine the number of alternating permutations E_{n-1} in S_{n-1}. Such refinements are the subject of the papers [FSch], [FS1], [FS2]. In particular, in [FSch] the authors give explicit bijections to explain these refinements. The recurrences of [FSch] for the numbers $a_i(n)$ can all be derived by counting orbits of maximal chains fixed by subgroups of the form $S_2^i \times S_1^{n-2i}$ in Π_n.

Since the rank-selected homologies $\beta_{n+1}(S)$ (in Π_{n+1}) refine the S_{n+1}-module α_{n+1}, the multiplicities $b_S(n+1)$ of the trivial representation in the homology $\beta_{n+1}(S)$ provide another refinement of the Euler number E_n into nonnegative integers indexed by subsets $S \subseteq \{1,2,\ldots,n-1\}$. More generally, if we write $b_\lambda(S)$ for the multiplicity of the irreducible indexed by the partition λ of n in the homology of the rank-selected subposet $\Pi_n(S)$, and $A_n(\lambda)$ for the multiplicity of the irreducible λ in α_n, then it follows that $\sum_S b_\lambda(S) = A_n(\lambda)$, where the sum runs over all subsets of $\{1,2,\ldots,n-2\}$.

We use Theorem 3.2 to compute the multiplicities of certain irreducibles $A_n(\lambda)$ in the permutation representation α_n.

Proposition 3.4

0. *([St3]) $A_n((n)) = E_{n-1}$.*
1. *The multiplicity of the trivial representation in the restriction of α_n to S_{n-1} is the Euler number E_n. Equivalently, $A_n((n-1,1)) = E_n - E_{n-1}$.*
2. *$A_n((2,1^{n-2})) = 1$ and $A_n((1^n)) = 0$ for all $n \geq 2$; $A_n(\lambda) \geq 2$ otherwise.*
3. *$A_n((3,1^{n-3})) = 2^{n-2}-1$ and $A_n((2^2,1^{n-4})) = 2^{n-2}-2$.*
4. *$A_n((2^r,1^{n-2r})) = \sum_{i=0}^{r-1} a_{r-i}(n) f^{(2^i,1^{n-2i})}$, $r \geq 1$, where $f^{(2^r,1^{n-2r})}$ is the number of standard Young tableaux of shape $(2^r,1^{n-2r})$.*

Proof

1. By Frobenius reciprocity, we compute

$$E_n =< \alpha_{n+1}, 1_{S_{n+1}} > =< 1_{S_2}\otimes \alpha_n\downarrow_{S_{n-1}}, 1_{S_2}\otimes 1_{S_{n-1}} >$$
$$=< \alpha_n, 1_{S_n} > + < \alpha_n, (n-1,1) > .$$

2. That the sign representation does not appear is clear from the orbit decomposition of Theorem 3.2. Computing the restriction of $(2,1^{n-2})$ to $S_2 \times S_{n-2}$, we have, by Frobenius reciprocity,

$$A_n(2,1^{n-2}) =< \alpha_{n-1}\downarrow_{S_{n-1}}, (1^{n-2}) >= A_{n-1}(2,1^{n-3}) + A_{n-1}(1^{n-1}) = A_{n-1}(2,1^{n-3}).$$

The result now follows by induction. The second statement is clear from the Pieri rule.

3. Using the Pieri rule to compute $(3, 1^{n-3})\downarrow_{S_{n-2}\times S_2}$, we find

$$\begin{aligned} A_n((3,1^{n-3})) &=< \alpha_{n-1}\downarrow_{S_{n-2}}, (2,1^{n-4}) + (1^{n-2}) > \\ &=< \alpha_{n-1}, (2,1^{n-4})\uparrow^{S_{n-1}} > + \alpha_{n-1}, (1^{n-2})\uparrow^{S_{n-1}} > \\ &=< \alpha_{n-1}, (3,1^{n-4}) + (2^2,1^{n-5}) + 2\,(2,1^{n-3}) + (1^{n-1}) > \\ &= A_{n-1}((3,1^{n-4})) + A_{n-1}((2^2,1^{n-5})) + 2, \quad \text{by 2.} \qquad (A) \end{aligned}$$

On the other hand, computing $(2^2, 1^{n-4})\downarrow_{S_{n-2}\times S_2}$, we find

$$\begin{aligned} A_n((2^2,1^{n-4})) &=< \alpha_{n-1}\downarrow_{S_{n-2}}, (2,1^{n-4}) > \\ &=< \alpha_{n-1}, (3,1^{n-4}) + (2^2,1^{n-5}) + (2,1^{n-3}) > \\ &= A_{n-1}((3,1^{n-4})) + A_{n-1}((2^2,1^{n-5})) + 1, \quad \text{by 2.} \qquad (B) \end{aligned}$$

Subtracting (B) from (A), we have

$$A_n((3,1^{n-3})) = A_n((2^2,1^{n-4})) + 1. \qquad (C)$$

Now (A) and (C) together give the recurrence

$$A_n((3,1^{n-3})) = 2\ A_{n-1}((3,1^{n-4})) + 1,$$

from which the result follows.

4. Follows from the Littlewood-Richardson rule. □

The preceding proposition suggests that the irreducible $(3, 1^{n-3})$ appears exactly once in each rank-selected homology corresponding to a nonempty subset S. This is, however, false, as can be seen by examining the top homology π_n itself: in π_6, this irreducible appears twice. There is consequently a different refinement of 2^{n-2}, also indexed by subsets. We also conclude that the irreducible $(2, 1^{n-2})$ appears exactly once in π_n, and never in any $\beta_n(S)$ for $S \neq \{1, 2, \ldots, n-2\}$.

In view of Part 1. of Propostion 3.4, it is also of enumerative interest to determine the restrictions of the rank-selected homology modules of Π_n to S_{n-1}. For, denoting by $b'_n(S)$ the multiplicity of the trivial representation in this restricted rank-selected homology module $\beta_n(S)\downarrow^{S_n}_{S_{n-1}}$, one has a *second* refinement of the Euler number E_n into nonnegative integers, now indexed by subsets $S \subseteq \{1, 2, \ldots, n-2\}$. Note that $b'_n(S)$ is always greater than or equal to $b_n(S)$.

It follows from Proposition 3.2 that the character values of the chain representation α_n are positive integers which are supported on involutions, and that they satisfy the following recurrence. (This recurrence was obtained by a different calculation in [St3]). Write $\chi_n(i)$ for the value of the character on the involution with i transpositions. Then for $0 \leq i \leq \lfloor\frac{n+1}{2}\rfloor$,

$$\chi_{n+1}(i) = \binom{n-2i+1}{2}\chi_n(i) + i\,\chi_n(i-1).$$

By computing the multiplicity $< \alpha_n, 1_{S_n} >$ using the character values, we find that

$$E_{n-1} = \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor} \frac{\chi_n(i)}{2^i\, i!\,(n-2i)!}.$$

The rational numbers in the summand are the numbers $g(i, n)$ in [St3, Proposition 7.9].

4. The multiplicity of the trivial representation

Using the results and methods of the first two sections, one can also obtain information about the multiplicity of the trivial representation (and other irreducibles in specific instances) in the rank-selected homology. We have already seen (Corollary 2.3) how to deduce Hanlon's result that the trivial representation does not appear in $V_n(\underline{r})$, for any $r \geq 2$. We shall present two other criteria for a rank-selected subposet whose homology module does not contain the trivial module. We start with a lemma that will be crucial to our computations. Again, we refer the reader to the notational conventions of Remark 1.4.0.

Lemma 4.1 *Let f be a symmetric function of degree k, let $\lambda \vdash n$, $\ell(\lambda) = k$, $\lambda = \prod_i i^{m_i}$. As usual let G_λ be the subgroup $\times_i S_{m_i}[S_i]$ of S_n.*

(1) Consider the (possibly virtual) S_n-module V_λ whose Frobenius characteristic is

$$f\big\downarrow_{\times_i S_{m_i}}[\pi_1 \otimes (-\pi_2) \otimes \ldots \otimes (-1)^{i-1}\pi_i) \otimes \ldots]\big\uparrow_{G_\lambda}^{S_n}.$$

Then the multiplicity of the trivial module in V_λ is 0 unless $2k \geq n$ and λ is a partition $(1^{m_1}, 2^{m_2})$, whose parts are 1 or 2, in which case it equals

$$(-1)^{m_2} < f\ ,\ h_{m_1}\ e_{m_2} >_{S_k} = (-1)^{m_2} < f\ ,\ H_{m_2} + H_{m_2-1} > .$$

Here we have denoted by H_r the Schur function corresponding to the hook $(1^r, k-r)$.

(2) Now let V_λ be the (possibly virtual) S_n-module whose characteristic is

$$f\big\downarrow_{\times_i S_{m_i}}[h_1 \otimes h_2 \otimes \ldots]\big\uparrow_{G_\lambda}^{S_n}.$$

Then the multiplicity of the trivial representation in V_λ is

$$< f, h_{m_1}\ h_{m_2}\ \ldots\ h_{m_i}\ \ldots >_{S_k}.$$

Proof

(1) By Frobenius reciprocity, $< V_\lambda, 1_{S_n} >$ equals

$$< f\big\downarrow_{\times_i S_{m_i}}[\pi_1 \otimes (-1)\pi_2 \otimes \ldots]\ ,\ h_{m_1}[h_1] \otimes h_{m_2}[h_2] \otimes \ldots >_{G_\lambda}. \qquad (A)$$

Suppose $f\big\downarrow_{\times_i S_{m_i}}$ decomposes into irreducibles as $\sum_{\bar\mu} c_{\bar\mu}\left(\otimes_i s_{\mu^i}\right)$, where $\bar\mu$ is the vector of partitions $(\mu^1, \mu^2, \ldots)$, $\mu^i \vdash m_i$. Continuing the computation in (A), we have

$$< V_\lambda, 1_{S_n} > = < \sum_{\bar\mu} c_{\bar\mu}\ \left(\otimes_i s_{\mu^i}[(-1)^{i-1}\pi_i]\right)\ ,\ h_{m_1}[h_1] \otimes h_{m_2}[h_2] \otimes \ldots >_{G_\lambda}$$

$$= \sum_{\bar\mu} c_{\bar\mu} \prod_i < s_{\mu^i}[(-1)^{i-1}\pi_i], h_{m_i}[h_i] >_{S_{m_i}[S_i]}.$$

From the rule for expanding a plethysm $g[f_1 + f_2]$, it is clear that the nonzero terms in the above sum occur only for those i such that π_i contains the trivial representation. But this happens only for $i = 1, 2$, (Corollary 2.3). Hence $m_i = 0$ unless $i \leq 2$, and λ is of the form $(1^{m_1}, 2^{m_2})$. Since π_i is the trivial module for $i = 1, 2$, the multiplicity $< V_\lambda, 1_{S_n} >$ equals

$$\sum_{\bar\mu=(\mu^1,\mu^2)} c_{\bar\mu}\ < s_{\mu^1}[\pi_1],\ h_{m_1}[h_1] >_{S_{m_1}[S_1]} < (-1)^{m_2}\omega(s_{\mu^2})[\pi_2],\ h_{m_2}[h_2] >_{S_{m_2}[S_2]}$$

by the rule of signs for the plethysm $f_1[-f_2]$

$$= (-1)^{m_2}\ < f,\ h_{m_1}\ e_{m_2} > = (-1)^{m_2}\ < f,\ s_{(1^{m_2}, m_1)} + s_{(1^{m_2-1}, m_1+1)} > .$$

Note that $\lambda = (1^{m_1}, 2^{m_2}) \vdash n$ forces $m_1 = 2k - n$ and $m_2 = n - k$. Hence such a partition exists only if $2k \geq n$.

