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Applications of the Hopf trace formula to computing homology representations

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0. Introduction.

The primary aim of this paper is to illustrate the use of a well-known technique of algebraic topology, the Hopf trace formula, as a tool in computing homology representations of posets. Inspired by a recent paper of Björner and Lovász ([BL]), we apply this tool to derive information about the homology representation of the symmetric group S_n on a class of subposets of the partition lattice Π_n. The majority of these subposets are not Cohen-Macaulay. Techniques for such computations have taken on an added significance because of recent developments in the theory of subspace arrangements, in particular the equivariant Goresky-MacPherson formula of [SWe1]. The latter formula reduces the calculation of the representation on the cohomology of the complement of a subspace arrangement to the problem of computing the homology representation on the intersection lattice of the arrangement.

Other methods for computing homology representations of posets arising from the partition lattice were introduced in [Su], and were applied to the k-equal partition lattice ([SWa]) and the corresponding subspace arrangement ([SWe1]).

Throughout the paper we will assume the reader is familiar with the ordinary representation theory of finite groups.

The first two sections of this paper are written with the non-specialist in mind. (Others may want to proceed directly to the discussion following Corollary 2.8.) Section 1 contains an elementary exposition of the Hopf trace formula, in the combinatorial context of a partially ordered set. This setting allows us to work with simplicial homology. We begin with the basic definitions of poset homology. The Hopf trace theorem is stated and proved. For a group G acting on an

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arbitrary poset P, the idea that the Möbius number of the fixed-point subposets gives information on the character values of the representation of G on the homology of P is illustrated with the example of S_n acting on the Boolean lattice of subsets of an n-element set. For Cohen-Macaulay posets, this approach was first pursued by Stanley ([St1]), and later by Calderbank, Hanlon and Robinson ([CHR]).

In Section 2 we focus on the action of the symmetric group, by permutation of coordinates, on products of posets. We show how, under certain conditions, the Hopf trace formula can be used to give a new, simple proof of the existence of sign twists in the action of S_n on the homology of a product of n copies of a poset P. These sign twists predict, for example, the exterior algebra structure of the cohomology of the complement of the complexified hyperplane arrangement of type A_{n-1} (see Arnold [A]). This action is of added interest because in many of the recently studied examples of hyperplane arrangements, and more generally subspace arrangements (see [Bj3], [BjWe] and [SWe1]), lower intervals in the intersection lattice L_n of the arrangement (where n is the dimension of the ambient space) are isomorphic to products of copies of the lattice L_t for smaller parameters t. From the formula of Goresky and MacPherson ([GM]), the homology of these lower intervals essentially determines the cohomology of the complement of the arrangement. From the recent generalisation of this formula in [SWe1], the group action on the cohomology of the complement is in turn determined (completely, in the complexified case) by the group action on the homology of the lower intervals. The paper of Björner [Bj3] is an excellent introduction to the combinatorial and topological aspects of subspace arrangements.

Another product, called the reduced product, is introduced and discussed in Section 2. The reduced product is a proper subposet of the ordinary product of posets (with unique minimal and maximal elements), and is exemplified by lower intervals in the d-divisible partition lattice (see [Bj3], [CHR] and [Wa]). The corresponding d-divisible orbit arrangement of [Bj3] is thus one example of a subspace arrangement in which reduced products arise. The homology representation for lower intervals in this case was determined in [Su], and applied to the cohomology of the arrangement and its orbit space in [SWe1]. A more general class of subposets of the partition lattice whose lower intervals admit a reduced product structure, is discussed in Section 4.

The discussion in Section 2 leads to a general theorem (Theorem 2.10) on computing the S_n-action on the homology of ordinary and reduced n-fold products of posets. We begin by using the Hopf trace formula to give elementary and self-contained proofs (see Propositions 2.3 and 2.7) of these results in the case when the poset is Cohen-Macaulay (or, in slightly more generality, in the case when the homology of the poset is concentrated in a single degree). The general case requires topological arguments, and is proved in [SWe2]. However, even in the case of posets with homology occurring in arbitrary degrees, the Hopf trace formula can be used to determine the sign twists on the virtual module consisting of the *alternating* sum, by degree, of the homology modules of the poset. This, the main result of this section, is recorded in Theorem 2.10.

These results are illustrated with applications to the ordinary partition lattice, (see [St1]) and also to the d-divisible lattice, and the subposets with block sizes congruent to 1 mod k, (whose homology representations were first studied in [CHR]).

In the next two sections of the paper, we use Theorem 2.10 together with the acyclicity of Whitney homology established in [Su], to derive new formulas for the homology representation of the symmetric group on subposets of the partition lattice with specified block sizes. In the case of arbitrary posets, the plethystic generating function determines the representation on the alternating sum of the top homology modules of the poset (one side of the Hopf trace formula). In this part of the paper we assume some familiarity with the theory of symmetric functions, as in [M].

In Section 3, following Björner and Lovász, we consider the class of posets of partitions with block sizes restricted to a fixed subset T containing 1. Again these posets exemplify the phenomenon of having lower intervals isomorphic to products. The formulas (3.1) and (3.2) of Theorem 3.5 may be viewed as group-equivariant generalisations of the formulas of [BL] for the Möbius function. In the case of the k-equal lattice, they specialise to an elegant identity (equation (3.3)) for the alternating sum of the homology modules.

Finally in Section 4 we consider the case of partitions with block sizes restricted to a fixed subset T^*, where now T^* does *not* contain 1. In this case the lower intervals are reduced products, and again Theorem 2.10 immediately gives a plethystic recurrence for the alternating sum of the top homology modules. This in turn gives generating functions for the Möbius number and the characteristic polynomial.

A major limitation of the formulas of Theorems 3.5 and 4.2 is that, in the case where homology occurs in arbitrary degrees, they fail to isolate the contribution to the homology of the poset by degree. For the k-equal partition lattice $\Pi_{(k,1^{n-k})}$, the homology representations by degree were determined in [SWa] by using a topological decomposition of the order complex provided by a lexicographic shelling of the poset. The lattice $\Pi_{(k,1^{n-k})}$ is the first known example of the extension of Björner's theory of shellability to nonpure posets; the shelling appears in [BWa] (see also [SWa]).

We conclude the paper in Section 5 with yet another identity (equation (5.1)) for the alternating sum of homology modules of partitions with restricted block sizes. Remarkably, in the case of the k-equal lattice, this formula predicts precisely the result of [SWa, Theorem 3.2] for the homology representation *in each degree*. This calculation suggests a general formula for the characteristic of the homology representation in degree r. For arbitrary posets, as in the k-equal case, it seems unlikely that the homology representation in each degree can be determined without additional topological information.

1. The Hopf trace formula.

We begin with some definitions. For more information on partially ordered sets (posets) and Möbius numbers see [St2]. All posets in this paper are assumed to be finite.

The unique minimal (respectively maximal) element of a poset P, when it exists, will be denoted $\hat{0}$ (respectively $\hat{1}$).

The proper part of a poset P with minimal element $\hat{0}$ and maximal element $\hat{1}$, denoted $\bar{P}$, is defined to be the subposet $P\backslash\{\hat{0},\hat{1}\}$. The order complex $\Delta(Q)$ of an arbitrary poset Q is the simplicial complex whose oriented i-dimensional simplices $[x_1, \ldots, x_{i+1}]$ are precisely the i-chains $(x_1 < \ldots < x_{i+1})$ of Q. (Points are 0-simplices, a line segment joining two points x_1 and x_2 is a 1-simplex and so on; the empty simplex has dimension -1.)

The general reference for concepts from algebraic topology is [Mu]. All homology in this paper will be over the field of complex numbers.

Let P be a poset with $\hat{0}$ and $\hat{1}$. We define the reduced homology of P in degree i, written $\widetilde{H}_i(P)$, to be the reduced homology in degree i of the simplicial complex $\Delta(\bar{P})$ of the proper part of P. (The order complex of P itself can be contracted to a point, and is therefore not of interest topologically. In particular its reduced homology vanishes identically.) Note that an oriented i-simplex in this simplicial complex corresponds to an $(i+2)$-chain $(\hat{0} = x_0 < x_1 \ldots < x_{i+1} < x_{i+2} = \hat{1})$ in P. In particular the 1-chain $(\hat{0} < \hat{1})$ corresponds to the empty simplex, of dimension (-1), in $\Delta(\bar{P})$.

For a simplicial complex, and in particular for a poset P, it is not hard to give a precise definition of homology. For $i \geq 1$, define $C_i(P)$ to be the (complex) vector space with basis the i-chains of P. Note that $C_1(P)$ is the one-dimensional vector space spanned by the empty chain $(\hat{0} < \hat{1})$. Define, for $i \geq 0$, the boundary operator $\partial_i : C_{i+2}(P) \mapsto C_{i+1}(P)$ by

$$\begin{aligned}&\partial_i(\hat{0} = x_0 < x_1 \ldots < x_{i+1} < x_{i+2} = \hat{1})\\ &= \sum_{j=1}^{i+1}(-1)^{j-1}(\hat{0} = x_0 < x_1 \ldots < \widehat{x_j} < \ldots < x_{i+1} < x_{i+2} = \hat{1}),\end{aligned}$$

where the hat denotes suppression of the element in the chain.

The simplicial boundary operator is defined in exactly the same way, with oriented simplices replacing the chains. (See [Mu, p. 26] for a rigorous definition of oriented simplices.) Readers unfamiliar with these concepts may find it instructive to see what the boundary operator does to an oriented 2-simplex $[x_1, x_2, x_3]$ (a closed triangle). Here one has

$$\partial_2([x_1, x_2, x_3]) = [x_2, x_3] - [x_1, x_3] + [x_1, x_2] = [x_2, x_3] + [x_3, x_1] + [x_1, x_2].$$

That is, intuitively, the image of ∂_2 is precisely the boundary of the oriented simplex, in the geometric sense.

Let r be the length of the longest chain in P from $\hat{0}$ to $\hat{1}$, so that $C_i(P) = 0$ if $i > r$. It is also easy to check that $\partial_i \circ \partial_{i+1} = 0$ for $0 \leq i \leq r-1$. Hence for $0 \le i \le r-1$, we can define the reduced homology $\widetilde{H}_i(P)$ of P in degree i to be the quotient of the kernel of ∂_i by the image of ∂_{i+1}. Define the reduced homology in degree -1 to be the quotient of $C_1(P)$ by the image of ∂_0. Note that ∂_0 is surjective if P has more than two elements, and hence $\widetilde{H}_{-1}(P)$ is zero unless P is the 2-element poset $\{\hat{0}, \hat{1}\}$, in which case it is defined to be $C_1(P)$.

Finally define the top homology of P, in degree r, to be $\widetilde{H}_{r-2}(P) = \ker \partial_{r-2}$. Note that, according to our definitions, the homology classes of $\widetilde{H}_i(P)$ are represented by linear combinations of chains of length $i+2$ (i.e., simplices of dimension i).

For more information on the Hopf trace formula in algebraic topology, see [Mu]. Let g be an automorphism of the poset P, or more generally, any ordering-preserving map of P into itself, that is, $x < y$ in P implies $g \cdot x < g \cdot y$. Then g induces an action on the chains of P via the formula $g : C_i(P) \mapsto C_i(P)$, for all $i \ge 1$,

$$g \cdot (\hat{0} = x_0 < x_1 < \ldots < x_i = \hat{1}) = (\hat{0} = g \cdot x_0 < g \cdot x_1 < \ldots < g \cdot x_i = \hat{1}).$$

In particular it is easy to check that, (for $i \ge 0$), $\partial_i \circ g = g \circ \partial_i$, that is, g commutes with the boundary operator.

THEOREM 1.1. *(The Hopf trace formula for posets) Let P be a poset with $\hat{0}$ and $\hat{1}$, let $r \ge 1$ be the length of the longest chain from $\hat{0}$ to $\hat{1}$. Let g be an order-preserving map $g : P \mapsto P$. Then g is an endomorphism both of the vector space of i-chains $C_i(P)$ and of the ith homology space $\widetilde{H}_i(P)$. We then have*

$$\sum_{i=1}^{r} (-1)^i \operatorname{tr}(g, C_i(P)) = \sum_{i=1}^{r} (-1)^i \operatorname{tr}(g, \widetilde{H}_{i-2}(P)). \tag{1.1}$$

Proof. If $r = 1$ the statement follows trivially from the definitions, so assume $r \ge 2$. Since g commutes with the boundary operator, the subspaces $\ker \partial_i$ and $\operatorname{im} \partial_i$ are invariant under g. In particular g acts on $\widetilde{H}_{i-2}(P)$, for $i \ge 1$. The hypothesis $r \ge 2$ guarantees that $\operatorname{im} \partial_0 = C_1(P)$.

We make two observations.

(i) Note that g commutes with the canonical isomorphism $C_i(P)/\ker \partial_{i-2} \simeq \operatorname{im} \partial_{i-2}$ induced by ∂_{i-2}. It follows that the trace of the map induced by g on the quotient $C_i(P)/\ker \partial_{i-2}$ equals the trace of g on $\operatorname{im} \partial_{i-2}$.