(2) The argument follows the same lines as above. □

Theorem 4.2 *Let U be a subset of the rank-set $[1, n-2]$ of Π_n. Denote by $S = \{k_1 < k_2 \ldots\}$ the complement of U in $[1, n-2]$. If $|S| \geq 2$ and $2k_1 = 2\min(S) \geq n-1$, or if $|S| = 1$ and $2k_1 = 2\min(T) \geq n$,* *then the trivial representation does not appear in the homology of the rank-selected subposet $\Pi_n(U)$. (For $|S| = 1$, i.e., for the rank-set $[1, n-2]\backslash\{k_1\}$, when $2k_1 = n-1$, the multiplicity is exactly 1).*

Proof

To avoid trivialities we assume $n \geq 3$.

Let $S = \{k_0, k_1, \ldots, k_t\}$ subset of ranks, of size $t+1 \geq 1$. Let $T = S\backslash\{k_0\}$. (So $|T| = t \geq 0$.) For brevity we write simply $\tilde{H}(S)$ for the homology of the subposet $\Pi_n(S)$ specified by the rank-set S. Applying Theorem 1.10 to the subposet consisting of the ranks in the complement T^c of T, we find

$$\tilde{H}(S^c) \oplus \tilde{H}(T^c) = \bigoplus_{x \text{ of rank } k_0} \left(\tilde{H}(\hat{0}, x)_{T^c} \otimes \tilde{H}(x, \hat{1})_{T^c}\right) \uparrow_{\mathrm{stab}(x)}^{S_n}, \qquad (A)$$

where as usual the sum is over one x of each type.

Let x be of rank k_0 and type $\lambda = \prod_i i^{m_i}$. By the usual arguments (Theorem 1.4), since $(x, \hat{1})$ is poset-isomorphic to the rank-selected subposet $\Pi_{n-k_0}(\{k_1 - k_0, k_2 - k_0, \ldots, k_t - k_0\}^c)$, its homology as a $\mathrm{stab}(x) = G_\lambda$-module, is $\beta_{n-k_0}(\{k_1 - k_0, k_2 - k_0, \ldots, k_t - k_0\}^c) \downarrow_{\times_i S_{m_i}} [h_1 \otimes h_2 \otimes \ldots]$. Furthermore, the interval $(\hat{0}, x)$ in $\Pi_n(T^c)$ clearly coincides with the interval $(\hat{0}, x)$ in Π_n, so that its G_λ-module structure is given by $(-1)^{k_0} h_{n-k_0} \downarrow_{\times_i S_{m_i}} [\pi_1 \otimes (-\pi_2) \otimes \ldots]$. By Remark 2.10.1, the inner tensor product of these two G_λ-modules is given by

$$(-1)^{k_0} \beta_{n-k_0}(\{k_1 - k_0, k_2 - k_0, \ldots, k_t - k_0\}^c) \downarrow_{\times_i S_{m_i}} [\pi_1 \otimes (-\pi_2) \otimes \ldots].$$

Writing $T - k_0$ for the set $\{k_1 - k_0, k_2 - k_0, \ldots, k_t - k_0\}$, the right-hand side of (A) is thus

$$\sum_{\substack{\lambda \vdash n \\ \ell(\lambda) = n-k_0, \lambda = \prod_i i^{m_i}}} (-1)^{k_0} \beta_{n-k_0}((T - k_0)^c) \downarrow_{\times_i S_{m_i}} [\pi_1 \otimes (-\pi_2) \otimes \ldots] \uparrow_{G_\lambda}^{S_n}.$$

By Lemma 4.1 (1), the multiplicity of the trivial representation is zero unless $2(n - k_0) \geq n$, i.e., for $n < 2k_0$. Otherwise it is $(-1)^{k_0}(-1)^{m_2} < \beta_{n-k_0}((T-k_0)^c)\ ,\ h_{m_1}\ e_{m_2} >$, where $m_1 = n - 2k_0$ and $m_2 = k_0$. We now consider the remaining cases.

(i) Let $n = 2k_0$. The multiplicity is thus the same as the number of occurrences of the sign representation in $\beta_{n-k_0}((T-k_0)^c)$, and this is zero for $k_0 \geq 2$, by Proposition 3.4 (2.) (Recall that $n \geq 3$.)

(ii) Let $n = 2k_0 + 1$. The multiplicity is now

$$< \beta_{k_0+1}((T-k_0)^c), (1^{k_0+1}) > + < \beta_{k_0+1}((T-k_0)^c), (1^{k_0-1}, 2) > .$$

The first inner product is zero as in (i), since $k_0 + 1 \geq 2$. The second is zero by (Proposition 3.4 (2.), and Corollary 2.3 (ii)) provided the set $T - k_0$, and hence the set T, is nonempty. Otherwise it evaluates to 1. (The irreducible $(1^{k_0-1}, 2)$ appears only once in the representation on the chains of Π_{k_0+1}, and this occurrence is accounted for by its appearance in the homology of the full partition lattice). The right-hand side of (A) is thus 1. For the left-hand side, when T is empty, we already know from Corollary 2.3 (1) that the trivial representation does not appear in the homology of $\Pi_n(T^c)$. Hence the result. $\square$

The next result yields another class of ranks for which the trivial representation does not appear.

Theorem 4.3 *Consider the rank-selected subposet $\Pi_n(S)$, where the set S of ranks chosen is of the form $[1, r] \cup \{a\}$, for $a \geq r \geq 2$. Then the trivial representation does not occur in $\tilde{H}(\Pi_n(S))$, provided $a < \binom{r+2}{2}$ or $a > n - r - 1$. In particular, this holds if $n < (r+4)(r+1)/2$.*

Equivalently, a necessary condition for the occurrence of the trivial module is $\binom{r+2}{2} \leq a \leq n - r - 1$.

Proof

We start by showing that the trivial representation does not occur if one of the following conditions is satisfied:

(1) $n < (r+4)(r+1)/2$;
(2) $a \leq 2(r+1)$, or $a > n-r-1$;
(3) $a < \binom{r+2}{2}$.

Again, we use the rank-deletion trick and Theorem 1.10, starting with the subposet $Q = [1, r+1] \cup \{a\}$, and deleting the rank $r+1$ to get the subposet $P = [1, r] \cup \{a\}$. We have

$$\tilde{H}(P) \oplus \tilde{H}(Q) = \bigoplus_{\substack{\lambda \vdash n \\ \ell(\lambda) = n-r-1}} \left(\tilde{H}(\hat{0}, x_\lambda)_Q \otimes \tilde{H}(x_\lambda, \hat{1})_Q \right) \uparrow_{G_\lambda}^{S_n}, \qquad (A)$$

where x_λ is of type $\lambda = \prod_i i^{m_i}$.

Since $(\hat{0}, x_\lambda)_Q$ coincides with $(\hat{0}, x_\lambda)_{\Pi_n}$, the G_λ-representation on its homology is given by

$$(-1)^{r+1} h_{n-r-1} \downarrow_{\times_i S_{m_i}} [\pi_1 \otimes (-\pi_2) \otimes \ldots].$$

Next, the interval $(x_\lambda, \hat{1})_Q$ is isomorphic to the subposet $\Pi_{n-r-1}(\{a-r-1\})$ of Π_{n-r-1} consisting of the single rank $(a-r-1)$; by Example 2.15 (1.), the homology representation $\beta_{n-r-1}(\{a-r-1\})$ of S_{n-r-1} is given by the degree $(n-r-1)$-term in $h_{n-a}[H] - h_{n-r-1}$. Here we have written $H = \sum_{i \geq 1} h_i$.

Now as a G_λ-module, the homology representation on $(x, \hat{1})_Q$ is given by

$$\beta_{n-r-1}(\{a-r-1\}) \downarrow_{\times_i S_{m_i}} [h_1 \otimes h_2 \otimes \ldots].$$

The right-hand side of (A) is thus the sum, over all partitions $\lambda = \prod_i i^{m_i}$ of n into $(r+1)$ parts, of terms of the form

$$(-1)^{r+1} \beta_{n-r-1}(\{a-r-1\}) \downarrow_{\times_i S_{m_i}} [\pi_1 \otimes (-\pi_2) \otimes \ldots] \uparrow_{G_\lambda}^{S_n}.$$

By Lemma 4.1 (1), the multiplicity of the trivial representation in the right-hand side of (A) is

$$
\begin{aligned}
&= < \beta_{n-r-1}(\{a-r-1\}),\ h_{n-2r-2}\ e_{r+1} >_{S_{n-r-1}} \qquad (B) \\
&= < h_{n-a}[H]|_{\deg\ n-r-1},\ h_{n-2r-2}\ e_{r+1} >_{S_{n-r-1}} - < h_{n-r-1},\ h_{n-2r-2}\ e_{r+1} >_{S_{n-r-1}}, \\
&= < h_{n-a}[H]|_{\deg\ n-r-1},\ h_{n-2r-2}\ e_{r+1} >_{S_{n-r-1}}, \qquad \text{since } r+1 \geq 2 \\
&= < \sum_{\substack{\mu = \prod_i i^{n_i} \vdash n-r-1 \\ \ell(\mu) = n-a}} \prod_j h_{n_j}[h_j],\ h_{n-2r-2}\ e_{r+1} >_{S_{n-r-1}}, \\
&= \sum_{\substack{\mu = \prod_i i^{n_i} \vdash n-r-1 \\ \ell(\mu) = n-a}} < \prod_j h_{n_j}[h_j],\ H_r + H_{r+1} >_{S_{n-r-1}}, \qquad (C)
\end{aligned}
$$

where we have written H_k for the Schur function corresponding to the hook shape $(1^k, n-r-1-k)$.

From the Littlewood-Richardson rule it is clear that the Schur function H_k corresponding to a hook appears in a product of symmetric functions $\prod_i f_i$ if and only if for every i there is a hook Schur function H_{k_i} such that $< f_i, H_{k_i} >$ is nonzero. We claim that

Lemma 4.4 *The only hook that can appear in the Schur function expansion of the plethysm $h_m[h_n]$, is a trivial one H_0, corresponding to the row shape (mn).*

Proof

From [M],

$$s_\lambda[1-q] = \sum_{\mu \subseteq \lambda} s_\mu[-q]\ s_{\lambda/\mu}[1] = \sum_{\mu \subseteq \lambda} s_{\mu'}[q]\ (-1)^{|\mu|} s_{\lambda/\mu}[1],$$

by the usual rule of signs. (As in [M], μ' denotes the conjugate shape of μ.) Noting that the Schur function in one variable is zero only for partitions of more than one part, we see that the nonzero terms in the above sum occur for $\mu=(1^r)\subseteq\lambda$ such that λ/μ is a horizontal strip (no two squares in a column). Clearly this is possible only if $\lambda=(1^r,n-r)$. Hence $s_\lambda[1-q]$ equals

$$\begin{cases}(-q)^r+(-q)^{r+1} & \text{if } \lambda=(1^r,n-r) \text{ for some r;}\\ 0 & \text{if otherwise.}\end{cases}$$

That is, plethysm with $(1-q)$ isolates Schur functions corresponding to hooks.

Now suppose $h_m[h_n]=\sum_{\lambda\vdash mn} c_{m,n}(\lambda)\ s_\lambda$. From above, the plethysm of the left-hand side with $(1-q)$ is $(1-q)$, while the same operation on the right-hand side yields $\sum_{r\geq 0} c_{m,n}(1^r,mn-r)\ (-q)^r(1-q)$. The lemma follows. □

Continuing the computation in (B), the multiplicity of the trivial representation in the right-hand side of (A) is now

$$\sum_{\substack{\mu=\prod_i i^{n_i}\vdash n-r-1\\ \ell(\mu)=n-a}} <\prod_j h_{jn_j}, H_r+H_{r+1}>_{S_{n-r-1}}.$$

Denote by $k(\mu)$ the number of *distinct* parts of μ. From the Pieri rule, it is clear that for each μ, the corresponding inner product is 0 if $k(\mu)<r+1$, 1 if $k(\mu)=r+1$, and 2 if $k(\mu)\geq r+2$. The part of the Schur function expansion corresponding to hooks in the product $\prod_j h_{jn_j}$ is $H_{k(\mu)-1}+H_{k(\mu)-2}+\ldots$. Hence the above sum evaluates to

$$|\{\mu\vdash n-r-1:\ell(\mu)=n-a,k(\mu)=r+1\}|+|\{\mu\vdash n-r-1:\ell(\mu)=n-a,k(\mu)=r+2\}|. \qquad (D)$$

Note first that for any partition μ, we have the inequalities

(i) $|\mu|\geq\sum_{i\geq 1}^{k(\mu)} i=\binom{k(\mu)+1}{2}$, or equivalently, $(k(\mu)+\frac12)^2\leq 2|\mu|+\frac14$, and

(ii) $k(\mu)\leq\ell(\mu)$. Also,

(iii) if ν is the partition that remains when we strip off the first column (of length $\ell(\mu)$) from μ, then $k(\nu)\leq k(\mu)\leq k(\nu)+1$.

Applying (i) to the partitions μ in (D), we find that $k(\mu)<r+1$ if $2(n-r-1)<(r+1)^2+(r+1)$, i.e., if $n<(r+4)(r+1)/2$. This gives the bound (1).

It follows from (ii) that if $n-a<r+1$, i.e., $a>n-r-1$, the trivial representation does not occur in the homology of P or of Q.

Now apply (iii) to the partitions μ in (D) : note that $|\nu|=a-r-1$.

If $|\nu|=a-r-1<r$, then $k(\nu)\leq|\nu|<r$, and thus $k(\mu)<r+1$, and the multiplicity in the right-hand side of (A) is zero. If $|\nu|=a-r-1=r$ then $k(\nu)\leq|\nu|\leq r$. Suppose $k(\nu)=r$. Then ν is a partition of r with r distinct parts, impossible unless $r=1$, and then $a=3$. Hence if $r\geq 2$, $a\leq 2r+1$ implies that the trivial representation does not occur in the homology of $P=[1,r]\cup\{a\}$, or of $Q=[1,r+1]\cup\{a\}$.

Note that if $|\nu|=a-r-1=r+1$, this also forces $k(\nu)\leq r$. In fact for $r\geq 2$, it forces $k(\nu)\leq r-1$, because the inequality (i) shows that $2|\nu|+\frac14=2(r+1)+\frac14\leq(r-\frac12)^2$ for $r\geq 2$. From (iii), we conclude that if $r\geq 2$ and $a-r-1=r+1$, then the trivial module does not appear in the left-hand side of (A). We have now established the bounds in (2).

Finally, apply the inequality (i) to the partition ν as in (iii). We find that $k(\nu)<r$ if $2|\nu|=2(a-r-1)<r^2+r$, i.e., if $a<r+1+\binom{r+1}{2}$. This gives the bound (3).

Note that conditions (2) and (3) are equivalent to requiring simply that $a>n-r-1$ or $a<\binom{r+2}{2}$. Note also that $n-r-1<\binom{r+2}{2}$ iff $n<(r+1)(r+4)/2$. Thus another way of expressing this condition is to require that a not be in the interval $[\binom{r+2}{2},n-r-1]$, whenever this interval is nonempty. □

Remark 4.3.1

For fixed n and $r\leq n-2$, assume $n\geq(r+1)(r+4)/2$, and denote by I_r the (possibly empty) interval $[\binom{r+2}{2},n-r-1]$. Note that these intervals are nested: $I_{r+1}\subset I_r$. The criterion says that if $a\notin I_r$, then $b_n([1,r]\cup\{a\})+b_n([1,r+1]\cup\{a\})=0$. (Recall the notation $b_n(S)$ of Section 3). Because of the nesting, we in fact have that $b_n([1,s]\cup\{a\})+b_n([1,s+1]\cup\{a\})=0$ for all $s\geq r$. Now suppose r_a is the least integer such that $a\notin I_{r_a}$. Our arguments show that $b_n([1,r_a]\cup\{a\})+b_n([1,r_a+1]\cup\{a\})=0$, but $b_n([1,r_a-1]\cup\{a\})+b_n([1,r_a]\cup\{a\})>0$. Hence $b_n([1,r_a-1]\cup\{a\})\neq 0$.

These bounds are tight in the sense above; for instance, if $r=2$, then $n\geq 9$, for nonzero multiplicity, and the "critical"interval is $I_a=[6,n-3]$. Thus if $n=9$ and $a=6$, the multiplicity is 1 for the rank-set $[1,2]\cup\{a\}$. Similarly for $n=10,\ r=2$, and $a=6,7$.

Next we collect two useful computations.

Lemma 4.5

(1) *Let H_k be the Schur function for the hook $(1^k,m-k)$. Then $<\pi_m,H_k+H_{k-1}>$ is given by the expression*
$$\frac{1}{m}\sum_{d|(k,m)}\mu(d)\,(-1)^{m-k-\frac{m-k}{d}}\binom{\frac{m}{d}}{\frac{k}{d}}.$$

(2) *Here we compute the multiplicity of a two-rowed shape in π_m. For $2k\leq m$, $<\pi_m,s_{(m-k,k)}>$ is given by the expression*
$$\frac{1}{m}\sum_{d|(k,m)}\mu(d)\,(-1)^{m-\frac{m}{d}}\binom{\frac{m}{d}}{\frac{k}{d}}-\frac{1}{m}\sum_{d|(k-1,m)}\mu(d)\,(-1)^{m-\frac{m}{d}}\binom{\frac{m}{d}}{\frac{k-1}{d}}.$$

In particular if $k=2$, this reduces to
$$\frac{1}{m}\sum_{d|(2,m)}\mu(d)\,(-1)^{m-\frac{m}{d}}\binom{\frac{m}{d}}{\frac{2}{d}}-1=\frac{m-3}{2}-\frac{1}{2}(-1)^{\frac{m}{2}}\delta_{m,\,\mathrm{even}},$$

where $\delta_{m,\,\mathrm{even}}$ is 1 if m is even, and 0 otherwise.

Proof

(1) To evaluate $<\pi_m,H_k+H_{k-1}>$, we use the formula of Example 1.6(i) for $\mathrm{ch}(\pi_m)$ and the plethystic trick of Lemma 4.3. Note that for all n, the plethysm of the power-sum p_n with $(1-q)$ is
$$p_n[1-q]=(1-q)[p_n]=1-q[p_n]=1-p_n[q]=1-q^n.$$
Consequently for any partition μ with m_i parts of size i, we have $p_\mu[1-q]=\prod_i(1-q^i)^{m_i}$. Write χ^λ for the irreducible character of S_m indexed by the partition λ, and for brevity write χ^r when λ is the hook $(1^r,m-r)$. Using the identity $p_\mu=\sum_{\lambda\vdash m}\chi^\lambda\, s_\lambda$, and the calculation of Lemma 4.4, we find that
$$\prod_i(1-q^i)^{m_i}=\sum_{r=0}^{m+1}(-q)^r\left(\chi^r(\mu)+\chi^{r-1}(\mu)\right),$$
for partitions μ of m. (Here we set $\chi^{-1}=0=\chi^{m+1}$.) This implies in particular that $\chi^r(\mu)+\chi^{r-1}(\mu)$ is the coefficient of $(-q)^r$ in $\prod_i(1-q^i)^{m_i}$.

Computing the inner product of π_m with the sum of hook Schur functions H_k+H_{k-1}, we obtain that $<\pi_m,H_k+H_{k-1}>$ equals
$$\frac{1}{m}\sum_{d|m}\mu(d)(-1)^{m-\frac{m}{d}}\left(\chi^k(d^{\frac{m}{d}})+\chi^{k-1}(d^{\frac{m}{d}})\right)=\frac{1}{m}\sum_{d|m}\mu(d)(-1)^{m-\frac{m}{d}}\left(\text{coeff. of }(-q)^k\text{ in }(1-q^d)^{\frac{m}{d}}\right)$$

But the coefficient of $(-q)^k$ in $(1-q^d)^{\frac{m}{d}}$ is $(-1)^k\binom{\frac{m}{d}}{\frac{k}{d}}(-1)^{\frac{k}{d}}$, and in particular it is nonzero only if d divides k. Substituting this into (A), we obtain the required formula.

(2) Just as plethysm with $(1-q)$ isolates hooks, plethysm with $(1+q)$ isolates partitions with at most two parts.

That is, $s_\lambda[1+q]=s_\lambda(1,q)$ is zero unless $\lambda=(m-k,k)$, in which case it is $(q^k-q^{m-k+1})/(1-q)$. On the other hand, $p_\mu[1+q]=\prod_i(1+q^i)^{m_i}$, where m_i is the multiplicity of i in μ. We therefore have the identity

$$(1-q)\prod_i(1+q^i)^{m_i}=\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\chi^k(\mu)\ (q^k-q^{m-k+1}),$$

where we have now written χ^k for the irreducible S_m-character corresponding to the partition $(m-k,k)$. Computing $<\pi_m, s_{(m-k,k)}>$, we find that it equals

$$\frac{1}{m}\sum_{d|m}\mu(d)(-1)^{m-\frac{m}{d}}\ \chi^k(d^{\frac{m}{d}})=\frac{1}{m}\sum_{d|m}\mu(d)(-1)^{m-\frac{m}{d}}\ \left(\text{coeff. of } q^k \text{ in } (1-q)(1+q^d)^{\frac{m}{d}}\right)$$

This gives the desired expression. ◻

Remark 4.5.1

The expression in (1) is also the number of standard Young tableaux of hook shape $(k,1^{m-k})$ or $(k+1,1^{m-k-1})$ whose major index is congruent to 1 modulo m. (See [KW]).