(ii) If ϕ is an endomorphism of a vector space V which leaves a subspace W invariant, then it is easy to see that the trace of the induced map on the quotient space V/W is the difference of the traces on V and W.

This gives the equality of traces $\operatorname{tr}(g, C_i(P)) = \operatorname{tr}(g, \ker \partial_{i-2}) + \operatorname{tr}(g, \operatorname{im} \partial_{i-2})$, and hence the left-hand side of (1.1) equals

$$
\begin{aligned}
&-\operatorname{tr}(g, \operatorname{im}\, \partial_0) + \sum_{i=2}^{r} (-1)^i \left(\operatorname{tr}(g, \ker \partial_{i-2}) + \operatorname{tr}(g, \operatorname{im}\, \partial_{i-2})\right) \\
&= \sum_{i=2}^{r} (-1)^i \left(\operatorname{tr}(g, \ker \partial_{i-2}) - \operatorname{tr}(g, \operatorname{im}\, \partial_{i-1})\right).
\end{aligned}
$$

A second application of observation (ii) completes the proof. □

Now take g to be the identity automorphism. Then the left-hand side of (1.1) is the Möbius number $\mu(P)$ of P; it is also the reduced Euler characteristic of the order complex of the proper part of P (see [Mu], [St2]). Hence we have, as a special case of (1.1),

$$\mu(P) = \sum_{i \geq 1} (-1)^i \dim \widetilde{H}_{i-2}(P). \tag{1.2}$$

More generally, since g acts by permuting the chains of P, the trace of g on $C_i(P)$ is the number of chains fixed by g. But a chain is fixed by g if and only if its elements are fixed point-wise by g. Hence the left-hand side is simply the Möbius number $\mu(P^g)$ of the subposet P^g of P consisting of elements fixed point-wise by g.

Thus the Hopf trace formula may be stated in the form

$$\mu(P^g) = \sum_{i \geq 1} (-1)^i \operatorname{tr}(g, \widetilde{H}_{i-2}(P)). \tag{1.3}$$

Some remarks about classical terminology from topology are in order. The right-hand side of (1.3) is called the (reduced) Lefschetz number of the map g on the order complex $\Delta(\bar{P})$, while the left-hand side is the reduced Euler characteristic of the fixed-point subcomplex $\Delta(\overline{P^g})$. The formula (1.1) was first stated, in these terms, by Baclawski and Björner (see [BB, Theorem 1.1]), who used it to study fixed points of maps on posets.

When g is an element of a group G of automorphisms of P, the equation (1.3) may be viewed as a group-equivariant version of (1.2): the trace is the character of a representation.

The preceding formulas are most useful when the homology of the poset vanishes in all but one degree r. In this case (1.3) becomes $\operatorname{tr}(g, \widetilde{H}_{r-2}(P)) = (-1)^r \mu(P^g)$, and hence gives a formula for the trace of g on the unique nonvanishing homology of P in terms of the Möbius number of the fixed-point subposet P^g.

An important case when this occurs is the following. (Again, see [St2] for more details.) We say that a poset P with $\hat{0}$ and $\hat{1}$ is Cohen-Macaulay if for every interval $[x, y]$ in P, the order complex of the open interval (x, y) has nonvanishing homology only in the highest possible degree. It can be shown that this implies that P and hence every interval $[x, y]$ is ranked, in which case the nonvanishing homology of the interval can occur only in degree $\operatorname{rank}(y) - \operatorname{rank}(x) - 2$. In particular the homology of P itself can occur only in degree $r-2$, where r is the rank of P.

One well-known example of this phenomenon is the rank n Boolean lattice B_n of subsets of $\{1, 2, \ldots, n\}$. The order complex $\Delta(\overline{B_n})$ triangulates the unit sphere S^{n-2} in $(n-1)$-dimensional space. (It is the barycentric subdivision of the boundary of an $(n-1)$-simplex.) From elementary algebraic topology one knows that (for $n \geq 0$) the reduced homology of a sphere S^n is nonvanishing only in degree n, in which case (over the complex field) it is a vector space of dimension 1. Since all proper intervals in B_n look like smaller Boolean lattices, it follows that B_n is Cohen-Macaulay, with unique nonvanishing homology in the top degree $n-2$. Hence, as a first application of the Hopf trace formula, we find that $\mu(B_n) = (-1)^{n-2} \dim \widetilde{H}_{n-2}(B_n)$. Since the latter dimension is 1, one has the well-known result that $\mu(B_n) = (-1)^n$.

The symmetric group S_n acts on B_n by permuting the elements, and thereby induces an action on the one-dimensional top homology. The Hopf trace formula can be used to determine this action.

PROPOSITION 1.2. *The symmetric group S_n acts on the homology of B_n like the sign representation.*

Proof. Let τ be the transposition $(1,2)$. Since the action is one-dimensional, it suffices to determine the trace of τ on the homology. The right-hand side of (1.3) equals $(-1)^n \operatorname{tr}(\tau, \widetilde{H}_{n-2}(B_n))$. The left-hand side of (1.3) is the Möbius number of the subposet of B_n fixed by τ; this is clearly isomorphic to B_{n-1}. Hence the left-hand side equals $\mu(B_{n-1}) = (-1)^{n-1}$.

It follows that the trace of τ on the top homology of B_n is -1, as claimed. $\square$

In the next section we shall continue to pursue the theme of converting an enumerative formula involving the Möbius number to a group-equivariant statement, via the Hopf trace formula. One enumerative result on the Möbius function which has proved to yield useful representation-theoretic formulas (see [Su, Theorem 1.10]) is the following result of Baclawski. (See [St1] for another application.)

PROPOSITION 1.3. *([Ba2, Lemma 4.6]) Let P be a poset with $\hat{0}$ and $\hat{1}$. Let Q be a subposet containing $\hat{0}$ and $\hat{1}$. Then*

$$\begin{aligned}&\mu(Q) - \mu(P)\\ &= \sum (-1)^t \mu_P(\hat{0}, z_1)\mu_P(z_1, z_2) \ldots \mu_P(z_t, \hat{1}),\end{aligned}$$

where the sum runs over all chains $\hat{0} < z_1 < \ldots < z_t < \hat{1}$ not in Q.

2. Products of posets.

Recall from [St2] that if P and Q are posets, then the product poset $P \times Q$ is defined to be the Cartesian product of P and Q together with the partial order defined by

$$(x_1, x_2) \leq (y_1, y_2) \iff x_1 \leq y_1 \text{ in } P \text{ and } x_2 \leq y_2 \text{ in } Q.$$

Also recall that the Möbius number of $P \times Q$ is $\mu(P)\mu(Q)$.

If P and Q are posets, each with $\hat{0}$ and $\hat{1}$, then there is a canonical homeomorphism (see [Wal, Theorem 5.1 (d)])

$$\Delta(\overline{P \times Q}) \simeq \Delta(\bar{P}) * \Delta(\bar{Q}) * S^0,$$

where the $*$ denotes the topological join (see [Mu]) and S^0 is the (two-point) unit sphere in one dimension. Hence essentially by the Künneth formula, (see [Mu, p. 373, Exercise 3]), for all $t \geq 1$, one has the following vector space isomorphism in homology

$$\widetilde{H}_{t-2}(P \times Q) \simeq \bigoplus_{\substack{i+j=t \\ i,j \geq 1}} \left(\widetilde{H}_{i-2}(P) \otimes \widetilde{H}_{j-2}(Q) \right).$$

In particular this isomorphism shows that products of Cohen-Macaulay posets are also Cohen-Macaulay.

Readers familiar with algebraic topology will know that the group-equivariant isomorphism of Proposition 2.1 below is, more generally, a consequence of the naturality of maps in the Künneth isomorphism (see [Mu]). We give a proof which illustrates the use of the Hopf trace formula when the poset has homology concentrated in a single degree.

Proposition 2.1. *Let P and Q be posets, each with $\hat{0}$ and $\hat{1}$, and let G_P, G_Q, be groups of automorphisms of P and Q respectively. Assume that the homology of P (respectively Q) is concentrated in a single dimension $r-2$ (respectively $s-2$). Then the homology of the product $P \times Q$ is concentrated in dimension $r+s-2$, and the vector spaces $\widetilde{H}_{r+s-2}(P \times Q)$ and $\widetilde{H}_{r-2}(P) \otimes \widetilde{H}_{s-2}(Q)$ are isomorphic as $(G_P \times G_Q)$-modules.*

Proof. The fact that the homology of the product is concentrated in degree $r+s-2$ follows from the Künneth formula, which, from the preceding remarks, also gives the isomorphism.

To show, via the Hopf trace formula, that the isomorphism of $\widetilde{H}_{r+s-2}(P \times Q)$ with $\widetilde{H}_{r-2}(P) \otimes \widetilde{H}_{s-2}(Q)$ commutes with the action of $(G_P \times G_Q)$, we apply (1.3) to the poset $(P \times Q)$ and the group element $g = (\sigma, \rho)$, where $\sigma \in G_P$ and $\rho \in G_Q$. The right-hand side of (1.3) is clearly $(-1)^{r+s}$ times the trace of g on the homology of $P \times Q$. Since g fixes an element (x, y) in $P \times Q$ if and only if $x \in P$ is fixed by σ and $y \in Q$ is fixed by ρ, the fixed-point subposet $(P \times Q)^g$ of the product is the product of fixed-point subposets $P^\sigma \times Q^\rho$. Hence the left-hand side of (1.3) is precisely the Möbius function $\mu((P \times Q)^g) = \mu(P^\sigma \times Q^\rho)$. This in turn equals the product of Möbius functions $\mu(P^\sigma)\mu(Q^\rho)$. By another application of (1.3) to P, σ, and Q, ρ, we have

$$(-1)^{r+s} \operatorname{tr}(g, \widetilde{H}_{r+s-2}(P \times Q)) = (-1)^r \operatorname{tr}(\sigma, \widetilde{H}_{r-2}(P)) \cdot (-1)^s \operatorname{tr}(\rho, \widetilde{H}_{s-2}(Q)),$$

from which it follows that

$$\operatorname{tr}(g, \widetilde{H}_{r+s-2}(P \times Q)) = \operatorname{tr}\left(g, \widetilde{H}_{r-2}(P) \otimes \widetilde{H}_{s-2}(Q)\right).$$

This establishes the claim. $\square$

Denote by P^n the product of P with itself n times. Then S_n acts on P^n by permuting the coordinates. If G is a group of automorphisms of P, then the wreath product $S_n[G]$ is a group of automorphisms of P^n. We wish to determine the representation of S_n on the homology of P^n. If V is an S_n-module and W is a G-module, we denote by $V[W]$ the corresponding module for $S_n[G]$. As a vector space $V[W]$ is simply the tensor product of the nth tensor power of W with V. The action of G is just the diagonal action on the n-fold tensor product $W^{\otimes n}$. For a precise description of the S_n-action, see [JK]. In this section we will be concerned only with the case in which V is a one-dimensional representation of S_n, that is, either the trivial representation or the sign representation. Thus $V[W]$ is isomorphic to $W^{\otimes n}$ as a vector space. An element σ in S_n acts on $W^{\otimes n}$ by permuting the tensor positions according to σ, and multiplying by c, where c is 1 if the representation V of S_n is trivial, and equals the sign of σ otherwise.

We record first the following general lemma.

Lemma 2.2. *Let P be a poset with $\hat{0}$ and $\hat{1}$. Let τ be the automorphism of $P \times P$ which permutes coordinates, and let Q be a subposet of $P \times P$, containing $(\hat{0}, \hat{0})$ and $(\hat{1}, \hat{1})$, such that*

(i) Q is invariant under τ and

(ii) Q contains all the diagonal elements (x, x).

Then one has the equation

$$\mu(P) = \sum_{i \geq 1} (-1)^i \operatorname{tr}(\tau, \widetilde{H}_{i-2}(Q). \tag{2.1}$$

Proof. Apply the Hopf trace formula (1.3) to the poset Q. By definition, $\tau(x, y) = (y, x)$. The key observation is that by definition of Q, all elements of the form (x, x) are in Q. Hence the fixed-point subposet Q^τ coincides with $(P \times P)^\tau$, which in turn is clearly poset-isomorphic to P. $\square$

Proposition 2.3. *Let G be a group of automorphisms of a poset P with $\hat{0}$ and $\hat{1}$. Assume that the homology W_r of P is concentrated in a single dimension r. Denote by sgn_n the sign representation of S_n. Then the homology of the product P^n is concentrated in dimension $n(r+2)-2$, and there is an isomorphism of $S_n[G]$-modules between $\widetilde{H}_{n(r+2)-2}(P^n)$ and $V_r[W_r]$ where V_r is the representation of S_n equal to $(\operatorname{sgn}_n)^r$. That is, V_r is the sign representation if r is odd, and is trivial if r is even.*

Proof. By Proposition 2.1, there is a $G^n = (\underbrace{G \times \ldots \times G}_{n})$-isomorphism between $\widetilde{H}_{n(r+2)-2}(P^n)$ and $W_r^{\otimes n}$. Hence, to show that the $S_n[G]$-action is as stated, we compute the trace of a transposition $\tau \in S_n$; we assume $n = 2$. Since τ permutes the two copies of P, we have $\tau(x, y) = (y, x)$.