Our work on determining the $(S_{n-2}\times S_2)$-module structure of the modules $V_n(\underline{r})$ can be used to compute explicitly the multiplicities $<\mathrm{ch}(V_n(\underline{r})),\ h_{n-k}\,e_k>=<\beta_n([1,r]),\ h_{n-k}\,e_k>$ in the special cases when $k=2$ or $n-k=2$. We do this in

Lemma 4.6 *Let $\delta_{r,\,\text{even}}$ be 1 if r is even and 0 otherwise. Let $\delta_{r\le a}$ be 1 if $r\le a$, and 0 otherwise.*
(1) $<\beta_n([1,r]),\ h_{n-2}\,e_2>$ equals

$$\begin{cases} r+1-\frac{1}{2}(1+(-1)^{\frac{r}{2}}) & \textit{if } r\le n-3 \textit{ is even;}\\ r+1-\frac{1}{2}(1+(-1)^{\frac{r-1}{2}}) & \textit{if } r\le n-3 \textit{ is odd;}\\ \frac{n-1}{2}+\frac{1}{2}(-1)^{\frac{n}{2}}\delta_{n,\,\text{even}} & \textit{if } r=n-2.\end{cases}$$

(2) If $r\le n-3$, $<\beta_n([1,r]),\ h_{n-2}\,h_2>$ equals

$$\delta_{r\le 1}\delta_{2(r+1)\le n}+\begin{cases}(r+1)-\frac{1}{2}(1+(-1)^{\frac{r+2}{2}}) & \textit{if } r \textit{ is even;}\\ (r+1)-\frac{1}{2}(1+(-1)^{\frac{r+1}{2}}) & \textit{if } r \textit{ is odd}\\ .\end{cases}$$

If $r=n-2$, $<\beta_n([1,r]),\ h_{n-2}\,h_2>=\frac{n-1}{2}-\frac{1}{2}(-1)^{\frac{n}{2}}\delta_{n,\,\text{even}}$.
(3) $<\beta_n([1,r]),\ h_{n-2}\,h_1^2>$ equals

$$\delta_{r\le 1}\delta_{2(r+1)\le n}+\begin{cases}2r+1, & \textit{if } r\le n-3;\\ n-1, & \textit{if } r=n-2.\end{cases}$$

(4)

$$<\beta_n([1,r]),\ e_{n-2}\,h_2>=\begin{cases}\frac{r+1}{2}+\frac{1}{2}\delta_{r,\,\text{even}}, & \textit{if } n-3\le r\le n-2;\\ 0, & \textit{otherwise.}\end{cases}$$

Proof
(1) Using the decomposition of Proposition 2.5, the multiplicity $<\beta_n([1,r]),\ h_{n-2}\,e_2>$ equals

$$\sum_{\substack{\lambda\vdash n,\lambda=1^{m_1}2^{m_2}\ldots,\\ \sum m_i=n-r-1}}\ \sum_{\substack{j\ge 2\\ m_j\ge 1}}<\prod_{i\ne j}\prod_{i \text{ odd}}h_{m_i}[\pi_i]\prod_{i \text{ even}}e_{m_i}[\pi_i]\ \delta_{m_j-1}[\pi_j]\ D_{(1^2)}\pi_j\ ,\ h_{n-2}>,$$

$$=\sum_{\substack{\lambda\vdash n\\ \sum m_i=n-r-1}}\ \sum_{\substack{j\ge 2\\ m_j\ge 1}}\left(\prod_{i\ne j}\prod_{i \text{ odd}}<h_{m_i}[\pi_i]\ ,\ h_{m_i}[h_i]>\prod_{i \text{ even}}<e_{m_i}[\pi_i]\ ,\ h_{m_i}[h_i]>\right)$$

$$<\delta_{m_j-1}[\pi_j],\ h_{m_j-1}[h_j]><D_{(1^2)}\pi_j\ ,\ h_{j-2}>.$$

By the usual argument, the terms in the first factor above are nonzero iff

$$i \geq 3 \text{ implies } m_i = 0, \text{ and } i = 2 \text{ implies } m_i \leq 1. \qquad \text{(A)}$$

The second factor $< \delta_{m_j-1}[\pi_j],\ h_{m_j-1}[h_j] >$ is similarly nonzero iff ($j \geq 3$ and $m_j - 1 = 0$), or ($j = 2$ and $m_j - 1 \leq 1$). In all these cases the nonzero value is 1.

Hence the required multiplicity is

$$\sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=n-r-1}} \sum_{j \geq 3, m_j=1} < D_{(1^2)}\pi_j\ ,\ h_{j-2} > + \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=n-r-1}} \sum_{j=2, m_j \leq 2} < D_{(1^2)}\pi_j\ ,\ h_{j-2} >, \qquad (B)$$

where λ is subject to the conditions in (A) for the parts not equal to j.

The second sum in (B) is clearly zero if it is nonempty, i.e., if $m_2 \neq 0$,, so that, incorporating the conditions in (A), we are left with

$$\sum_{\substack{\lambda \vdash n,\ \ell(\lambda)=n-r-1 \\ \lambda=(1^{m_1},2,j), j \geq 3}} < D_{(1^2)}\pi_j\ ,\ h_{j-2} > + \sum_{\substack{\lambda \vdash n,\ \ell(\lambda)=n-r-1 \\ \lambda=(1^{m_1},j), j \geq 3}} < D_{(1^2)}\pi_j\ ,\ h_{j-2} >,$$

which in turn reduces to

$$< D_{(1^2)}\pi_{r+1}, h_{r-1} > \delta_{r \leq n-3}\delta_{2 \leq r} + < D_{(1^2)}\pi_{r+2}, h_r > \delta_{r \leq n-2}\delta_{1 \leq r}. \qquad (C)$$

The constraints on r for the first sum, for example, arise because the existence of those partitions forces $3 \leq j = r+1$ and $0 \leq m_1 = n-r-3$. Note however that the constraints $\delta_{2 \leq r}$ in the first term, and $\delta_{1 \leq r}$ in the second, are in fact redundant.

The expression (C) can now be evaluated in the form stated most directly by using Lemma 2.7, but also using Lemma 4.5.

(2) Replace $D_{(1^2)}$ with $D_{(2)}$ in the preceding argument, until step (B) above, which now reads: Hence the required multiplicity is

$$\sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=n-r-1}} \sum_{j \geq 3, m_j=1} < D_{(2)}\pi_j\ ,\ h_{j-2} > + \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=n-r-1}} \sum_{j=2, m_j \leq 2} < D_{(2)}\pi_j\ ,\ h_{j-2} >, \qquad (D)$$

where λ is subject to the conditions in (A) for the parts not equal to j.

The second sum in (D) is now equal to 1 for $m_j = m_2 = 1, 2$. Hence partitions with parts of size at most 2, and at most two parts of size 2, contribute an additional term. The multiplicity is

$$\begin{aligned} &< D_{(2)}\pi_{r+1}, h_{r-1} > \delta_{r \leq n-3} + < D_{(2)}\pi_{r+2}, h_r > \delta_{r \leq n-2} \\ &+ |\{\lambda \vdash n,\ \ell(\lambda) = n-r-1,\ \lambda = (1^{m_1}, 2^{m_2}), 1 \leq m_2 \leq 2\}|. \end{aligned}$$

The third sum above is nonzero, and then equal to 1, iff $m_2 = r+1 \leq 2$ and $m_1 = n-2r-2 \geq 0$. Hence, using Corollary 2.3(1) and Lemma 4.5 (2), the multiplicity is

$$\begin{aligned} &< \pi_{r+1}, h_2 h_{r-1} > \delta_{r \leq n-3} + < \pi_{r+2}, h_2 h_r > \delta_{r \leq n-2} + \delta_{r \leq 1}\delta_{2(r+1) \leq n} \\ &= 1 + < \pi_{r+1}, s_{(r-1,2)} > \delta_{r \leq n-3} + 1 + < \pi_{r+2}, s_{(r,2)} > \delta_{r \leq n-2} + \delta_{r \leq 1}\delta_{2(r+1) \leq n} \\ &= \delta_{r \leq 1}\delta_{2(r+1) \leq n} \\ &+ \delta_{r \leq n-3} \begin{cases} \frac{r}{2}, & \text{if } r+1 \text{ is odd;} \\ \frac{r+1}{2} - \frac{1+(-1)^{\frac{r+1}{2}}}{2}, & \text{if } r+1 \text{ is even;} \end{cases} + \delta_{r \leq n-2} \begin{cases} \frac{r+2}{2} - \frac{1+(-1)^{\frac{r+2}{2}}}{2}, & \text{if } r+2 \text{ is even;} \\ \frac{r+1}{2}, & \text{if } r+2 \text{ is odd.} \end{cases} \end{aligned}$$

Combining these terms, we obtain the desired expression.

(3) Follows from (1) and (2).

(4) As in (1), the multiplicity $< \beta_n([1,r]),\ e_{n-2}\ h_2 >$ equals

$$\sum_{\substack{\lambda\vdash n \\ \sum m_i = n-r-1}} \sum_{\substack{j\geq 2 \\ m_j \geq 1}} \left(\prod_{i\neq j} \prod_{i \text{ odd}} < h_{m_i}[\pi_i] \ , \ e_{m_i}[e_i] > \prod_{i \text{ even}} < e_{m_i}[\pi_i] \ , \ e_{m_i}[e_i] > \right)$$

$$< \delta_{m_j-1}[\pi_j], \ e_{m_j-1}[e_j] > < D_{(2)}\pi_j \ , \ e_{j-2} > .$$

The terms in the first factor are nonzero iff

i odd implies $i = 1$ and $m_i \leq 1$, and i even implies $m_i = 0$.

For the second factor, if $j \geq 2$, clearly we must have $m_j - 1 = 0$ for nonzero multiplicity. Hence the only contribution is from

the partition (n), forcing $n - r - 1 = 1$ and $j = n$, and giving the multiplicity $< \pi_n, h_2 e_{n-2} >$, or from the partition $(n-1, 1)$, forcing $n-r-1 = 2$ and $j = n-1$, and giving the multiplicity $< \pi_{n-1}, h_2 e_{n-3} >$.

That is, the required multiplicity is $< \pi_{r+2}, h_2 e_r >$ if $n-3 \leq r \leq n-2$, and 0 otherwise. This quantity may now be evaluated using Lemma 4.5 (1). □

Theorem 4.7

(1) *Let Q_k denote the subposet $\Pi_n - \{\text{elements at rank } k\}$. The trivial representation does not occur in the homology of Q_k iff $n \leq 2k$. Otherwise, let $C_{m,k}$ denote the number of m-cycles in S_m with a unique descent, in position k. Then the trivial representation appears with nonzero multiplicity equal to*

$$\frac{1}{n-k} \sum_{d|(k,n-k)} \mu(d)\, (-1)^{n-\frac{n}{d}} \binom{\frac{n-k}{d}}{\frac{k}{d}} = \begin{cases} C_{m,k} + C_{\frac{m}{2},\frac{k}{2}}, & \text{if } \frac{m}{2} \text{ and } \frac{k}{2} \text{ are both odd integers;} \\ C_{m,k}, & \text{otherwise.} \end{cases}$$

(It follows for instance that if $k = 1$ or $n = 2k+1$, the multiplicity is always 1).

(2) *The trivial representation does not occur in the homology of the subposets obtained by taking a segment of consecutive ranks $[1, r]$, and then deleting a rank k, if $n \leq 2k (\leq 2r)$, or if $n = 2k+1$ and $r \leq n-3$. Otherwise, for $k = 2$ and $r \geq 3$ we have:*

$$b_n([1,r]\backslash\{2\}) = \begin{cases} r-1 & \text{if } r \leq n-3 \text{ and } r \equiv 0 \bmod 4 \text{ or } r \equiv 1 \bmod 4; \\ r-2 & \text{if } r \leq n-3 \text{ and } r \equiv 2 \bmod 4 \text{ or } r \equiv 3 \bmod 4; \\ \frac{n-3}{2} + \frac{1}{2}(-1)^{\frac{n-2}{2}} \delta_{n,\text{ even}} & \text{if } r = n-2. \end{cases}$$

For $n = 2k+2$ we have, for all r such that $2k \geq r \geq k+1 \geq 2$,

$$b_{2k+2}([1,r]\backslash\{k\}) = \begin{cases} \frac{r-k+1}{2} + \frac{1}{2}\delta_{r-k,\text{ even}} = (r-k+1) - C_{r-k+2,2}, & \text{if } r = 2k-1 \text{ or } r = 2k; \\ 0, & \text{if } r \leq 2k-2; \end{cases}$$

Proof

(1) The first statement follows also from Theorem 4.2. From Theorem 2.10 and the first part of Lemma 4.1, the multiplicity is

$$b_n([1, n-2]\backslash\{k\}) = (-1)^k (-1)^{m_2} < \pi_{n-k} \ , \ h_{m_1}\, e_{m_2} > = < \pi_{n-k} \ , \ H_k + H_{k-1} >,$$

since $m_1 + m_2 = n-k$ and $m_1 + 2m_2 = n$ imply $m_1 = n-2k$, $m_2 = k$. Setting $m = n-k$ in the expression of Lemma 4.5(1), we obtain the required formula. The fact that it is nonzero for $n > 2k$ follows from Remark 2.4.1 and Remark 4.5.1, which says that the first expression in (1) is also the multiplicity of the hook shapes $(k, 1^{n-2k})$ or $(k+1, 1^{n-2k-1})$ in π_{n-k}. The simpler combinatorial interpretation in terms of cyclic permutations follows from Gessel and Reutenauer's formula for the number $C_{m,k}$:

$$C_{m,k} = \frac{1}{m} \sum_{d|(m,k)} \mu(d) \binom{\frac{m}{d}}{\frac{k}{d}}.$$

It is clear that this formula coincides with ours if m or k is odd. The rest is established by analysing the remaining cases.

(2) The first part is just a special case of Theorem 4.2. For the second part, using Lemma 4.1(1) and Corollary 2.12, we have, provided $r \geq 2$,

$$b_n([1,r]\backslash\{k\}) = <\beta_{n-k}([1,r-k])\ ,\ h_{n-2k}\ e_k> = <\beta_{n-k}([1,r-k])\ ,\ H_k + H_{k-1}> .$$

Note that this calculation is valid only if $r-k \geq 1$. Now use Lemma 4.6 (1) and (4), noting that from part (1), $C_{m,2} = \frac{m-1}{2} - \frac{1}{2}\delta_{m,\text{ even}}$.

□

Using the main recurrence, Theorem 2.13, we also have

Theorem 4.8

(1) The multiplicity of the trivial representation in the homology of the subposet consisting of a segment of consecutive ranks, beginning with rank 2, is always 1;

(2) The multiplicity of the trivial representation in the homology of the subposet consisting of a segment of consecutive ranks, beginning with rank 3, that is, in $\beta_n([3,r])$, is given below.
First let $r = n-2$. Then $b_n([3,n-2])$ is

$$\begin{cases} \lfloor \frac{n}{2} \rfloor - 1 & \text{if } n \text{ is odd or if } n \equiv 0 \bmod 4; \\ \frac{n}{2} - 2 & \text{if } n \equiv 2 \bmod 4. \end{cases}$$

Now assume $r \leq n-3$. Then $b_n([3,r])$ is

$$\begin{cases} r-1 & \text{if } r \equiv 2 \bmod 4 \text{ or } r \equiv 3 \bmod 4; \\ r-2 & \text{if } r \equiv 0 \bmod 4 \text{ or } r \equiv 1 \bmod 4. \end{cases}$$

(3) For $n \geq 6$, $b_n([4,n-2])$ is given by the equation

$$b_n([3,n-2]) + b_n([4,n-2]) = \begin{cases} \frac{n(n-3)}{6}, & \text{if } 3|n; \\ \frac{(n-1)(n-2)}{6}, & \text{otherwise.} \end{cases}$$

Proof

Corollary 2.14 and Lemma 4.1 (2) together give the following recurrence for the multiplicity in an interval $[s,r]$:

$$b_n([s,r]) + b_n([s+1,r]) = \sum_{\substack{\lambda \vdash n, \ell(\lambda)=n-s \\ \lambda = \prod_i i^{m_i}}} <\beta_{n-s}([1,r-s]), h_{m_1} h_{m_2} \ldots> . \qquad (A)$$

(1) Evaluating (A) for $s=1$, we have

$$b_n([1,r]) + b_n([2,r]) = <\beta_{n-1}([1,r-1]), h_1\ h_{n-2}> = <\beta_{n-1}([1,r-1]), h_{n-1} + s_{(n-2,1)}> .$$

But by Corollary 2.3 (i), the latter multiplicity is 1.

(2) Evaluating (A) for $s=2$, we see that $b_n([2,r]) + b_n([3,r])$ equals

$$<\beta_{n-2}([1,r-2]), h_1\ h_{n-3}> + <\beta_{n-2}([1,r-2]), h_2\ h_{n-4}> = 1 + <\beta_{n-2}([1,r-2]), h_2\ h_{n-4}> .$$

Hence, from (1) above, $b_n([3,r]) = <\beta_{n-2}([1,r-2]), h_2\ h_{n-4}>$.
Now use Lemma 4.6 (2).

(3) This time (A) gives, for $b_n([3,r]) + b_n([4,r])$, $(4 \leq r \leq n-2)$, the expression

$$< \beta_{n-3}([1,r-3]), h_1\ h_{n-4} > + < \beta_{n-3}([1,r-3]), h_1^2\ h_{n-5} > + < \beta_{n-3}([1,r-3]), h_3\ h_{n-6} > .$$

The first inner product is 1 by Corollary 2.3(i), and the second may be evaluated using Lemma 4.6 (3). Hence

$$b_n([3,r]) + b_n([4,r]) = 1 + \begin{cases} 2r-5, & \text{if } r \leq n-3; \\ n-4, & \text{if } r = n-2 \end{cases} + < \beta_{n-3}([1,r-3]), h_3\ h_{n-6} > .$$

We are able to evaluate the third inner product only for $r = n-2$, in which case it is equal to $< \pi_{n-3}, h_3\ h_{n-6} >$. Using Lemma 2.7, this equals

$$\frac{1}{n-3} \sum_{d|(n-3,3,n-6)} \mu(d)(-1)^{n-3-\frac{n-3}{d}} \binom{\frac{n-3}{d}}{\frac{3}{d}}.$$

The result follows. $\square$

We close with a brief discussion of the restricted multiplcities $b'_n(S)$. Recall that these constitute a refinement of the Euler number E_n by subsets of $[1, n-2]$. (The numbers $b_n(S)$, for subsets S of $[1, n-2]$, refine the Euler number E_{n-1}). From the main theorem 2.13 we can at least deduce the following connection between the two refinements:

Proposition 4.9 *Let $S \subseteq [1, n-2]$, and assume $1 \notin S$. Denote by $S-1$ the subset of $[1, n-3]$ formed by subtracting 1 from each element of S. Then*

$$b_n(S \cup \{1\}) + b_n(S) = b'_{n-1}(S-1).$$

Proof
We use the rank-deletion technique on the rank-selected subposet $\Pi_n(S \cup \{1\})$. The left-hand side is then equal to the multiplicity of the trivial representation in the degree n term of

$$\beta_{n-1}(S-1)[\sum_{i \geq 1} h_i].$$

Lemma 4.1(2) now shows that this is the same as $< \beta_{n-1}(S-1), h_{n-2}h_1 >$, which in turn equals $b'_{n-1}(S-1)$ by reciprocity. $\square$

Finally we remark that from Theorems 2.1 and 2.2 we can compute $b'_n(S)$ when S is an initial or final segment of consecutive ranks. We record these and other assorted results in

Proposition 4.10
(0) Let $1 \leq r \leq n-2$. Then $b_n(\{r\}) = p(n, n-r) - 1$, and $b'_n(\{r\}) = \sum_{i=n-r}^{n-1} p(i, n-r-1)$, (where $p(m,k)$ denotes the number of integer partitions of m into k parts).
(1) Let $S = \{1, r\}$, $1 < r \leq n-2$. Then $b_n(S) = b'_{n-1}(\{r-1\}) - b_n(\{r\})$.
(2) Let S be the interval $[1,r]$ of ranks in Π_n. Then $b_n(S) = 0$, and $b'_n(S) = 1$ for all $r = 1, \ldots, n-2$.
(3) Let S be the interval $[n-1-r, n-2]$ of ranks in Π_n. Then for all $r = 1, \ldots, n-2$,

$$b'_n(S) = \binom{n-2}{r}.$$

(4) Let $S = [1, n-2] \backslash \{k\}$. Then

$$b'_n(S) = \sum_{r=0}^{min(n-k-1,k)} \binom{n-k-1}{r} - 1.$$

In particular, if $2k > n-1$, then $b'_n(S) = 2^{n-k-1} - 1$ (while $b_n(S) = 0$ by Theorem 1.10).

(5) *Let $n \geq 6$, $S = [2, n-3]$. Then $b'_n(S) = 2n - 7$.*

Proof

(3) Follows from Theorem 2.1 (2); $b'_n(S)$ is the number of orbits in the decomposition, and this evaluates to

$$\sum_{\substack{\lambda \vdash (n-1), \ell(\lambda)=r+1 \\ \lambda = 1^{m_1} 2^{m_2} \ldots}} \binom{r+1}{m_1, m_2, \ldots}.$$

By a well-known combinatorial identity, this is just $\binom{n-2}{r}$.

(4) We use Theorem 2.9. Taking the derivative of $\beta_n([1, n-2] \backslash \{k\})$ with respect to p_1, we have that the restriction to S_{n-1} is the degree $n-1$ term in

$$(-1)^k {p_1}^{n-k-1} [\pi_1 - \pi_2 + \ldots] \; (1 - p_1 + p_1^2 - \ldots) - p_1^{n-1}.$$

Since, for the trivial representation, we need only consider the contributions of $\pi_i = h_i, i \leq 2$, the required multiplicity is the same as that of the degree $n-1$ term of

$$(-1)^k (h_1 - h_2)^{n-k-1} \; (1 - h_1 + h_1^2 - \ldots) - h_1^{n-1}.$$

This can be written down precisely; it is

$$(-1)^k \sum_{i=0}^{k} (-1)^i h_1^i \sum_{j=0}^{n-k-1} (-1)^j h_2^j \binom{n-k-1}{j} h_1^{n-k-1-j} \; - \; h_1^{n-1},$$

where we must have $i + (n-k-1-j) + 2j = n-1$, i.e., $i + j = k$. From this the result follows.