Apply equation (2.1) of the preceding lemma to the product $Q = P \times P$. We obtain

$$\mu(P) = \sum_{i \geq 1} (-1)^i \operatorname{tr}(\tau, \tilde{H}_{i-2}(P \times P)). \tag{2.2}$$

If the homology of P is concentrated in a single degree r, (for instance when P is Cohen-Macaulay), then the homology of $P \times P$ is concentrated in degree $2(r+2)-2$, by the Künneth formula, and equation (2.2) becomes

$$\operatorname{tr}(\tau, \tilde{H}_{2r+2}(P \times P)) = \mu(P) = (-1)^r \dim(W_r).$$

But the latter term is precisely the trace of τ on the wreath product module $V_r[W_r]$, where V_r is the one-dimensional representation $(\mathrm{sgn}_2)^r$ of S_2. □

Recall that the partition lattice Π_n is defined to be the poset of set partitions of an n-element set, ordered by refinement. (Thus the least element is the finest partition with n blocks, in which every element is in its own block, and the greatest element is the partition with one block of size n.) Proposition 2.3 can be applied to compute the group action on the homology of lower intervals of the partition lattice. In order to state this and other similar results, we need some notation. Write π_n for the representation of S_n on the top homology of the partition lattice Π_n, and h_n, e_n respectively for the trivial and sign representations of S_n.

If x is a set partition in Π_n with m_i blocks of size i, then the type of x is defined to be the integer partition λ of n, $\lambda = \prod_i i^{m_i}$. The stabiliser of x under the action of S_n, which we sometimes denote by $\mathrm{stab}(x)$, is the normaliser G_λ of the Young subgroup S_λ; it is a direct product of wreath product groups. Finally if V_i is an S_{a_i}-module, we write $V_1[V_2]$ for the wreath product; this is a module for the wreath product group $S_{a_1}[S_{a_2}]$ (the normaliser of the product of a_1 copies of S_{a_2}).

Part (i) of Corollary 2.4 below can be deduced by combining the work of Orlik and Solomon with Lehrer and Solomon's determination of the S_n-module structure of the cohomology of the braid arrangement for A_{n-1} (see [LS, Theorem 4.5]). A direct proof was given in [Su, Theorem 1.8]. Part (ii) below also follows from recent work of Hanlon and Wachs ([HW, Lemma 3.1]).

Corollary 2.4.

(i) Let x be an element of the partition lattice Π_n. Assume that x is of type $\lambda = \prod_i i^{m_i}$. Then the representation of the stabiliser G_λ of x on the homology of the interval $[\hat{0}, x]$ in Π_n is given by

$$h_{m_1}[\pi_1] \otimes e_{m_2}[\pi_2] \otimes \ldots \otimes h_{m_{2i-1}}[\pi_{2i-1}] \otimes e_{m_{2i}}[\pi_{2i}] \otimes \ldots .$$

(ii) Let P_n be the poset $\Pi_n^{(1,k)}$ of partitions of n with block sizes congruent to 1 mod k, (of rank $[\frac{n}{k}]$) and let β_n denote the representation of S_n on the top homology of $\Pi_n^{(1,k)}$. Let $x \in P_n$ be of type $\lambda = \prod_i (1 + ka_i)^{m_{a_i}}$. Then the representation of G_λ on the homology of $[\hat{0}, x]$ in P_n is given by

$$f_{m_{a_1}}[\beta_{1+ka_1}] \otimes \ldots \otimes f_{m_{a_j}}[\beta_{1+ka_j}] \otimes \cdots,$$

where $f_{m_{a_j}}$ is the one-dimensional representation of $S_{m_{a_j}}$ equal to $h_{m_{a_j}}$ if a_j is even, and equal to $e_{m_{a_j}}$ if a_j is odd.

Proof. First note that the zero-rank one-point poset Π_1, contributes only the trivial representation to the homology. Thus $\pi_1 = h_1$.

(i) First we note the well-known fact Π_n is a geometric lattice, and hence Cohen-Macaulay (see [Bj1]). Since the lattice Π_n has rank $n-1$, it has nonvanishing homology only in degree $n-3$. Observe that a lower interval $[\hat{0}, x]$ in Π_n is isomorphic to a product of copies of Π_j for parameters j smaller than n. This is precisely the situation where Propositions 2.3 and 2.1 apply.

(ii) Note that if n is odd, then the poset $P_n = \Pi_n^{(1,k)}$ has its own greatest and least elements, while for even n the $\hat{1}$-element must be artificially adjoined. Also the rank of P_n is $\lfloor \frac{n}{k} \rfloor$. This poset was first studied in [CHR], where the representation β_n was determined. The fact that it is a Cohen-Macaulay poset follows from [Bj1, Theorem 6.1]: Björner showed that the poset is ranked and locally semimodular.

Clearly a lower interval $[\hat{0}, x]$ in P_n, for x of the given type, is indeed isomorphic to a direct product of posets P_{1+ka_j}, and the result follows as before. Note that the rank of P_{1+ka_j} is precisely a_j. $\square$

These "sign twists" were also noticed by this author for lower intervals of another much-studied subposet of the partition lattice, the d-divisible lattice Π_n^d consisting of partitions in Π_{nd} formed by all partitions whose block sizes are divisible by d. This lattice was first considered by Calderbank, Hanlon and Robinson, who determined the representation of S_{nd} on the top homology. The d-divisible lattice again turns out to be Cohen-Macaulay ([CHR], [Wa]; see also [Sa]). For this lattice it is not quite true that lower intervals are isomorphic to products.

To see exactly what happens, write $P_n = \Pi_n^d$. Then P_n is a lattice of rank n, with rank function $\mathrm{rank}(x) = n+1$ minus the number of blocks of x. For $x \in P_n$, the lower interval $[\hat{0}, x]$ is in fact a proper subposet of the appropriate product of posets P_j for $j < n$. The difficulty is introduced by the $\hat{0}$-element, which must be artificially appended.

The action of the stabiliser subgroup $\mathrm{stab}(x) = G_\lambda$ on the homology of the lower interval $[\hat{0}, x]$ in Π_n^d was determined in [Su, Theorem 5.3] by a somewhat intricate argument. Our aim now is to give a simpler proof, by establishing an analogue of Proposition 2.3 for an appropriate class of posets which will include the d-divisible lattice.

The general situation which describes the "almost-product"structure of lower intervals of Π_n^d is as follows. Let P be any poset with $\hat{0}$ and $\hat{1}$, and let $\bar{P} = P \backslash \{\hat{0}, \hat{1}\}$ denote the proper part of P. We define the *reduced product* of two posets P_1 and P_2 to be the poset obtained by attaching to the product $(\bar{P}_1 \cup \{\hat{1}_1\}) \times (\bar{P}_2 \cup \{\hat{1}_2\})$ the minimal element $\{(\hat{0}_1, \hat{0}_2)\}$.

We denote the reduced product by $P_1 \dot{\times} P_2$; note that it is a poset with unique maximal element $(\hat{1}_1, \hat{1}_2)$ and unique minimal element $(\hat{0}_1, \hat{0}_2)$. It is clear how to extend this construction to an arbitrary number of posets. Also note that $P_1 \dot{\times} P_2$ is in fact the subposet of $P_1 \times P_2$ obtained by removing all elements (x_1, x_2) such that x_i equals $\hat{0}_i$ for exactly one i. More generally, a reduced product of n posets P_i (with minimal elements $\hat{0}_i$ and maximal elements $\hat{1}_i$) is the subposet of $P_1 \times \ldots \times P_n$ obtained by excluding all elements $(x_1, \ldots, x_n)$ having at least one coordinate x_i equal $\hat{0}_i$, and at least one coordinate x_j different from $\hat{0}_j$.

The last observation makes it clear that a lower interval $(\hat{0}, x)$ in the d-divisible partition lattice Π_n^d is a reduced product of d-divisible lattices Π_j^d for parameters $j < n$. A similar observation holds for the subposets $\Pi_n^{(i,k)}$ consisting of partitions in Π_n with all blocks sizes congruent to i mod k, where $2 \leq i \leq k-1$. These posets, however, are not always Cohen-Macaulay. We defer the discussion of their homology representations to Section 4, where we shall have more general tools at our disposal.

Proposition 2.5. *For $i = 1, 2$ let P_i be Cohen-Macaulay posets of rank r_i, and let $Q = P_1 \dot{\times} P_2$ be the reduced product of P_1 and P_2. Then $Q = P_1 \dot{\times} P_2$ is Cohen-Macaulay of rank $r_1 + r_2 - 1$, and*

$$\widetilde{H}_{r_1+r_2-3}(P_1 \dot{\times} P_2) \simeq \widetilde{H}_{r_1+r_2-2}(P_1 \times P_2).$$

Proof. The statement about the rank is easily verified.

To show that Q is Cohen-Macaulay we need to show that any interval $[x, y]$ of Q has nonvanishing homology only in the top dimension. Note that an interval $[x, y]$ in Q coincides with the interval $[x, y]$ in $P_1 \times P_2$ if $x \neq (\hat{0}_1, \hat{0}_2)$. Since ordinary products of Cohen-Macaulay posets are well-known to be Cohen-Macaulay (see Section 1), it is enough to consider intervals $[x, y]$ where $x = (\hat{0}_1, \hat{0}_2)$. Since such intervals are clearly in turn reduced products $I_1 \dot{\times} I_2$ of intervals I_j of P_j, $j = 1, 2$, it suffices to give the argument for the whole poset Q.

A theorem of Quillen [Qui, Proposition 1.9] states that if Q_i, $i = 1, 2$, are posets, and Q_i' is obtained from Q_i by attaching a unique maximal element $\hat{1}_i$, (topologically the order complex $\Delta(Q_i')$ of Q_i' is a cone over the order complex of Q_i) then the order complex of chains of the poset $Q_1' \times Q_2' \backslash \{(\hat{1}_1, \hat{1}_2)\}$ is homeomorphic to the join of the order complexes of Q_1 and Q_2. It now follows from the Künneth theorem of algebraic topology (see [Mu, Exercise 3, p. 373]) that

$$\widetilde{H}_{j+1}(\Delta(Q_1' \times Q_2' \backslash \{(\hat{1}_1, \hat{1}_2)\})) = \bigoplus_{p_1+p_2=j} \widetilde{H}_{p_1}(\Delta(Q_1)) \otimes \widetilde{H}_{p_2}(\Delta(Q_2)).$$

Now observe that our definition of reduced product coincides with Quillen's construction, if we take for Q_i the proper part of the Cohen-Macaulay poset P_i. Recall from Section 1 that the homology of the poset P is defined to be the homology of the order complex of its proper part $\bar{P}$. Hence we have

$$\widetilde{H}_{j+1}(P_1 \dot{\times} P_2) = \bigoplus_{p_1+p_2=j} \widetilde{H}_{p_1}(P_1) \otimes \widetilde{H}_{p_2}(P_2).$$

Using the fact that P_i is Cohen-Macaulay, we obtain that the reduced product has nonvanishing homology only in degree $(r_1 + r_2 - 3)$, in which case the homology is isomorphic to

$$\widetilde{H}_{r_1-2}(P_1) \otimes \widetilde{H}_{r_2-2}(P_2).$$

Finally (again by the Künneth theorem), this latter expression is precisely the unique nonvanishing homology of the ordinary product poset $P_1 \times P_2$, in degree $r_1 + r_2 - 2$. $\square$

The isomorphism of the preceding proposition can be made group-equivariant, presumably by a more careful examination of Quillen's homeomorphism. However, for the benefit of readers who are less topologically inclined, and in order to keep our treatment self-contained, we shall give a more elementary proof by using the Hopf trace formula.

Proposition 2.6. *For $i = 1, 2$, let P_i be Cohen-Macaulay posets of rank r_i, and let Q be the reduced product of P_1 and P_2. Then the homology of Q (in degree $r_1 + r_2 - 3$) is isomorphic to the homology of $P_1 \times P_2$, and hence to $\widetilde{H}_{r_1-2}(P_1) \otimes \widetilde{H}_{r_2-2}(P_2)$; furthermore, if G_i is a group of automorphisms of P_i, then $G_1 \times G_2$ acts on Q, and the two homology modules are isomorphic as $(G_1 \times G_2)$-modules.*

Proof. In the present case Proposition 1.3 gives

$$\begin{aligned} &\mu(Q) - \mu(P_1 \times P_2) \\ &= \sum (-1)^t \mu_{P_1\times P_2}((\hat{0}_1, \hat{0}_2), z_1)\mu_{P_1\times P_2}(z_1, z_2) \ldots \mu_{P_1\times P_2}(z_t, (\hat{1}_1, \hat{1}_2)), \end{aligned} \tag{2.3}$$

where the sum runs over all chains $(\hat{0}_1, \hat{0}_2) < z_1 < \ldots < z_t < (\hat{1}_1, \hat{1}_2)$ *not* in Q.