(5) Using Theorem 2.13, we have the formula

$$\beta_n([2, n-3]) + \beta_n([1, n-3]) = h_1 \, \beta_{n-1}([1, n-4]).$$

By parts (1) and (2) of Theorem 2.2, this gives

$$\begin{aligned} &\beta_n([2, n-3]) \\ &= \pi_n + h_2(\pi_{n-2} + h_1 \pi_{n-3} + h_1^2 \pi_{n-4} + \ldots + h_1^{n-5} \pi_3 + h_1^{n-4} \pi_2) - \sum_{1 \leq k < n-k} \pi_k \pi_{n-k} - f_n; \end{aligned}$$

where the last term f_n is nonzero only if n is even, in which case it is $h_2[\pi_{\frac{n}{2}}]$ if $\frac{n}{2}$ is odd, and equals $e_2[\pi_{\frac{n}{2}}]$ if $\frac{n}{2}$ is even.

The restriction to S_{n-1} is:

$$\begin{aligned} &h_1^{n-1} + h_1 (\sum_{i=0}^{n-4} h_1^i \, \pi_{n-2-i}) + h_2 \left(h_1^{n-3} + \sum_{i=1}^{n-4} (i \, h_1^{i-1} \pi_{n-2-i} + h_1^{n-3}) \right) \\ &- \sum_{1 \leq k < n-k} (h_1^{k-1} \pi_{n-k} + h_1^{n-k-1} \pi_k) - \delta_{n, \text{even}} \; \pi_{\frac{n}{2}} \; h_1^{\frac{n}{2}-1}. \end{aligned}$$

It is now easy to count the contribution to the trivial representation term by term; for $n \geq 6$, we obtain $2n - 7$, as stated. □

Table 1 contains values for $b_n(S)$ and $b'_n(S)$ for $3 \leq n \leq 9$. (These values were obtained by programming our main recurrence for rank-selected homology, Theorem 2.13, using Stembridge's Maple package for symmetric functions). Some remarks concerning these tables follow.

Remark 4.10.1 We observe that all the non-occurrences of the trivial representation in Table 1 are accounted for by the criteria of this section, which we may summarise as follows: We have seen that $b_n(S)$ is zero if

(i) $S = [1, r]$ ([H2]), (see also Corollary 2.3 (i)) or

(ii) $S = [1, \lfloor \frac{n+1}{2} \rfloor] \cup T$, (Theorem 4.2) or

(iii) $S = [1, r] \cup \{a\}$, where $a \notin [\binom{r+2}{2}, n - r - 1]$, (Theorem 4.3) or

(iv) $S = [1, r] \backslash \{k\}$, for n even and $k = \frac{n}{2} - 1$, provided $k = \frac{n}{2} - 1 \leq r \leq n - 4$, (Theorem 4.7 (2)).

Remark 4.10.2

(i) As noted in [St3], for a fixed set S, the values $b_n(S)$ stabilise at $n = 2\max(S)$. That is, $b_m(S) = b_{2\max(\mathrm{S})}(S)$ for all $m \geq 2\max(S)$. In particular, if $b_{2\max(\mathrm{S})}(S) = 0$ then $b_n(S)$ is zero for all $n \geq 2\max(\mathrm{S})$.

(ii) However, given a set S of ranks, it is possible that the trivial representation does not occur in $\beta_n(S)$, for some n, but that it occurs later for some $m > n$. (In this case necessarily $n < 2\max(\mathrm{S})$). This is shown by the following example, based on the results of Theorem 4.3.

Let S be the set of ranks $[1, 3] \cup \{11\}$, so that $r = 3$, and $a = 11$. If $n = 14$, then a is not in the interval $[\binom{r+2}{2} = 10, n - r - 1 = 10]$, and hence $b_{14}(S) = 0$. Now let $n = 15$. We claim that $b_{15}(S) > 0$. Note first that the "critical "interval, as in Remark 4.3.1, is now $[10, 11]$, which contains $a = 11$. Hence, by (A) and (D) of the proof of Theorem 4.3,

$$\begin{aligned} b_{15}([1, 3] \cup \{11\}) + b_{15}([1, 4] \cup \{11\}) =& |\{\mu \vdash 11 : \ell(\mu) = 4, k(\mu) = r + 1 = 4\}| \\ &+ |\{\mu \vdash 11 : \ell(\mu) = 4, k(\mu) = r + 2 = 5\}|, \end{aligned}$$

which is clearly nonzero (recall that $k(\mu)$ is the number of distinct parts of μ).

However, the interval $[\binom{4+2}{2}, 15 - 4 - 1]$ is empty, so $b_{15}([1, 4] \cup \{11\}) = 0$. Therefore $b_{15}(S) \neq 0$, even though $b_{14}(S) = 0$.

(iii) The reader may similarly verify that for the rank-set $S = [1, 3] \cup \{10\}$, one has $b_{14}(S) = 1$, while $b_{15}(S) = 0$.

Based on our results so far, we can formulate the following

Conjecture 4.11 *If the multiplicity $b_n(S)$ is zero, then the rank-set S must contain the rank 1.*

We are also led to make the following curious conjecture, which, if true, would give a completely numerical characterisation of the lower truncations $[1, r]$ of the partition lattice:

Conjecture 4.12 *The restricted multiplicity $b'_n(S)$ is never zero, and is equal to 1 iff $S = [1, r]$ for some $1 \leq r \leq n - 2$.*

5. The d-divisible lattices Π_n^d.

In this section we discuss briefly how our techniques carry over to the d-divisible subposets Π_n^d of Π_{nd}. Fix $d \geq 1$. As a subposet, Π_n^d consists of all partitions x in Π_{nd} all of whose block sizes are divisible by d. As mentioned in Example 1.6 (ii), this subposet is Cohen-Macaulay, and its S_{nd}-homology representation, with characteristic π_n^d, was determined in [CHR]. In [Wa], Wachs gives a simple and ingenious basis for the homology module.

From the arguments of Sections 1 and 2, it is not hard to see that the general recurrence for arbitrary rank-selection, Theorem 2.13, translates into the following for the rank n lattice Π_n^d :

Theorem 5.1 *Let S be a subset of the n nontrivial ranks in Π_n^d; let s_1 be the least element of S. Denote by $\beta_n^d(S)$ the Frobenius characteristic of the S_{nd}-homology module arising from the rank-selected subposet $\Pi_n^d(S)$. Then*

$$\beta_n^d(S) + \beta_n^d(S\backslash\{s_1\}) = \pi_{n+1-s_1}(S - s_1)[\sum_{j\geq 1} h_{jd}].$$

Recall also from Example 1.6 (ii) the defining equation for the characteristics π_n^d :

$$\sum_{n\geq 1}(-1)^n \pi_n^d = \sum_{i\geq 1}(-1)^i \pi_i[\sum_{j\geq 1} h_{jd}].$$

Taking partial derivatives with respect to p_1, it is easy to write down from this the defining equation for the characteristics $\frac{\partial}{\partial p_1}\pi_n^d$ of the representations restricted to S_{nd-1} :

Proposition 5.2 *([CHR, proof of Theorem 6.4]) We have*

$$\sum_{n\geq 1}(-1)^n \frac{\partial}{\partial p_1}\pi_n^d = -\frac{\sum_{j\geq 1} h_{jd-1}}{1+\sum_{j\geq 1} h_{jd}}.$$

In [CHR], Theorem 6.3, the authors show that the right-hand side is in fact the generating function for the Foulkes representations (indexed by border-strips with n rows, of sizes $d-1, d, \ldots d$). Wachs uses her new basis to give another proof of this fact, in [Wa].

The remainder of this section is devoted to showing that Theorem 1.8, and consequently Theorem 1.7, generalises in a natural way to the case $d > 1$. For notation and background we refer to Remark 1.4.0 and the proof of Theorem 1.8.

Theorem 5.3 *Let x be a partition in Π_n^d, of type $\lambda = d^{n_1}(2d)^{n_2}\ldots(sd)^{n_s}\ldots$ Then the top homology of the interval $(\hat{0}, x)$ is a $G_\lambda = \times_i S_{n_i}[S_{di}]$-module isomorphic to*

$$\iota_{n_1}[\pi_1^d] \otimes \epsilon_{n_2}[\pi_2^d] \otimes \ldots \otimes \iota_{n_{2i-1}}[\pi_{2i-1}^d] \otimes \epsilon_{n_{2i}}[\pi_{2i}^d] \otimes \ldots,$$

where ι_m and ϵ_m denote the trivial and the sign representations respectively.

In particular, the characteristic of the rth Whitney homology of Π_n^d, for $r = 1, \ldots, n$, is given by

$$\sum_{\substack{\lambda\vdash nd,\ \ell(\lambda)=n+1-r \\ \lambda=(d^{m_1},(2d)^{m_2},\ldots)}} h_{m_1}[\pi_1^d]e_{m_2}[\pi_2^d]\ldots e_{m_{2i}}[\pi_{2i}^d]h_{m_{2i+1}}[\pi_{2i+1}^d]\ldots$$

We shall prove this result by computing the homology of the interval $(\hat{0}, x)$ as an alternating sum, over $y \neq \hat{0}$ lying below x, of the homologies $\tilde{H}(y, x)$. Since (for $y \neq \hat{0}$) the interval (y, x) has the same combinatorial structure in Π_{nd} and in Π_n^d, our strategy is to use the results for the partition lattice Π_{nd} to write down a formula for the module structure of $\tilde{H}(y, x)$. We collect this information in the following lemma, omitting the proof.

Lemma 5.4 *Let $x \in \Pi_n^d$ be of rank i, and type $\lambda = d^{n_1}(2d)^{n_2}\ldots(sd)^{n_s}\ldots$ Let $G_\lambda = \times_i S_{n_i}[S_{di}]$ be the stabiliser of x. Let b_j be the number of blocks of an element of rank j (thus b_i equals $n+1-i$ if $d>1$, and $n-i$ if $d=1$). Write $H(d)$ for the sum $\sum_{j\geq 1} h_{jd}$. For $0 \leq j \leq i$, set $W_{j,x} = \sum_{y:\, y\leq x,\, \mathrm{rank}(y)=j} \tilde{H}(y,x)$. Then the G_λ-module structure of $W_{j,x}$ is given by:*

$$\sum_{\substack{\alpha=(\alpha^{(1)},\ldots,\alpha^{(s)}\ldots) \\ \ell(\alpha^{(s)})=n_s,\ \sum_s |\alpha^{(s)}|=b_j,\ \alpha_1^{(s)}\leq ds}} \prod_s \prod_k f_{m_k(\alpha^{(s)})}\big[\pi_k[H(d)]|_{\deg\, ds}\big],$$

where the sum runs over sequences of partitions $\{\alpha^{(s)}\}$, each $\alpha^{(s)}$ having at most n_s parts, with largest part at most ds, and $m_k(\alpha^{(s)})$ denotes the multiplicity of the part k in $\alpha^{(s)}$. The characteristic $f_{m_k(\alpha^{(s)})}$ equals $h_{m_k(\alpha^{(s)})}$ if k is odd, and $e_{m_k(\alpha^{(s)})}$ if k is even.