Let (g, h) be an element of $G_1 \times G_2$. Recall that, because Q is Cohen-Macaulay, the trace of (g, h) on the top homology of Q is given by the Möbius number of the fixed-point subposet $Q^{(g,h)}$ of Q, times $(-1)^{\operatorname{rank} Q}$. We shall apply (2.3) to the appropriate fixed-point subposets, thereby converting the enumerative statement to a statement about the character values. In this case, noting that the subposet of $P_1 \times P_2$ fixed by (g, h) is in fact the product $P_1^g \times P_2^h$, the equation (2.3) becomes

$$\begin{aligned} &\mu(Q^{(g,h)}) - \mu(P_1^g \times P_2^h) \\ &= \sum (-1)^t \mu_{P_1^g\times P_2^h}((\hat{0}_1, \hat{0}_2), z_1)\mu_{P_1^g\times P_2^h}(z_1, z_2) \ldots \mu_{P_1^g\times P_2^h}(z_t, (\hat{1}_1, \hat{1}_2)), \end{aligned} \tag{2.4}$$

where the sum now runs over all chains $(\hat{0}_1, \hat{0}_2) < z_1 < \ldots < z_t < (\hat{1}_1, \hat{1}_2)$ which are in $P_1^g \times P_2^h$ but *not* in $Q^{(g,h)}$.

Since elements not in Q are of the form $(\hat{0}_1, y)$ or $(x, \hat{0}_2)$ where $x \in P_1$, $y \in P_2$, it is clear that every chain $c_1 = (\hat{0}_1 < x_1 < \ldots < x_t \leq \hat{1}_1)$ in P_1^g contributes exactly one chain in the above sum, namely the chain whose ith element is $(x_i, \hat{0}_2)$, here $1 \le i \le t$.

The summand in (2.4) corresponding to this chain is easily seen to be

$$\begin{aligned}&F_1^g(t; \hat{0}_1 < x_1 < \ldots < x_t \le 1_1)\\ &= (-1)^t \mu_{P_1^g}(\hat{0}_1, x_1)\mu_{P_1^g}(x_1, x_2)\ldots\mu_{P_1^g}(x_{t-1}, x_t)\mu_{P_1^g}(x_t, \hat{1}_1)\mu_{P_2^h}(\hat{0}_2, \hat{1}_2);\end{aligned}$$

here we have used the fact that, in the product poset $P_1 \times P_2$, the fixed-point Möbius number $\mu_{P_1\times P_2{}^{(g,h)}}((a_1, a_2), (b_1, b_2))$ equals $\mu_{P_1^g}(a_1, a_2)\mu_{P_2^h}(b_1, b_2)$.

By symmetry, it is clear that similarly every chain $c_2 = (\hat{0}_2 < y_1 < \ldots < y_t \le \hat{1}_2)$ in P_2^h contributes exactly one chain in the above sum, namely the chain whose ith element is $(\hat{0}_1, y_i)$, and the corresponding summand is given by the expression

$$\begin{aligned}&F_2^h(t; \hat{0}_2 < y_1 < \ldots < y_t \le 1_2)\\ &= (-1)^t \mu_{P_2^h}(\hat{0}_2, y_1)\mu_{P_2^h}(y_1, y_2)\ldots\mu_{P_2^h}(y_{t-1}, y_t)\mu_{P_2^h}(y_t, \hat{1}_2)\mu_{P_1^g}(\hat{0}_1, \hat{1}_1).\end{aligned}$$

Clearly these chains account for all the terms in the right-hand side of (2.4). Hence the right-hand side of (2.4) is in fact

$$\sum_{t=1}^{r_1}\sum F_1^g(t; \hat{0}_1 < x_1 < \ldots < x_t \le 1_1) + \sum_{t=1}^{r_2}\sum F_2^h(t; \hat{0}_2 < y_1 < \ldots < y_t \le 1_2),$$

where each x_i is fixed by g, and each y_i is fixed by h.

Next observe that for fixed t, terms of the form $F_1^g(t; \hat{0}_1 < x_1 < \ldots < x_t = 1_1)$ cancel out terms of the form $F_1^g(t-1; \hat{0}_1 < x_1 < \ldots < x_{t-1} < 1_1)$. A similar observation applies to the expressions F_2^h. By following this cancellation from $t = r_i$ (chains of maximum length in P_i) down to $t = 1$, we find that equation (2.4) reduces to

$$\begin{aligned}\mu(Q^{(g,h)}) - \mu(P_1^g \times P_2^h) &= F_1^g(1; \hat{0}_1 < x_1 = \hat{1}_1) + F_2^h(1; \hat{0}_2 < y_1 = \hat{1}_2)\\ &= -2\mu_{P_1^g}(\hat{0}_1, \hat{1}_1)\mu_{P_2^h}(\hat{0}_2, \hat{1}_2).\end{aligned}$$

and hence

$$\mu(Q^{(g,h)}) = -\mu(P_1^g \times P_2^h).$$

Finally recall that the ranks of Q and $P_1 \times P_2$ differ by exactly one; this absorbs the extra minus sign in the above equation, giving

$$\mathrm{tr}((g, h), \widetilde{H}_{r_1+r_2-3}(Q)) = \mathrm{tr}((g, h), \widetilde{H}_{r_1+r_2-2}(P_1 \times P_2)),$$

as claimed. □

Remark 2.6.1. By specialising the equation $\mu(Q^{(g,h)}) = -\mu(P_1^g \times P_2^h)$ to the case when g and h are the respective identity elements of the groups G_i, we find the Möbius number of the reduced product to be

$$\mu(Q) = -\mu(P_1)\mu(P_2).$$

Now we examine the group action on the homology of the reduced product poset Q. Note how the conclusion about the sign twists differs from Proposition 2.3.

PROPOSITION 2.7. *Let P be a Cohen-Macaulay poset with homology W_r concentrated in degree r, and let $\dot{P}^n$ be the n-fold reduced product poset $\underbrace{P\dot{\times}\ldots\dot{\times}P}_{n}$. Then the action of $S_n[G]$ on the homology of $\dot{P}^n$ is given by $(\mathrm{sgn}_n)^{r-1}[W_r]$.*

Proof. In view of Proposition 2.6, as in the proof of Proposition 2.3 it suffices to compute the trace of a transposition τ, and hence to consider the case $n=2$. Noting that $\dot{P}^2$ satisfies the hypotheses of Lemma 2.2, we need only observe that in this case, $\dot{P}^2$ has homology in degree $2r+1$, (while $P\times P$ has homology in degree $2r+2$). Hence from equation (2.1) of Lemma 2.2, the trace of τ (which permutes the two copies of P) on the homology of $\dot{P}^2$ equals $-\mu(P)$. The result now follows from Proposition 2.6. $\square$

The analogue of Corollary 2.4 for the d-divisible lattice now follows immediately from the preceding proposition. Note that Proposition 2.3, used with the incorrect assumption that for x in Π_n^d, the interval $[\hat{0},x]$ is a product of Π_j^d for $j<n$, would predict the wrong result for the homology of the interval: we leave it to the reader to check that the predicted sign twists would be exactly the opposite of what they should be.

COROLLARY 2.8. *([Su, Theorem 5.3]) Let π_n^d denote the representation of S_n on the top homology of the rank n lattice Π_n^d. If $x\in\Pi_n^d$ is of type $\lambda=\prod_i(di)^{m_i}$, then the representation of the stabiliser G_λ on the homology of the interval $[\hat{0},x]$ in Π_n^d is*

$$h_{m_1}[\pi_1^d]\otimes e_{m_2}[\pi_2^d]\otimes\ldots\otimes h_{m_{2i-1}}[\pi_{2i-1}^d]\otimes e_{m_{2i}}[\pi_{2i}^d]\otimes\ldots.$$

Next we consider what information can be extracted from the Hopf trace formula in the case when the poset is not Cohen-Macaulay. This will be useful in the next section, where we shall derive identities similar to Corollary 2.4 and Corollary 2.8 for a class of posets which are not necessarily Cohen-Macaulay. It turns out that in this case Lemma 2.2 can be interpreted as a statement about virtual homology modules.

Let P be a poset with least and greatest elements, and let $\mathrm{Alt}(P)$ denote the alternating sum, by degree, of the (reduced) top homology modules of P. That is, $\mathrm{Alt}(P)$ is the virtual module

$$\sum_{i\geq -1}(-1)^i\widetilde{H}_i(P).$$

By (1.2), the virtual dimension of $\mathrm{Alt}(P)$ is precisely the Möbius number $\mu(P)$ of P.

Now let G be a group of automorphisms of P. As above, let P^n and $\dot{P}^n$ denote respectively the n-fold product and the n-fold reduced product of P with itself. Then the homology modules $\widetilde{H}(P^n)$ and $\widetilde{H}(\dot{P}^n)$ are both $S_n[G]$-modules.

Consider first the case of the ordinary product. The Künneth isomorphism

$$\tilde{H}_{j+2(n-1)}(P^n) \simeq \bigoplus_{r_1+\ldots+r_n=j} \left(\tilde{H}_{r_1}(P) \otimes \ldots \otimes \tilde{H}_{r_n}(P)\right)$$

suggests that the action of G on the homology of P induces the G^n-action on $\mathrm{Alt}(P^n)$ via the isomorphism

$$\mathrm{Alt}(P^n) \simeq \mathrm{Alt}(P)^{\otimes n}, \tag{2.5}$$

This can be shown exactly as in Proposition 2.1, by interpreting the statement of the Hopf trace formula in the context of virtual modules: one has only to take care at each step to interpret the value of the Möbius function as the trace on a virtual module. (Alternatively, it follows from general considerations by using the Künneth formula.)

In the case of reduced products, it is instructive to examine the isomorphism in each degree given by the Künneth formula and Quillen's theorem (cf. proof of Proposition 2.5) more closely. One has the isomorphisms

$$\bigoplus_{r_1+\ldots+r_n=j} \left(\tilde{H}_{r_1}(P) \otimes \ldots \otimes \tilde{H}_{r_n}(P)\right) \simeq \tilde{H}_{j+2(n-1)}(P^n) \simeq \tilde{H}_{j+n-1}(\dot{P}^n).$$

This in turn gives the isomorphism of virtual modules

$$\begin{aligned} &(-1)^{n-1}(-1)^{j+n-1}\tilde{H}_{j+n-1}(\dot{P}^n) \\ \simeq &\sum_{r_1+\ldots+r_n=j} \left((-1)^{r_1}\tilde{H}_{r_1}(P) \otimes \ldots \otimes (-1)^{r_n}\tilde{H}_{r_n}(P)\right). \end{aligned}$$

Note the extra $(n-1)$th power of (-1) attached to the homology of the reduced product. Hence the correct statement for reduced products is that the action of G on the homology of P induces a G^n-action on the virtual module $(-1)^{n-1}\mathrm{Alt}(\dot{P}^n)$, via the isomorphism

$$(-1)^{n-1}\mathrm{Alt}(\dot{P}^n) \simeq \mathrm{Alt}(P)^{\otimes n}. \tag{2.6}$$

Again, one can give a self-contained and elementary proof of this isomorphism by mimicking the proof of Proposition 2.6 and interpreting the Möbius numbers as traces on virtual modules. Note in particular (cf. Remark 2.6.1) that $\mu(\dot{P}^n) = (-1)^{n-1}\mu(P)^n$.

We record the analogue of Proposition 2.1 for these virtual modules below, in the form most appropriate for the results of the following sections.

Proposition 2.9. *Let P_i, $i = 1, \ldots, k$, be posets each with $\hat{0}$ and $\hat{1}$, and let G_i be a group of automorphisms of P_i. Let $P = \times_i P_i^{m_i}$ for positive integers m_i, be the product of posets, and let $\dot{P}$ denote the reduced product $\dot{\times}_i \dot{P}_i^{m_i}$. If G^{m_i} denotes the m_i-fold direct product of the group G_i with itself, then there are isomorphisms of virtual $\times_i G_i^{m_i}$-modules*

$$Alt(P) \simeq \otimes_i Alt(P_i)^{\otimes m_i} \qquad \text{and} \qquad (-1)^{r-1}Alt(\dot{P}) \simeq \otimes_i Alt(P_i)^{\otimes m_i},$$

where $r = \sum_{i=1}^{k} m_i$.

Proof. By interpreting the fixed-point Möbius function as a trace on the virtual module in the proofs of Propositions 2.1 and 2.6, we find that for any posets P_1 and P_2, as in (2.5) and (2.6) there are $(G_1 \times G_2)$-equivariant homomorphisms

$$\mathrm{Alt}(P_1 \times P_2) \simeq \mathrm{Alt}(P_1) \otimes \mathrm{Alt}(P_2), \qquad (-1)\mathrm{Alt}(P_1 \dot{\times} P_2) \simeq \mathrm{Alt}(P_1) \otimes \mathrm{Alt}(P_2).$$

To obtain the sign in the statement about the reduced products, we compute, by iterating (2.6), that

$$\begin{aligned}(-1)^{r-1}\mathrm{Alt}(\dot{P}) &\simeq (-1)^{r-1}(-1)^{k-1} \otimes_{i=1}^{k} \mathrm{Alt}(\dot{P_i}^{m_i}) \\ &\simeq (-1)^{r+k} \otimes_{i=1}^{k} (-1)^{m_i-1}\mathrm{Alt}(P_i)^{\otimes m_i},\end{aligned}$$

which verifies the result. □

Now we shall show that the virtual modules $\mathrm{Alt}(P^n)$ and $(-1)^{n-1}\mathrm{Alt}(\dot{P}^n)$ are in fact $S_n[G]$-modules, and more precisely that

$$\mathrm{Alt}(P^n) \simeq f_n[\mathrm{Alt}(P)], \qquad (-1)^{n-1}\mathrm{Alt}(\dot{P}^n) \simeq g_n[\mathrm{Alt}(P)],$$

where f_n, g_n, are one-dimensional representations of S_n. In view of the preceding remarks, in order to determine the virtual $S_n[G]$-module structure it remains to determine the action of S_n in each case.

The analogues of Propositions 2.3 and 2.7 are as follows. Recall that h_n and e_n denote respectively the trivial and sign representations of S_n.

THEOREM 2.10. *Under the preceding hypotheses, the action of* $S_n[G]$

(i) on the virtual module $\mathit{Alt}(P^n)$ *is given by* $h_n[\mathit{Alt}(P)]$;

(ii) on the virtual module $(-1)^{n-1}\mathit{Alt}(\dot{P}^n)$ *is given by* $e_n[\mathit{Alt}(P)]$.

Proof. By arguments as in Proposition 2.3, it suffices to consider the case $n = 2$. As before, let τ be the transposition in S_2, which acts by switching coordinates in $P \times P$. Equation (2.1) of Lemma 2.2 gives

$$\mathrm{tr}(\tau, \mathrm{Alt}(P^2)) = \mu(P) = \dim(\mathrm{Alt}(P)). \tag{2.7}$$

(i) Since $\mathrm{Alt}(P^2)$ is isomorphic to $\mathrm{Alt}(P) \otimes \mathrm{Alt}(P)$, the result follows.

(ii) This time we have a shift in dimension by 1, as observed in (2.6) above. That is, as virtual modules $(-1)\mathrm{Alt}(\dot{P}^2)$ is isomorphic to $\mathrm{Alt}(P) \otimes \mathrm{Alt}(P)$. Hence, applying Lemma 2.2 to the poset $\dot{P}^2$, from (2.1) we have

$$\mathrm{tr}(\tau, -\mathrm{Alt}(\dot{P}^2)) = (-1) \cdot \mathrm{tr}(\tau, \mathrm{Alt}(\dot{P}^2)) = \mathrm{sgn}(\tau)\dim(\mathrm{Alt}(P)), \tag{2.8}$$

as claimed. □

Proposition 2.3 can be generalised in a more explicit form, to posets with homology occurring in arbitrary degrees. This result appears in [SWe2, Theorem 1.1]. Unfortunately the Hopf trace formula can no longer be used in the proof, because it fails to isolate the contribution to homology by degree. The proof in [SWe2] is topological.

3. Partitions with block sizes restricted to T : the case $1 \in T$.

In this section we use the result on the sign twists of the symmetric group S_n on a product of posets, Theorem 2.10, to derive plethystic identities for the homology representation of other posets. The posets of this section were first considered systematically by Björner and Lovász ([BL]), who determined generating functions for the Möbius numbers and the characteristic polynomial. We recover these results as a special case of our representation-theoretic equation. Our general plethystic equation (3.1) specialises to an intriguing identity (3.3) for the homology representation of the k-equal lattice.

Let T be a subset of the positive integers such that $1 \in T$. Let k denote the least element of the set $T\backslash\{1\}$. For $m \geq k$, define $\Pi_{m,T}$ to be the poset of partitions in Π_m with block sizes restricted to the set T, with a $\hat{1}$-element artificially adjoined if $m \notin T$. If $m = 1$, define $\Pi_{1,T}$ to be the one-point poset with nonzero reduced homology only in degree 0. By convention the homology of $\Pi_{1,T}$ is the trivial module. It follows that $\Pi_{m,T}$ is the 2-element poset $\{\hat{0}, \hat{1}\}$ if and only if there are no partitions of m with block sizes in T, other than the trivial partition into m blocks.

We shall use the product poset theorems of the previous sections to deduce information about the homology representations of the subposets $P_m = \Pi_{m,T},\ m \geq k$.

Recall from [M] that the Frobenius characteristic map, denoted ch, is a one-to-one correspondence from the ring of homogeneous symmetric functions of degree m onto the representation ring of S_m. In particular the dimension of a (possibly virtual) S_m-module W_m is the coefficient of the square-free term in the symmetric function $\mathrm{ch}(W_m)$. The characteristics of the trivial and sign representations of S_m are respectively the homogeneous symmetric function h_m and the elementary symmetric function e_m. Note that this is consistent with the notational conventions made in the remarks following Proposition 2.3. Recall also the involution ω on the ring of symmetric functions which takes h_m to e_m; applying ω corresponds, in the representation ring of S_m, to tensoring with the sign representation of S_m.

If μ is an integer partition of n, write $\mu \vdash n$. Also let $\ell(\mu)$ denote the number of parts of μ. If $x \in \Pi_{n,T}$ has type μ, denote by G_x the stabiliser of x in S_n. Then G_x is the normaliser $N(S_\mu)$ of the Young subgroup S_μ in S_n, hence is a direct product of wreath product groups as described in Section 2.

Write $\pi^r_{n,T}$ for the Frobenius characteristic of the S_n-action on the homology of $\Pi_{n,T}$ in degree r. Also write $\pi_{n,T}$ for the characteristic of the alternating sum of homology modules $\sum_{r\geq -1}(-1)^r \widetilde{H}_r(\Pi_{n,T})$, so that

$$\pi_{n,T} = \sum_{r\geq -1} (-1)^r \pi^r_{n,T}.$$

In particular $\pi^s_{1,T}$ equals h_1 if $s = 0$, and zero otherwise. For $n \geq k$ and

$2 \leq r < n$, define $V_r(\Pi_{n,T})$ to be the virtual module

$$V_r(\Pi_{n,T}) = \sum_{\substack{x \in \Pi_{n,T} \\ x \text{ has } r \text{ blocks}}} \sum_{k \geq -1} (-1)^k \widetilde{H}_k(\hat{0}, x).$$

Define $V_n(\Pi_{n,T})$ to be the trivial S_n-module. Also set $V_1(\Pi_{n,T})$ to be the alternating sum of the top homology modules of $\Pi_{n,T}$, i.e.,

$$V_1(\Pi_{n,T}) = \sum_{k \geq -1} (-1)^k \widetilde{H}_k(\hat{0}, \hat{1}).$$

Our conventions imply that the Frobenius characteristic of $V_1(\Pi_{n,T})$ is $\pi_{n,T}$, and that the Möbius number $\mu(\Pi_{n,T})$ is the coefficient of any square-free monomial in $\pi_{n,T}$.

The product poset theorem may now be reformulated in an S_n-equivariant way as follows.

THEOREM 3.1. *Let $1 \in T$, $n \geq k$. The Frobenius characteristic of the representation of S_n on the (virtual) module $V_r(\Pi_{n,T})$ is given by the degree n term in the plethysm*

$$h_r \left[h_1 + \sum_{m \in T, m \neq 1} \pi_{m,T} \right],$$

provided $2 \leq r \leq n$ or $r = 1$ and $n \in T$.

Proof. We shall apply Proposition 2.9 and Theorem 2.10. Note that by the preceding remarks, the characteristic of $V_n(\Pi_{n,T})$ is h_n, and this agrees with the degree n term in the plethysm of the formula.

If $r = 1$ and $n \in T$ the statement follows from the preceding definitions, so we assume $2 \leq r < n$.

By definition, as an S_n-module, $V_r(\Pi_{n,T})$ is isomorphic to $\oplus_{\mu \vdash n, \ell(\mu) = r} W_\mu$, where, for any $x \in \Pi_{n,T}$ of type μ, W_μ is the alternating sum of induced modules

$$W_\mu = \sum_{k \geq -1} (-1)^k \widetilde{H}_k(\hat{0}, x) \Big\uparrow_{G_x}^{S_n} = \mathrm{Alt}((\hat{0}, x)) \Big\uparrow_{G_x}^{S_n},$$

in the terminology of Theorem 2.10.

Suppose the part i occurs in μ with multiplicity m_i. In particular, $m_i > 0$ implies that $i \in T$. From Proposition 2.9 and Theorem 2.10 (i), $\mathrm{Alt}((\hat{0}, x))$ is G_x-isomorphic to $\otimes_i h_{m_i}[\pi_{i,T}]$. Hence W_μ has Frobenius characteristic $\prod_i h_{m_i}[\pi_{i,T}]$.

From standard properties of the plethysm operation, we conclude that the characteristic of $V_r(\Pi_{n,T})$ is the degree n term in $h_r[\sum_{i \in T} \pi_{i,T}]$, as claimed. $\square$

Next we need to recall the definition of Whitney homology for a poset. This was originally defined by Baclawski ([Ba1]). We shall use the following equivalent definition of Whitney homology, due to Björner ([Bj2, Theorem 5.1]). This formulation brings out the usefulness of the concept for posets and subspace arrangements.

Definition 3.2. *The Whitney homology of a poset P in degree r is defined to be*

$$WH_r(P) = \bigoplus_{x \in P} \widetilde{H}_{r-2}(\hat{0}, x).$$

In [Su] the following important property, the acyclicity of Whitney homology for posets, was established using Baclawski's original definition.

Proposition 3.3. *([Su, Lemma 1.1]) Let P be a poset with least element $\hat{0}$ and greatest element $\hat{1}$; let G be the automorphism group of P. Assume that the Whitney homology groups are free in all degrees. Then each Whitney homology is a G-module. If r is the maximum length of a chain from $\hat{0}$ to $\hat{1}$ in P, then as a virtual sum of G-modules, one has*

$$WH_r(P) - WH_{r-1}(P) + \ldots + (-1)^{r-1} WH_1(P) + (-1)^r WH_0(P) = 0;$$

here $WH_0(P)$ is interpreted as the trivial G-module.

Proof. For this proof we make the mild assumption that the group G is finite. For the general case, see [Su, Lemma 1.1]. In keeping with the spirit of this paper, our proof uses the Hopf trace formula. In particular, this proof will show that the result is simply a group-equivariant version of the defining recursion for the Möbius function of a poset.

Let g be an element of the group G. It suffices to show that the trace of g on the alternating sum of the Whitney homology groups is zero. From Definition 3.2, it is clear that, as a virtual G-module, the alternating sum of Whitney homology groups in fact equals the sum of induced (virtual) modules

$$\sum_{x \in P \backslash G} \mathrm{Alt}((\hat{0}, x)) \big\uparrow_{G_x}^{G},$$

where we have used the notation of Theorem 2.10, and as usual G_x denotes the stabiliser of x. Also the sum runs over one element in each orbit of the action of G on P. Hence the alternating sum of traces $\sum_{i=0}^{r} (-1)^i \operatorname{tr}(g, WH_i(P))$ equals

$$\begin{aligned}\sum_{x \in P \backslash G} \operatorname{tr}(g, (\mathrm{Alt}(\hat{0}, x) \big\uparrow_{G_x}^{G})) &= \sum_{x \in P \backslash G} \sum_{\substack{u \in G \\ u^{-1} g u \in G_x}} \operatorname{tr}(u^{-1} g u, \mathrm{Alt}((\hat{0}, x))) \\ &= \sum_{x \in P \backslash G} \sum_{\substack{u \in G \\ u^{-1} g u \in G_x}} \mu_{P^{u^{-1} g u}}(\hat{0}, x),\end{aligned}$$

where we have used equation (1.3), and written P^h for the subposet of P consisting of points fixed by h. Since $u^{-1} g u \in G_x$ if and only if $g \in G_{u \cdot x}$, that is, if and only if $u \cdot x$ is fixed by g, we can replace the two sums in the latter expression by a single sum running over the fixed-point poset P^g.

Hence the alternating sum of traces equals $\sum_{y \in P^g} \mu_{P_g}(\hat{0}, y)$, which vanishes by definition of the Möbius function. $\square$

With the above notations, for the family of subposets $P_m = \Pi_{m,T}$ we immediately have

COROLLARY 3.4. *Let $1 \in T$. For fixed $m \in T$, $m \geq 2$, the sum of virtual S_m-modules*

$$\sum_{r \geq 1} V_r(\Pi_{m,T})$$

vanishes identically.