Example 5.5 It is best to illustrate with an example. We shall compute $W_{j,x}$ in Π_6 for $x = 123/456$ and for all $j = 1,\ldots,4$. In what follows it is understood that when we write $(f_a[g_1]\otimes f_b[g_2])\uparrow$ for representations $\mathrm{ch}^{-1}(g_i)$ of S_n, we mean the representation of $S_{a+b}[S_n]$ induced from $S_a[S_n]\times S_b[S_n]$.

Here $s=3$. Write B_1 for the block $/123/$, and B_2 for the block $/456/$.

For $j=4$, the only possibility is $\alpha^{(s)} = (1,1)$. (This corresponds of course to $y=x$.) Here the representation is $h_2[\pi_1[H]|_{\deg\,3}]$, the trivial representation of $S_2[S_3]$.

For $j=3$, we have to break up the blocks B_i into a total of 6-3=3 blocks. This can be accomplished only by splitting one of the blocks into two, and leaving the other one alone. Hence $\alpha^{(3)} = (2,1)$, and the representation is $(h_1[\pi_2[H]|_{\deg\,3}]\otimes h_1[\pi_1[H]|_{\deg\,3}])\uparrow$.

For $j=2$, the blocks B_i, $i=1,2$, have to be broken up into 4 blocks. The choices for $\alpha^{(3)}$ are thus $(3,1)$ or $(2,2)$. The contribution to the representation is consequently $(h_1[\pi_3[H]|_{\deg\,3}]\otimes h_1[\pi_1[H]|_{\deg\,3}])\uparrow + e_2[\pi_2[H]|_{\deg\,3}]$.

For $j=1$, the unique choice is $\alpha^{(3)} = (3,2)$, corresponding to 5 blocks. The representation is $(h_1[\pi_3[H]|_{\deg\,3}]\otimes h_1[\pi_2[H]|_{\deg\,3}])\uparrow$.

Using the fact that $h_n[g_1+g_2] = h_n[g_1]+h_n[g_2]+\sum_{1\leq i\neq n-i} h_i[g_1]h_{n-i}[g_2]$, we observe that $W_{0,x} = W_{1,x}-W_{2,x}+W_{3,x}-W_{4,x}$ equals

$$h_2\big[\pi_3[H]|_{\deg\,3}\big] - h_2\big[\pi_1[H]|_{\deg\,3} - \pi_2[H]|_{\deg\,3} + \pi_3[H]|_{\deg\,3}\big] + h_2\big[\pi_3[H]|_{\deg\,3}\big] = h_2[\pi_3],$$

by acyclicity of Whitney homology.

Before proceeding to the proof of Theorem 5.3, we do an example of the completely analogous calculation in Π_6^2 :

Example 5.6 Let x be the partition $123456/abcdef$ in Π_6^2. Note that the rank function in Π_n^d is $(n+1)$ minus the number of blocks, so that the rank of x is 5. Since all the block sizes must be multiples of 2, the partitions α^s are exactly the same as in the previous example. Write $H(2)$ for the sum $\sum_{j\geq 1} h_{2j}$.

At rank $j=5$, $\alpha^{(3)} = (1,1)$, yielding $h_2[\pi_1[H(2)]|_{\deg\,6}]$.

At rank $j=4$, $\alpha^{(3)} = (2,1)$, yielding $(h_1[\pi_2[H(2)]|_{\deg\,6}]\otimes h_1[\pi_1[H(2)]|_{\deg\,6}])\uparrow$.

At rank $j=3$, $\alpha^{(3)} = (3,1)$ or $(2,2)$ giving

$$(h_1[\pi_3[H(2)]|_{\deg\,6}]\otimes h_1[\pi_1[H(2)]|_{\deg\,6}])\uparrow + e_2[\pi_2[H(2)]|_{\deg\,6}].$$

At rank $j=2$, $\alpha^{(3)} = (3,2)$, giving $(h_1[\pi_3[H(2)]|_{\deg\,6}]\otimes h_1[\pi_2[H(2)]|_{\deg\,6}])\uparrow$.

At rank $j=1$, $\alpha^{(3)} = (3,3)$, giving $h_2[\pi_3[H(2)]|_{\deg\,6}]$.

Note that in contrast to the preceding case, the shift in rank by 1 gives us one extra term.

The acyclicity of Whitney homology now gives $\tilde{H}(\hat{0},x) = W_{1,x}-W_{2,x}+W_{3,x}-W_{4,x}+W_{5,x}$. It can then be verified that this is precisely $h_2\big[(\pi_3[H(2)]-\pi_2[H(2)]+\pi_1[H(2)])_{\deg\,6}\big]$. By the defining equation for π_n^d, this is $h_2[\pi_3^2]$.

Proof of Theorem 5.3:

It remains to extract the general computational principle behind this example. Note that now we have $\sum_s |\alpha^{(s)}| = n+1-j$.

From the lemma, $W_{j,x}$ is given by a sum over all t-tuples of integer partitions $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(t)})$ such that $\ell(\alpha^{(s)}) = n_s$, $\sum_s |\alpha^{(s)}| = n+1-j$, and $\alpha_1^{(s)} \le ds$, of products of the form

$$(-1)^{\sum_s (|\alpha^{(s)}|-n_s)} \prod_s \left(\prod_k h_{m_k(\alpha^{(ds)})}[(-1)^{k-1} g_{k,ds}] \right) \uparrow^{S_{n_s}[S_{ds}]},$$

where we have written $g_{k,ds}$ for the degree (ds) term in $\pi_k[H(d)]$, and used the fact that $\sum_{k \text{ even}} m_k(\alpha^{(s)}) \equiv |\alpha^{(s)}| - n_s (mod\ 2)$.

Note that each factor indexed by s is a representation of $S_{n_s}[S_{ds}]$.

Now substitute into $\tilde{H}(\hat{0}, x) = \sum_{j\ge 1}^{i} (-1)^{j-1} W_{j,x}$, and use the equality $\sum_s |\alpha^{(s)}| = n+1-j$ for the terms in $W_{j,x}$. We also substitute $\ell(\lambda)$ for $\sum_s n_s$.

This gives, for the characteristic of $\tilde{H}(0,x)$, the expression

$$(-1)^{n-\ell(\lambda)} \prod_s \left\{ \sum_{\substack{\alpha^{(s)} = \prod_k k^{m_k(\alpha^{(s)})} \\ \ell(\alpha^{(s)}) = n_s}} \prod_k h_{m_k(\alpha^{(ds)})}[(-1)^{k-1} g_{k,ds}]. \right\} \uparrow^{S_{n_s}[S_{ds}]}$$

Now recall the observation made in Example 5.5 on expanding a plethysm of the form $h_n[\sum_i g_i]$ in the case when the g_i all have the same degree: The expansion is a sum over all finite sequences $\{m_i\}$ whose terms add up to n, of terms of the form $\prod_i h_{m_i}[g_i]$. The product indexed by s in the last line above falls in this category; we can think of the sum as being indexed by all sequences $\{m_k = m_k(\alpha^{(ds)})\}_k$ whose terms add up to n_s. The expression in braces can therefore be rewritten as

$$h_{n_s}[\textstyle\sum_k (-1)^{k-1} g_{k,ds}].$$

By definition of the $g_{k,ds}$, and using the defining equation for the π_n^d, this gives

$$\mathrm{ch}(\tilde{H}(\hat{0},x)) = (-1)^{n-\ell(\lambda)} \textstyle\prod_s h_{n_s}[(-1)^{s-1}\pi_s^d],$$

which in turn can be expanded into

$$\prod_{s \text{ odd}} h_{n_s}[\pi_s^d] \prod_{s \text{ even}} e_{n_s}[\pi_s^d],$$

as required. □

An alternative proof, which mimics the proof of Theorem 1.8, can be given using Wachs' new cubical basis for the homology module. Recall the observation made in Remark 1.7.1, that the S_n-equivariant structure of the Whitney homology of the partition lattice, (i.e., the Orlik-Solomon algebra for A_{n-1}), up to sign, coincides with the (1^n)-component of the exterior algebra of the free Lie algebra. By applying the involution ω to the preceding formula, we find that the characteristic of $WH_r(\Pi_n^d)$ tensored with the sign is given by the degree nd term in

$$\begin{cases} (-1)^{r-1} h_{n-r+1}[\sum_{k\ge 1} (-1)^{k-1} \omega(\pi_k^d)], & \text{if } d \text{ is even;} \\ e_{n-r+1}[\sum_{k\ge 1} \omega(\pi_k^d)], & \text{if } d \text{ is odd.} \end{cases}$$

Hence if d is odd, the S_{nd}-equivariant structure of the Whitney homology of Π_{nd}, up to sign, coincides with the (1^{nd}) component of the exterior algebra over a graded space whose kd-graded component is the S_{kd}-module $\omega(\pi_k^d)$, thereby generalising the well-known result in the case $d = 1$.

$n,\ S \subseteq [1, n-2]$	$b_n(S)$	$b'_n(S)$
$4,\ \{1\}$	0	1
$4,\ \{2\}$	1	2
$4,\ \{1,2\}$	0	1
$5,\ \{1\}$	0	1
$5,\ \{2\}$	1	3
$5,\ \{3\}$	1	3
$5,\ \{1,2\}$	0	1
$5,\ \{1,3\}$	1	3
$5,\ \{2,3\}$	1	3
$5,\ \{1,2,3\}$	0	1
$6,\ \{1\}$	0	1
$6,\ \{2\}$	1	3
$6,\ \{3\}$	2	5
$6,\ \{4\}$	2	4
$6,\ \{1,2\}$	0	1
$6,\ \{1,3\}$	1	5
$6,\ \{1,4\}$	1	5
$6,\ \{2,3\}$	1	5
$6,\ \{2,4\}$	3	10
$6,\ \{3,4\}$	1	6
$6,\ \{1,2,3\}$	0	1
$6,\ \{1,2,4\}$	0	3
$6,\ \{1,3,4\}$	2	6
$6,\ \{2,3,4\}$	1	4
$6,\ \{1,2,3,4\}$	0	1
$7,\ \{1\}$	0	1
$7,\ \{2\}$	1	3
$7,\ \{3\}$	2	6
$7,\ \{4\}$	3	8
$7,\ \{5\}$	2	5
$7,\ \{1,2\}$	0	1
$7,\ \{1,3\}$	1	5
$7,\ \{1,4\}$	2	10
$7,\ \{1,5\}$	2	7
$7,\ \{2,3\}$	1	5
$7,\ \{2,4\}$	5	19
$7,\ \{2,5\}$	5	17
$7,\ \{3,4\}$	2	11
$7,\ \{3,5\}$	6	21
$7,\ \{4,5\}$	3	10

Table 1: the multiplicity of the trivial module in $\tilde{H}(\Pi_n(S))$ and in $\tilde{H}(\Pi_n(S))\downarrow^{S_n}_{S_{n-1}}$