When $1 \in T$, the symmetric functions $\pi_{m,T}$ encoding the alternating sum of the top homology modules of $\Pi_{m,T}$ are completely determined by the following generating function:

THEOREM 3.5. *Let $1 \in T$, and let k be the least element of the set $T \backslash \{1\}$. Then*

$$\sum_{i \geq 1} h_i \left[\sum_{m \in T} \pi_{m,T} \right] = \sum_{1 \leq i < k} h_i - \sum_{\substack{m \notin T \\ m > k}} \pi_{m,T}. \tag{3.1}$$

Proof. By Theorem 3.1 and Corollary 3.4, for $n \geq k$, the degree n term in the plethysm

$$\sum_{i \geq 1} h_i [\sum_{m \in T} \pi_{m,T}]$$

vanishes if $n \in T$, and equals $-\pi_{n,T}$ if $n \notin T$. If $1 \leq n < k$, it is clear that the degree n term in the plethysm is h_n. The identity follows. □

Now we apply this identity to some familiar examples. The results of Examples 3.6 and 3.7 are due respectively to Stanley and to Calderbank, Hanlon and Robinson (see [CHR]). They may also be derived by applying the acyclicity of Whitney homology to the dual poset; see [Su, Example 1.6].

EXAMPLE 3.6. Take T to be the set of all positive integers. In this case $\Pi_{n,T} = \Pi_n$. Since the partition lattice is Cohen-Macaulay, it has nonvanishing homology only in the top dimension $n-3$, and hence $\pi_{n,T} = (-1)^{n-3} \pi_{n,T}^{n-3}$. Writing π_n for the characteristic of the S_n action on the top homology of Π_n, in this case we obtain the following well-known generating function identity for π_n :

$$\sum_{i \geq 1} h_i \left[\sum_{n \geq 1} (-1)^{n-1} \pi_n \right] = h_1.$$

Equivalently, the alternating sum $\sum_{n \geq 1} (-1)^{n-1} \pi_n$ is the plethystic inverse of the sum $\sum_{i \geq 1} h_i$.

Using the last observation, one has the following alternative formulation of equation (3.1):

$$\sum_{m \in T} \pi_{m,T} = \sum_{i \geq 1} (-1)^{i-1} \pi_i \left[\sum_{i=1}^{k-1} h_i - \sum_{\substack{m \notin T \\ m > k}} \pi_{m,T} \right]. \tag{3.2}$$

Example 3.7. Let $k_0 \geq 2$, and let T be the set of all positive integers congruent to 1 mod k_0. Note that $k = \min(T \backslash \{1\}) = k_0 + 1$. Now $\Pi_{n,T}$ is just the poset $\Pi_n^{(1,k_0)}$ of [CHR], which was discussed in Section 2. In particular it is Cohen-Macaulay, and thus $\pi_{n,T}$ equals $(-1)^{\lfloor \frac{n}{k_0} \rfloor}\beta_n$, where β_n is the characteristic of the top homology of $\Pi_n^{(1,k_0)}$. From Theorem 3.5 we deduce the generating function for the β_n (note that $\beta_1 = h_1$):

$$\sum_{i \geq 1} h_i \left[\sum_{j \geq 0} (-1)^j \beta_{k_0 j+1}\right] = \sum_{i=1}^{k_0} h_i - \sum_{\substack{n \geq k_0+1 \\ n \not\equiv 1 \mod k_0}} (-1)^{\lfloor \frac{n}{k_0} \rfloor} \beta_n.$$

If λ is an integer partition of n of length r with all parts congruent to $1 \mod k_0$, then $n \equiv r \mod k_0$. Hence, noting that $\beta_i = h_i$ for $1 \leq i \leq k_0$, and separating terms according to congruence mod k_0 on either side, we obtain the following formulas of Calderbank, Hanlon and Robinson ([CHR, Theorem 4.7]):

$$\sum_{i \geq 0} h_{k_0 i+1} \left[\sum_{j \geq 0} (-1)^j \beta_{k_0 j+1}\right] = h_1,$$

and, for fixed $r \neq 1$, $0 \leq r \leq k_0 - 1$,

$$\sum_{n \equiv r \mod k_0} h_n \left[\sum_{j \geq 0} (-1)^j \beta_{k_0 j+1}\right] = - \sum_{n \equiv r \mod k_0} (-1)^{\lfloor \frac{n}{k_0} \rfloor} \beta_n.$$

From these formulas one obtains generating functions for the Möbius number. (A general method is explained in Corollary 3.11 below.) Let $M_r^{(1,k_0)}(x)$ denote the exponential generating function for the Möbius number of $\Pi_n^{(1,k_0)}$ when $n \equiv r$ mod k_0. Also let $G_{r,k_0}(x) = \sum_{n \equiv r \mod k_0} \frac{x^n}{n!}$. Then the preceding identities give the formulas

$$G_{1,k_0}(M_1^{(1,k_0)}(x)) = x; \qquad M_r^{(1,k_0)}(x) = -G_{r,k_0}(M_1^{(1,k_0)}(x)).$$

For more explicit solutions to these equations (for instance when $k_0 = 2$), see [St3, Chapter 5, Exercise 23].

Next we consider an example of a non-Cohen-Macaulay poset in the family $\Pi_{n,T}$.

Example 3.8. Let $k \geq 3$, and take $T = \{i : i \geq k\}$. Clearly $\Pi_{n,T}$ is the k-equal lattice of [BLY] and [BWe], denoted $\Pi_{(k,1^{n-k})}$ in [SWa] and [SWe1]. Its homotopy type and Betti numbers were determined by Björner and Welker in [BWe]. In particular $\Pi_{(k,1^{n-k})}$ is not Cohen-Macaulay. For simplicity write $\pi_{n,k}^r$ for the characteristic of the S_n-action on the homology in degree r of $\Pi_{(k,1^{n-k})}$. As in Example 3.6, let π_n denote the characteristic $\pi_{n,2}^{n-3}$, i.e., π_n is the characteristic of the top homology of Π_n.

Equation (3.2) now gives the generating function

$$h_1 + \sum_{n\geq k} \sum_{r} (-1)^r \pi^r_{n,k} = \sum_{i\geq 1} (-1)^{i-1} \pi_i \left[h_1 + \ldots + h_{k-1}\right] \tag{3.3}$$

The homology representations $\pi^r_{n,k}$ in degree r were determined in [SWa]. See also Section 5.

Let $S_{\leq k-1}(n,i)$ denote the number of nonempty set partitions of n into i blocks of sizes $1, \ldots, k-1$.

Corollary 3.9. *[BLY, Corollary 5.3] The Möbius number of $\Pi_{(k,1^{n-k})}$ is given by*

$$\mu(\Pi_{(k,1^{n-k})}) = \sum_{i\geq 1} (-1)^{i-1} (i-1)! S_{\leq k-1}(n,i).$$

Proof. It suffices to compute the (virtual) dimension of the S_n-module encoded by the degree n term in each side of the identity (3.3) of Example 3.8. Note that on the left we obtain the alternating sum of the Betti numbers of $\Pi_{(k,1^{n-k})}$, which is indeed the Möbius number. On the right-hand side one computes the dimension using [Su, Lemma 1.5]. □

It was shown in [BWe], where the lattice $\Pi_{(k,1^{n-k})}$ was first studied for its topological properties, that the homology of $\Pi_{(k,1^{n-k})}$ is nonvanishing only in dimensions $n-3-t(k-2)$. In particular, if k is even, this means that the degree n term in the alternating sum of homology modules is in fact $(-1)^{n-3}$ times a representation of S_n. Hence we have the curious observation

Corollary 3.10. *Suppose $k \geq 3$ is even. Then the degree n term in the plethysm*

$$\sum_{i\geq 1} (-1)^{i-1} \pi_i \left[h_1 + \ldots + h_{k-1}\right]$$

is $(-1)^{n-3}$ times the Frobenius characteristic of a (nonvirtual) representation of S_n (namely, the sum of the homology modules of $\Pi_{(k,1^{n-k})}$). In particular, if $k \geq 3$ is even, the Möbius number of the k-equal lattice $\Pi_{(k,1^{n-k})}$ is never zero.

On the other hand for odd k, it is easy to check that the alternating sum of homology modules of $\Pi_{(k,1^{n-k})}$ usually contains irreducibles with both positive and negative multiplicity. When $k=3$, the smallest example occurs for $n=6$. One computes that $\pi_{6,3} = \pi^2_{6,3} - \pi^1_{6,3} = (3,1^3) - (5,1) - (3,3)$, where the integer partitions index the corresponding irreducibles. More generally, Björner and Lovász ([BL]) show that when k is odd, the Möbius number of $\Pi_{(k,1^k)}$ is zero, and hence the case $n = 2k$ provides a counter-example for odd k.

Theorem 3.5, or more precisely equation (3.2), also contains much of the enumerative information of the poset. In order to keep our treatment self-contained, we now show how the results of [BL] on the Möbius function and the characteristic polynomial follow by specialising the equations (3.1) and (3.2). The methods described below are used in the next section to obtain new formulas for a different class of posets. It should be pointed out that if one were interested in computing only the Möbius number and the characteristic polynomial (as defined below), there are more traditional combinatorial methods, most notably Stanley's theory of exponential structures and the compositional formula (see [St3, Theorem 5.1.4]). In fact the plethystic identities of this section and the next may be viewed as representation-theoretic analogues of the enumerative formulas that would arise by using the techniques of [St3, Chapter 5].

Denote by $p_T(x)$ the power-series

$$\sum_{j=1}^{k-1}\frac{x^j}{j!}-\sum_{\substack{j\notin T\\ j>k}}\mu_{j,T}\frac{x^j}{j!}.$$

Corollary 3.11. *([BL, Corollary 3.5]) Let T be a subset of the positive integers containing 1, and let $k=\min(T\backslash\{1\})$. The Möbius numbers $\mu_{n,T}$ of the posets $\Pi_{n,T}$ satisfy the equation*

$$\sum_{n\in T}\mu_{n,T}\frac{x^n}{n!}=\log\left(1+p_T(x)\right).$$

(Recall that by convention, $\mu_{1,T}=1$.)

Proof. For the Möbius number one has to extract the square-free term in equation (3.2). Recall that the dimension of an S_m-module W_m is the coefficient of $p_1^m/m!$ (in the basis of power-sums) in the characteristic $\mathrm{ch}(W_m)$. (As in [M], $p_1=h_1$ denotes the first power-sum.) Consequently it suffices to restrict the equation (3.2) to the terms consisting of powers of p_1. Using, for example, the well-known fact (see [St1]) that the homology of Π_n has dimension $(n-1)!$, one sees that the term involving only p_1 in π_i is p_1^i/i. But $p_1^i[f]=f^i$ for any symmetric function f, and hence one has

$$p_1+\sum_{\substack{n\in T\\ n>1}}\mu_{n,T}\frac{p_1^n}{n!}=\sum_{i\geq 1}(-1)^{i-1}\frac{1}{i}\left(\sum_{i=1}^{k-1}\frac{p_1^i}{i!}-\sum_{\substack{n\notin T\\ n>k}}\mu_{n,T}\frac{p_1^n}{n!}\right)^i.$$

The result now follows upon replacing p_1 by x. □

Note that for the k-equal lattice, ($T=\{1,k,k+1,\ldots\}$), this gives the formula of [BLY, Theorem 5.5].

One can also derive an equation for the characteristic polynomial of the poset $\Pi_{n,T}$. For this calculation we first record the following generating function for the full partition lattice Π_n. By the Cohen-Macaulay property, $V_r(\Pi_n)$ is precisely $(-1)^{n-r}$ times the Whitney homology of Π_n in degree $n-r$. Let $\ell_m(t)$ denote the polynomial

$$\ell_m(t)=\frac{1}{m}\sum_{d|m}\mu(d)t^{\frac{m}{d}}.$$

In particular $\ell_1(t)=t$.

LEMMA 3.12. *([Su, proof of Theorem 2.2]) The sum* $\sum_{r=1}^{n} \mathrm{ch}(V_r(\Pi_n))t^r$ *is given by the degree n term in the symmetric function*

$$(-1)^n \prod_{m\geq 1} (1-(-1)^{m-1}p_m)^{\ell_m(t)}.$$

In particular, the term involving only powers of p_1 *is given by the degree n term in* $(1+p_1)^t$.

As in [BL], for $n \geq k$ define the characteristic polynomial of the poset $\Pi_{n,T}$ to be

$$\chi_{n,T}(t) = \sum_{x\in\Pi_{n,T}} \mu(\hat{0},x)t^{b(x)},$$

where $b(x)$ denotes the number of blocks of x, and by convention $b(\hat{1}) = 1$ when $n \notin T$. Observe that the Möbius number is obtained by evaluating, at $t=0$, the derivative of the characteristic polynomial with respect to t.

Now we use Corollary 3.12 to derive the Björner-Lovász formula for the characteristic polynomial $\chi_{n,T}(t)$. Denote by $G_T(x,t)$ the generating function

$$1 + \sum_{n=1}^{k-1} \frac{t^n x^n}{n!} + \sum_{n\geq k} \chi_{n,T}(t)\frac{x^n}{n!} - t\sum_{\substack{n\geq k\\ n\notin T}} \mu(\Pi_{n,T})\frac{x^n}{n!}.$$

COROLLARY 3.13. *([BL, Theorem 4.4 and Corollary 4.5]) Let T be a subset of positive integers containing 1, and let k be the least element of* $T\backslash\{1\}$. *Then one has*

$$G_T(x,t) = (1+p_T(x))^t = \left(1 + \left(\sum_{i=1}^{k-1}\frac{x^i}{i!} - \sum_{\substack{m\notin T\\ m\geq k}} \mu_{m,T}\frac{x^m}{m!}\right)\right)^t.$$

Proof. Write ρ_T for the symmetric function $\sum_{i=1}^{k-1} h_i - \sum_{\substack{m\notin T\\ m\geq k}} \pi_{m,T}$. Observe that for $n \geq k$, $\chi_{n,T}(t)$ is $n!$ times the square-free term in the sum $\sum_{r\geq 1} t^r\,\mathrm{ch}(V_r(\Pi_{n,T})$.

Using (3.2), (3.1) and Theorem 3.1, we have

$$1 + \sum_{n\geq 1}\sum_{r\geq 1} h_r\left[\sum_{m\in T}\pi_{m,T}\right] = 1 + \sum_{n\geq 1}\sum_{r\geq 1} t^r\,\mathrm{ch}(V_r(\Pi_n))\,[\rho_T]\,.$$

Now we extract the square-free term on each side. This in turn can be achieved by extracting the terms involving only the first power-sum p_1, as in Corollary 3.11. The expression involving only powers of p_1 on the left is precisely $G_T(p_1,t)$; on the right, using Lemma 3.12, it consists of the terms involving only p_1 in the expression

$$(1+p_1)^t\,[\rho_T] = (1+\rho_T)^t.$$

But the terms supported by powers of p_1 in ρ_T are given by $p_T(p_1)$, where $p_T(x)$ is the power-series of Corollary 3.12. Hence we have

$$G_T(p_1, t) = (1 + p_T(p_1))^t. \tag{3.4}$$

Replacing p_1 by x gives the assertion. □

By extracting the coefficient of x^n on both sides of this equation, one obtains an explicit expression for the characteristic polynomial of $\Pi_{n,T}$ as follows.

COROLLARY 3.14. *Let $n \in T$. Then $\chi_{n,T}(t)$ equals*

$$\sum_{s\geq 1} t(t-1)\dots(t-(s-1)) \sum_{\substack{\lambda\vdash n, \ell(\lambda)=s \\ m_i(\lambda)>0 \text{ only if } i=1 \text{ or } i\notin T}} r(\lambda) \prod_{i\geq k} (-\mu_{i,T})^{m_i},$$

where $r(\lambda)$ denotes the number of partitions in Π_n of type λ, and the inner sum runs over all integer partitions λ none of whose parts belong to T, except for parts equal to 1.

In particular for the k-equal lattice one has the following formula:

$$\chi_{n,T}(t) = \sum_{s\geq 1} t(t-1)\dots(t-(s-1))S_{\leq k-1}(n,s), \tag{3.5}$$

in agreement with the Möbius number formula of Corollary 3.9.

4. Partitions with block sizes restricted to T^* : the case $1 \notin T^*$.

Let T^* be a nonempty subset of the positive integers such that $1 \notin T^*$. Throughout this section we let d denote the least element of T^*. Note that $d \geq 2$. A wide class of subposets of partitions whose lower intervals are reduced products is obtained by considering partitions with block sizes in T^*. More precisely, for $n \geq d$, we define Π_{n,T^*} to be the poset of partitions of Π_n with block sizes in T^*, with a $\hat{0}$-element artificially adjoined; in addition, if $n \notin T^*$, we also append a $\hat{1}$-element. Hence for $n \geq d$, Π_{n,T^*} equals the 2-element poset $\{\hat{0}, \hat{1}\}$ if and only if either $n = \min(T^*) = d$ or there are no partitions in Π_n with block sizes in T^*. The d-divisible partition lattice ($d > 1$) is an example, with $T^* = \{id : i \geq 1\}$.

Define the symmetric functions π^r_{n,T^*}, π_{n,T^*} exactly as in the previous section, with T^* replacing T. Thus $\pi_{d,T^*} = -h_d$. Also for $1 < r < n$, define the virtual module $V_r(\Pi_{n,T^*})$ to be

$$V_r(\Pi_{n,T^*}) = \sum_{\substack{x\in\Pi_{n,T^*} \\ x \text{ has } r \text{ blocks}}} \sum_{k\geq -1} (-1)^k \tilde{H}_k(\hat{0}, x).$$

Define $V_n(\Pi_{n,T^*})$ to be the trivial S_n-module. Also set $V_1(\Pi_{n,T^*})$ to be the alternating sum of the top homology modules of Π_{n,T^*}, i.e.,

$$V_1(\Pi_{n,T^*}) = \sum_{k\geq -1} (-1)^k \tilde{H}_k(\hat{0}, \hat{1}).$$

In all other cases $V_r(\Pi_{n,T^*})$ is the zero module. Our conventions imply that the Frobenius characteristic of $V_1(\Pi_{n,T^*})$ is π_{n,T^*}, and that the Möbius number $\mu(\Pi_{n,T^*})$ is the coefficient of the square-free monomial in π_{n,T^*}.

We have, in analogy with Theorem 3.1,

THEOREM 4.1. *Let $1 \notin T^*$. Let $n \geq d$. The characteristic of the representation of S_n on the (virtual) module $V_r(\Pi_{n,T^*})$ is given by the degree n term in the plethysm*

$$-h_r\left[-\sum_{m\in T^*}\pi_{m,T^*}\right],$$

provided $2 \leq r \leq n-1$ and $V_r(\Pi_{n,T^*})$ is nonzero, or $r = 1$ and $n \in T^*$.

Proof. We refer the reader to the proof of Theorem 3.1 for notation. Let x be a partition of type μ, where $\mu = \prod_i i^{m_i}$ has r parts. From Proposition 2.9 and Theorem 2.10 (ii), $(-1)^{r-1}\mathrm{Alt}((\hat{0}, x))$ is G_x-isomorphic to $\otimes_i e_{m_i}[\pi_{i,T^*}]$. Hence the virtual module $(-1)^{r-1}W_\mu$ has Frobenius characteristic $\prod_i e_{m_i}[\pi_{i,T^*}]$. Just as in Theorem 3.1, we can conclude that the characteristic of $(-1)^{r-1}V_r(\Pi_{n,T^*})$ is the degree n term in $e_r[\sum_{i\in T^*}\pi_{i,T^*}]$. Now use the rule of signs for plethysm (see [M]), which says that if f and g are symmetric functions, and f is homogeneous, then $f[-g] = (-1)^{\deg f}\omega(f)[g]$. $\square$

Recall that when $r = n$, by convention $V_r(\Pi_{n,T^*})$ is the trivial S_n-module, and hence has characteristic equal to h_n. Using Corollary 3.4, we have the analogue of Theorem 3.5. The functions π_{m,T^*} encoding the alternating sum of the top homology modules of Π_{m,T^*} are completely determined by the following generating function.

THEOREM 4.2. *Let d be the least element of T^*. Then*

$$\sum_{i\geq 1} h_i\left[-\sum_{m\in T^*}\pi_{m,T^*}\right] = \sum_{i\geq d} h_i + \sum_{\substack{m\notin T^*\\ m>d}}\pi_{m,T^*}, \tag{4.1}$$

or equivalently,

$$-\sum_{m\in T^*}\pi_{m,T^*} = \sum_{i\geq 1}(-1)^{i-1}\pi_i\left[\sum_{i\geq d} h_i + \sum_{\substack{m\notin T^*\\ m>d}}\pi_{m,T^*}\right]. \tag{4.2}$$

Proof. Consider the plethysm

$$\sum_{r\geq 1}(-1)h_r\left[-\sum_{m\in T^*}\pi_{m,T^*}\right] \tag{4.3}$$

By the acyclicity of Whitney homology (Proposition 3.3), the sum of virtual modules $\sum_{r\geq 1} V_r(\Pi_{n,T})$ vanishes identically. Clearly all terms in (4.3) have degree at least d. Let $n \in T^*$. Then the degree n term in (4.3) is the characteristic of $\sum_{r\geq 1} V_r(\Pi_{n,T^*})$, and hence equals $-h_n = -\mathrm{ch}(V_n(\Pi_{n,T^*}))$ in this case. If

$n \notin T^*$, $n \geq d$, the degree n term is the characteristic of $\sum_{r\geq 2} V_r(\Pi_{n,T^*})$ and hence equals $-\operatorname{ch}(V_n(\Pi_{n,T^*}) + V_1(\Pi_{n,T^*})) = -\pi_{n,T^*} - h_n$. $\square$

We consider some examples. First we recall (see [St2]) that in any poset P, an element y is said to cover an element x if there is no element z such that $x < z < y$. For a poset P with $\hat{0}$, we shall say x is an atom if x covers $\hat{0}$.

EXAMPLE 4.3. Let $T^* = \{d\}$. Then (for $n \geq d$) $\Pi_{n,T^*} = \{\hat{0}, \hat{1}\}$ has homology in degree -1 if $n = d$ or if d does not divide n, and hence in these cases $\pi_{n,T^*} = -h_n$. If $n = kd$ and $k \geq 2$, then clearly the nontrivial elements of Π_{n,T^*} are the atoms of the d-divisible lattice Π_n^d. Hence the homology is concentrated in degree 0, and $\pi_{kd,T^*} = h_k[h_d] - h_{kd}$. It is clear that these facts are confirmed by the identity (4.1).

EXAMPLE 4.4. Let $T^* = \{n : n \geq d\}$. Note that Π_{m,T^*} is not Cohen-Macaulay in general. For $d = 2$ and $m = 6$, it is easy to see that it is not even ranked; there are two kinds of maximal chains, $(\hat{0} < x_1 < x_2 < \hat{1})$ where x_1 and x_2 are of types $(2,2,2)$ and $(4,2)$ respectively, and $(\hat{0} < y_1 < \hat{1})$ with y_1 of type $(3,3)$. Here (4.2) gives

$$-\sum_{m\in T^*} \pi_{m,T^*} = \sum_{i\geq 1} (-1)^{i-1}\pi_i \left[\sum_{i\geq d} h_i\right],$$

and hence, by extracting the square-free term of degree n, for $n \in T^*$ the Möbius number is

$$\mu_{n,T^*} = \sum_{i\geq 1} (-1)^i (i-1)! S_{\geq d}(n,i), \tag{4.4}$$

where $S_{\geq d}(n,i)$ is the number of partitions of n into i blocks all of size at least d. Note how these formulas compare with the formulas of Example 3.8 and Corollary 3.9 for the k-equal lattice. It would be interesting to see if a lexicographic shelling exists for this poset, as in [BWa].

The next example, the d-divisible partition lattice, is the most interesting for a variety of reasons. The representations were shown to have interesting properties in [CHR]; a nice basis for the top homology (this poset is Cohen-Macaulay) was given in [Wa]. It is also the intersection lattice of an orbit arrangement, see [Bj3].

EXAMPLE 4.5. Let $d \geq 2$, and let $T^* = \{id : i \geq 1\}$. Then Π_{n,T^*} is precisely the d-divisible lattice of Corollary 2.8. Since $\Pi_{n,T^*} = \{\hat{0}, \hat{1}\}$ if d does not divide n, and hence $\pi_{n,T^*} = -h_n$, (4.2) gives the following formula, equivalent to [CHR, Theorem 4.6]:

$$\sum_{m\geq 1} \pi_m^d (-1)^m = \sum_{i\geq 1} (-1)^i \pi_i \left[h_d + h_{2d} + \ldots\right].$$

In [Su, Example 1.6 (iii)] this formula was derived by applying the acyclicity of Whitney homology to the dual poset. Again by specialising one obtains, for the Möbius number, the formula

$$\mu_{n,T^*} = \sum_{i\geq 1}(-1)^i (i-1)! S^{(d)}(n,i),$$

where $S^{(d)}(n,i)$ is the number of partitions of n into i blocks all of size divisible by d. More explicitly, using his theory of exponential structures [St3, Theorem 5.1] and [St2, Proposition 3.16.4], Stanley showed that the Möbius number μ_{n,T^*} is in fact $(-1)^n$ times the number of permutations σ in the symmetric group S_{nd-1} such that $\sigma(i) > \sigma(i+1)$ if and only if i is divisible by d.

As a final example of reduced products, we consider the following natural extension of the preceding one, and also of Example 3.7 (partitions with block sizes congruent to 1 modulo k_0). This example is somewhat intriguing because while it seems to be closely related to both the d-divisible and the 1 mod k_0 cases, (cf. formula (4.6) below) it is not, in general, ranked, and therefore not Cohen-Macaulay. Again, as in Example 4.4, it would be interesting to see if a shelling exists for these posets.