$n,\ S \subseteq [1, n-2]$	$b_n(S)$	$b'_n(S)$
$7,\ \{1,2,3\}$	0	1
$7,\ \{1,2,4\}$	0	5
$7,\ \{1,2,5\}$	0	5
$7,\ \{1,3,4\}$	3	12
$7,\ \{1,3,5\}$	4	21
$7,\ \{1,4,5\}$	3	14
$7,\ \{2,3,4\}$	1	7
$7,\ \{2,3,5\}$	3	17
$7,\ \{3,4,5\}$	2	10
$7,\ \{1,2,3,4\}$	0	1
$7,\ \{1,2,3,5\}$	0	3
$7,\ \{1,2,4,5\}$	1	7
$7,\ \{1,3,4,5\}$	2	10
$7,\ \{2,3,4,5\}$	1	5
$7,\ \{1,2,3,4,5\}$	0	1
$8,\ \{1\}$	0	1
$8,\ \{2\}$	1	3
$8,\ \{3\}$	2	6
$8,\ \{4\}$	4	10
$8,\ \{5\}$	4	11
$8,\ \{6\}$	3	6
$8,\ \{1,2\}$	0	1
$8,\ \{1,3\}$	1	5
$8,\ \{1,4\}$	2	12
$8,\ \{1,5\}$	4	17
$8,\ \{1,6\}$	2	9
$8,\ \{2,3\}$	1	5
$8,\ \{2,4\}$	5	22
$8,\ \{2,5\}$	10	39
$8,\ \{2,6\}$	7	24
$8,\ \{3,4\}$	2	13
$8,\ \{3,5\}$	11	45
$8,\ \{3,6\}$	10	38
$8,\ \{4,5\}$	5	22
$8,\ \{4,6\}$	11	38
$8,\ \{5,6\}$	4	15
$8,\ \{1,2,3\}$	0	1
$8,\ \{1,2,4\}$	0	5
$8,\ \{1,2,5\}$	0	11
$8,\ \{1,2,6\}$	0	7
$8,\ \{1,3,4\}$	3	13
$8,\ \{1,3,5\}$	8	45
$8,\ \{1,3,6\}$	7	36
$8,\ \{1,4,5\}$	6	32
$8,\ \{1,4,6\}$	10	53
$8,\ \{1,5,6\}$	6	25
$8,\ \{2,3,4\}$	1	7

Table 1, continued

$n,\ S \subseteq [1, n-2]$	$b_n(S)$	$b'_n(S)$
$8,\ \{2,3,5\}$	5	33
$8,\ \{2,3,6\}$	5	30
$8,\ \{2,4,5\}$	10	50
$8,\ \{2,4,6\}$	19	90
$8,\ \{2,5,6\}$	12	54
$8,\ \{3,4,5\}$	3	20
$8,\ \{3,4,6\}$	8	46
$8,\ \{3,5,6\}$	13	57
$8,\ \{4,5,6\}$	4	0
$8,\ \{1,2,3,4\}$	0	1
$8,\ \{1,2,3,5\}$	0	5
$8,\ \{1,2,3,6\}$	0	5
$8,\ \{1,2,4,5\}$	2	15
$8,\ \{1,2,4,6\}$	2	25
$8,\ \{1,2,5,6\}$	2	17
$8,\ \{1,3,4,5\}$	4	21
$8,\ \{1,3,4,6\}$	9	48
$8,\ \{1,3,5,6\}$	10	59
$8,\ \{1,4,5,6\}$	6	30
$8,\ \{2,3,4,5\}$	1	9
$8,\ \{2,3,4,6\}$	3	24
$8,\ \{2,3,5,6\}$	7	42
$8,\ \{2,4,5,6\}$	8	44
$8,\ \{3,4,5,6\}$	3	15
$8,\ \{1,2,3,4,5\}$	0	1
$8,\ \{1,2,3,4,6\}$	0	3
$8,\ \{1,2,3,5,6\}$	0	7
$8,\ \{1,2,4,5,6\}$	2	14
$8,\ \{1,3,4,5,6\}$	2	15
$8,\ \{2,3,4,5,6\}$	1	6
$8,\ \{1,2,3,4,5,6\}$	0	1
$9,\ \{1\}$	0	1
$9,\ \{2\}$	1	3
$9,\ \{3\}$	2	6
$9,\ \{4\}$	4	11
$9,\ \{5\}$	5	15
$9,\ \{6\}$	6	15
$9,\ \{7\}$	3	7
$9,\ \{1,2\}$	0	1
$9,\ \{1,3\}$	1	5
$9,\ \{1,4\}$	2	12
$9,\ \{1,5\}$	5	23
$9,\ \{1,6\}$	5	25
$9,\ \{1,7\}$	3	11
$9,\ \{2,3\}$	1	5
$9,\ \{2,4\}$	5	22

Table 1, continued

$n,\ S \subseteq [1, n-2]$	$b_n(S)$	$b'_n(S)$
$9,\ \{2,5\}$	12	51
$9,\ \{2,6\}$	16	65
$9,\ \{2,7\}$	9	31
$9,\ \{3,4\}$	2	13
$9,\ \{3,5\}$	13	57
$9,\ \{3,6\}$	22	96
$9,\ \{3,7\}$	15	55
$9,\ \{4,5\}$	6	28
$9,\ \{4,6\}$	22	91
$9,\ \{4,7\}$	20	73
$9,\ \{5,6\}$	8	37
$9,\ \{5,7\}$	17	61
$9,\ \{6,7\}$	6	21
$9,\ \{1,2,3\}$	0	1
$9,\ \{1,2,4\}$	0	5
$9,\ \{1,2,5\}$	0	13
$9,\ \{1,2,6\}$	1	19
$9,\ \{1,2,7\}$	0	9
$9,\ \{1,3,4\}$	3	13
$9,\ \{1,3,5\}$	9	54
$9,\ \{1,3,6\}$	17	95
$9,\ \{1,3,7\}$	51	254
$9,\ \{1,4,5\}$	7	40
$9,\ \{1,4,6\}$	23	131
$9,\ \{1,4,7\}$	18	99
$9,\ \{1,5,6\}$	14	67
$9,\ \{1,5,7\}$	21	103
$9,\ \{1,6,7\}$	9	39
$9,\ \{2,3,4\}$	1	7
$9,\ \{2,3,5\}$	5	38
$9,\ \{2,3,6\}$	11	73
$9,\ \{2,3,7\}$	7	43
$9,\ \{2,4,5\}$	11	61
$9,\ \{2,4,6\}$	40	213
$9,\ \{2,4,7\}$	33	167
$9,\ \{2,5,6\}$	27	139
$9,\ \{2,5,7\}$	47	227
$9,\ \{2,6,7\}$	22	97
$9,\ \{3,4,5\}$	3	23
$9,\ \{3,4,6\}$	16	102
$9,\ \{3,4,7\}$	15	89
$9,\ \{3,5,6\}$	27	139
$9,\ \{3,5,7\}$	49	243
$9,\ \{3,6,7\}$	30	138
$9,\ \{4,5,6\}$	8	46
$9,\ \{4,5,7\}$	20	103
$9,\ \{4,6,7\}$	26	117
$9,\ \{5,6,7\}$	8	35

Table 1, continued

$n,\ S \subseteq [1,n-2]$	$b_n(S)$	$b'_n(S)$
$9,\ \{1,2,3,4\}$	0	1
$9,\ \{1,2,3,5\}$	0	5
$9,\ \{1,2,3,6\}$	0	11
$9,\ \{1,2,3,7\}$	0	7
$9,\ \{1,2,4,5\}$	2	17
$9,\ \{1,2,4,6\}$	5	59
$9,\ \{1,2,4,7\}$	3	43
$9,\ \{1,2,5,6\}$	5	45
$9,\ \{1,2,5,7\}$	6	69
$9,\ \{1,2,6,7\}$	3	31
$9,\ \{1,3,4,5\}$	4	24
$9,\ \{1,3,4,6\}$	17	105
$9,\ \{1,3,4,7\}$	15	89
$9,\ \{1,3,5,6\}$	23	147
$9,\ \{1,3,5,7\}$	41	251
$9,\ \{1,3,6,7\}$	24	141
$9,\ \{1,4,5,6\}$	12	73
$9,\ \{1,4,5,7\}$	26	157
$9,\ \{1,4,6,7\}$	31	177
$9,\ \{1,5,6,7\}$	12	65
$9,\ \{2,3,4,5\}$	1	9
$9,\ \{2,3,4,6\}$	5	47
$9,\ \{2,3,4,7\}$	5	43
$9,\ \{2,3,5,6\}$	15	98
$9,\ \{2,3,5,7\}$	25	171
$9,\ \{2,3,6,7\}$	17	107
$9,\ \{2,4,5,6\}$	16	102
$9,\ \{2,4,5,7\}$	39	231
$9,\ \{2,4,6,7\}$	50	281
$9,\ \{2,5,6,7\}$	24	131
$9,\ \{3,4,5,6\}$	5	32
$9,\ \{3,4,5,7\}$	11	81
$9,\ \{3,4,6,7\}$	22	132
$9,\ \{3,5,6,7\}$	23	135
$9,\ \{4,5,6,7\}$	6	35
$9,\ \{1,2,3,4,5\}$	0	1
$9,\ \{1,2,3,4,6\}$	0	5
$9,\ \{1,2,3,4,7\}$	0	5
$9,\ \{1,2,3,5,6\}$	0	15
$9,\ \{1,2,3,5,7\}$	0	25
$9,\ \{1,2,3,6,7\}$	0	17
$9,\ \{1,2,4,5,6\}$	5	33
$9,\ \{1,2,4,5,7\}$	9	71
$9,\ \{1,2,4,6,7\}$	9	83

Table 1, continued

$n,\ S \subseteq [1, n-2]$	$b_n(S)$	$b'_n(S)$
$9,\ \{1,2,5,6,7\}$	6	45
$9,\ \{1,3,4,5,6\}$	4	32
$9,\ \{1,3,4,5,7\}$	13	85
$9,\ \{1,3,4,6,7\}$	20	135
$9,\ \{1,3,5,6,7\}$	21	135
$9,\ \{1,4,5,6,7\}$	9	55
$9,\ \{2,3,4,5,6\}$	1	11
$9,\ \{2,3,4,5,7\}$	3	31
$9,\ \{2,3,4,6,7\}$	7	61
$9,\ \{2,3,5,6,7\}$	13	89
$9,\ \{2,4,5,6,7\}$	12	75
$9,\ \{3,4,5,6,7\}$	3	21
$9,\ \{1,2,3,4,5,6\}$	0	1
$9,\ \{1,2,3,4,5,7\}$	0	3
$9,\ \{1,2,3,4,6,7\}$	0	7
$9,\ \{1,2,3,5,6,7\}$	1	15
$9,\ \{1,2,4,5,6,7\}$	3	25
$9,\ \{1,3,4,5,6,7\}$	3	21
$9,\ \{2,3,4,5,6,7\}$	1	7
$9,\ \{1,2,3,4,5,6,7\}$	0	1

Table 1, continued

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