Example 4.6. Now fix $k_0 \geq 3$ and $2 \leq d \leq k_0 - 1$, and let T^* be the subset of positive integers congruent to d mod k_0. Then d is the least element of T^*, and $\Pi_{n,T^*} = \Pi_n^{(d,k_0)}$ is the poset of partitions with block sizes congruent to d mod k_0. This time $\Pi_{n,T^*} = \{\hat{0},\hat{1}\}$ if n is not congruent to a multiple of d mod k_0. The identity (4.2) becomes

$$\sum_{r\geq 1} h_r \left[- \sum_{m\in T^*} \pi_{m,T^*}\right] = \sum_{\substack{i\equiv qd \mod k_0 \\ q\geq 1}} h_i + \sum_{\substack{n\equiv qd \mod k_0 \\ q\geq 2}} \pi_{n,T^*}.$$

If λ is an integer partition of n with r parts, all congruent to d mod k_0, then $n \equiv qd$ mod k_0 if and only if $d(r-q) \equiv 0$ mod k_0. Write k_d for the quotient of k_0 by the greatest common divisor of d and k_0. Then it is clear that $d(r-q)\equiv 0$ mod k_0 if and only if $r \equiv q \mod k_d$.

Hence, separating terms by congruence mod k_0, we have the two identities

$$\sum_{r\equiv 1 \mod k_d} h_r \left[- \sum_{m\in T^*} \pi_{m,T^*}\right] = \sum_{n\equiv d \mod k_0} h_n, \tag{4.5}$$

and for $2 \leq q \leq k_d - 1$,

$$\sum_{\substack{r\geq 1 \\ r\equiv q \mod k_d}} h_r \left[- \sum_{m\in T^*} \pi_{m,T^*}\right] = \sum_{n\equiv qd \mod k_0} (h_n + \pi_{n,T^*}). \tag{4.6}$$

Using Example 3.7, (4.5) may be written

$$- \sum_{m\in T^*} \pi_{m,T^*} = \sum_{j\geq 0} (-1)^j \beta_{jk_d+1} \left[\sum_{n\equiv d \mod k_0} h_n \right], \tag{4.7}$$

where β_n is the character on the top homology of the poset $\Pi_n^{(1,k)}$ as in Example 3.7. Note the similarity between (4.7) and the formula of Example 4.5 for the d-divisible lattice.

As in Example 3.7, (4.5) and (4.6) give the following enumerative formulas for the Möbius numbers. (See [St3, Chapter 5] for techniques for solving such equations.) Let $M_r^{(d,k_0)}(x)$ be the exponential generating function for the Möbius number of Π_{n,T^*} as n ranges over all values congruent to r mod k_0. Define $G_{r,k_0}(x)$ as in Example 3.7. Then one has

$$M_d^{(d,k_0)}(x) = -M_1^{(1,k_0)}(G_{d,k_0}(x));$$
$$M_{qd}^{(d,k_0)}(x) = G_{q,k_d}\left(-M_d^{(d,k_0)}(x)\right) - G_{qd,k_0},\ 2 \leq q \leq k_d - 1.$$

Formula (4.6) can also be explained by observing that in general, y covers an element $x \neq \hat{0}$ in the poset $\Pi_n^{(d,k_0)}$ if and only if y is obtained from x by merging $1 + k_d$ blocks; this is the least value of t such that dt is congruent to $d \mod k_0$. In other words, for $x \neq \hat{0}$ the upper interval $[x, \hat{1}]$ is poset-isomorphic to the poset Π_n^{1,k_d} of Example 3.7 (in particular it is Cohen-Macaulay). Now (4.6) can be derived by using the acyclicity of the Whitney homology of the dual poset, as in [Su].

In general, atoms in $\Pi_n^{(d,k_0)}$ can have types of the form $((d+k_0)^a, d^b)$ for more than one choice of the pair (a, b). Hence the remarks of the preceding paragraph show that the poset is not ranked. An example is $\Pi_{14}^{(2,3)}$. Here the atoms are of types (2^7) (seven blocks of size 2) and $(5^2, 2^2)$ (two blocks of size 5 and two of size 2). Covering relations in the poset are described by merging 4 blocks at a time. There are thus two maximal chains of different lengths: the atom of type (2^7) is covered by an element of type $(8, 2^3)$, while the atom of type $(5^2, 2^2)$ is covered by $\hat{1}$.

Now we return to the case of arbitrary T^*. As in the previous section, generating functions for the Möbius number and the characteristic polynomial follow easily. Denote by $p_{T^*}(x)$ the power-series

$$\sum_{m \geq d} \frac{x^m}{m!} + \sum_{\substack{m \notin T^* \\ m > d}} \mu_{m,T^*} \frac{x^m}{m!}.$$

The formula below was also obtained by Svante Linusson, ([L, Proposition 1.2]), who used it to estimate the Möbius number of the lattice of Example 4.4, in connection with a problem in algorithmic complexity.

Corollary 4.7. *If $d > 1$ is the least element of T^*, then* $\sum_{m \in T^*} \mu_{m,T^*} \frac{x^m}{m!} = -\log(1 + p_{T^*}(x))$.

Proof. Use (4.2) and extract the terms involving only powers of p_1, as in the proof of Corollary 3.11. □

For $n \geq d$, define the characteristic polynomial of Π_{n,T^*} by

$$\chi_{n,T^*}(t) = \sum_{x \in \Pi_{n,T^*}} \mu(\hat{0}, x) t^{b(x)},$$

where $b(x)$ denotes the number of blocks of x if $x > \hat{0}$, and $b(\hat{0}) = n$ by definition. If $n \notin T^*$, we set $b(\hat{1}) = 1$. In particular this is a polynomial of degree n.

Denote by $G_{T^*}(x,t)$ the generating function

$$\sum_{n \geq d} \chi_{n,T^*}(t) \frac{x^n}{n!} - \sum_{n \geq d} \frac{t^n x^n}{n!} - t \sum_{\substack{n \geq d \\ n \notin T^*}} \mu(\Pi_{n,T^*}) \frac{x^n}{n!}.$$

The analogue of Corollary 3.13 is

COROLLARY 4.8. $1 - G_{T^*}(x,t) = (1 + p_{T^*}(x))^t$.

Proof. By the definition and Theorem 4.1, $G_{T^*}(x,t)$ is the square-free term in

$$\sum_{r \geq 1} t^r (-h_r) \left[- \sum_{m \in T^*} \pi_{m.T^*} \right].$$

The result follows by using Lemma 3.12, Theorem 4.1 and equation (4.1), exactly as in the proof of Corollary 3.13. □

For the poset of Example 4.4, with $T^* = \{n : n \geq d\}$, one has, for $n \geq d$, the formula

$$\chi_{n,T^*}(t) = -\sum_{s \geq 1} t(t-1)\dots(t-(s-1)) S_{\geq d}(n,s), \tag{4.8}$$

where $S_{\geq d}(n,s)$ is the number of partitions of n into s blocks whose sizes are all at least d.

For the d-divisible lattice, taking $T^* = \{id : i \geq 1\}$, this gives the expected formula

$$\chi_{nd,T^*} = -\sum_{s \geq 1} t(t-1)\dots(t-(s-1)) S^{(d)}(nd,s),$$

where $S^{(d)}(nd,s)$ is the number of partitions of nd into s blocks whose sizes are divisible by d.

Note also that as usual, the Möbius number is obtained by evaluating at $t = 0$ the derivative of the characteristic polynomial with respect to t.

5. Conclusion.

As noted in the Introduction, in the case where homology occurs in arbitrary degrees, the formulas of Theorems 3.5 and 4.2 fail to isolate the contribution to the homology of the poset by degree. For the k-equal partition lattice $\Pi_{(k,1^{n-k})}$, more precise topological information was used in [SWa] to determine the homology representations by degree, that is, the symmetric functions $\pi^r_{n,k}$.

We now indicate how the formula of [SWa, Theorem 3.2] for $\pi^r_{n,k}$ can be predicted from the identity (3.3). Write $H = 1 + \sum_{i \geq 1} h_i$. We omit the proof of the following

Proposition 5.1. *Let $k = \min(T\backslash\{1\})$. One has the identity*

$$\sum_{\substack{n\in T\\ n\neq 1}} \pi_{n,T} = \sum_{i\geq 1}(-1)^{t-1}\pi_t[\Gamma_T], \quad \textit{where} \tag{5.1}$$

$$\Gamma_T = -\tfrac{1}{H}\left(\textstyle\sum_{i\geq k} h_i + \sum_{\substack{n\notin T\\ n\geq k}} \pi_{n,T}\right).$$

Now let $T = \{1, i : i \geq k\}$. Then $\Pi_{n,T}$ is the k-equal lattice. In this case further manipulations show that Γ_T simplifies to the expression

$$\sum_{j\geq k}(-1)^{j-k+1}s_{(k,1^{j-k})},$$

where $s_{(k,1^{j-k})}$ is the Schur function indexed by the hook shape $(k, 1^{j-k})$. (Notation as in [M].) Rather surprisingly, it was shown in [SWa, Theorem 3.2] that the degree n term in the plethysm $(-1)^{t-1}\pi_t[\Gamma_T]$ is precisely $(-1)^r\pi^r_{n,k}$, for $r = n-3-t(k-2)$.

We are therefore led to make the following conjecture.

Conjecture. *Let T be any subset containing 1, such that the poset $\Pi_{n,T}$ is not Cohen-Macaulay. The degree n term in the plethysm $(-1)^{t-1}\pi_t[\Gamma_T]$ is precisely $(-1)^r$ times the characteristic $\pi^r_{n,T}$ of the homology of $\Pi_{n,T}$ in some degree $r = r(t)$ uniquely determined by t.*

The conjecture predicts that the degrees of nonvanishing homology are indexed by parameters t between 1 and $\lfloor \frac{n}{k} \rfloor$. Furthermore, the homology representation in the degree indexed by t is given by the plethysm of the partition lattice character π_t, together with a virtual character.

By an identical argument one can obtain an analogue of Proposition 5.1 for the posets considered in Section 4.

References

[A] V.I. Arnold, *The cohomology ring of the colored braid group*, Math. Notes **5** (1969), 138-140.

[Ba1] K. Baclawski, *Whitney numbers of geometric lattices*, Adv. in Math. **16** (1975), 125-138.

[Ba2] K. Baclawski, *Cohen-Macaulay Connectivity and Geometric Lattices*, Europ. J. Combinatorics **3** (1982), 293-305.

[BB] K. Baclawski and A. Björner, *Fixed points in partially ordered sets*, Adv. in Math. **31** (1979), 263-287.

[Bj1] A. Björner, *Shellable and Cohen-Macaulay Partially Ordered Sets*, Trans. Amer. Math. Soc. **260, No. 1** (1980), 159-183.

[Bj2] ———, *On the homology of geometric lattices*, Algebra Universalis **14** (1982), 107-128.

[Bj3] ———, *Subspace arrangements*, Proc. 1st European Congress Math. (Paris, 1992) (A. Joseph, R. Rentschler, eds.), to appear, Birkhäuser, Basel-Boston.

[BL] A. Björner and L. Lovász, *Linear decision trees, subspace arrangements and Möbius functions*, to appear (preprint, 1993), J. Amer. Math. Soc. **7** (1994).

[BLY] A. Björner, L. Lovász and A. Yao, *Linear decision trees: volume estimates and topological bounds* (1992), ACM Press, New York, 170-177.

[BWa] A. Björner and M.L. Wachs, *Shellable nonpure complexes and posets*, in preparation.

[BWe] A. Björner and V. Welker, *The homology of "k-equal"manifolds and related partition lattices*, preprint 1992, Adv. in Math. (to appear).

[CHR] A. R. Calderbank, P. Hanlon, and R. W. Robinson, *Partitions into even and odd block size and some unusual characters of the symmetric groups*, Proc. London Math. Soc. **53** (1986), no. 3, 288-320.

[GM] M. Goresky and R. MacPherson, *Stratified Morse Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 14, Springer, Berlin, Heidelberg, New York, 1988.

[HW] P. Hanlon and M. L. Wachs, *On Lie k-algebras*, preprint, 1994, Adv. in Math. (to appear).

[JK] G. D. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Encyclopedia of Mathematics, vol. 16, Addison-Wesley, Reading, MA, 1981.

[L] S. Linusson, *Partitions with restricted block sizes, Möbius functions and the k-of-each problem*, Abstracts of the 6th Conference on Formal Power Series and Algebraic Combinatorics (1994), DIMACS, Rutgers, NJ.

[LS] G. Lehrer and L. Solomon, *On the Action of the Symmetric Group on the Cohomology of the Complement of its Reflecting Hyperplanes*, J. Algebra **104** (1986), 410-424.

[M] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford University Press, 1979.

[Mu] J. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, 1984.

[Qui] D. Quillen, *Homotopy Properties of the Poset of Nontrivial p-Subgroups of a Group*, Adv. in Math. **28** (1978), 101-128.

[Sa] B. Sagan, *Shellability of Exponential Structures*, Order **3** (1986), 47-54.

[St1] R. P. Stanley, *Some Aspects of Groups Acting on Finite Posets*, J. of Comb. Theory (A) **32, No. 2** (1982), 132-161.

[St2] ———, *Enumerative Combinatorics*, vol. 1, Wadsworth and Brooks/Cole, 1986.

[St3] ———, *Enumerative Combinatorics*, manuscript, vol. 2.

[Su] S. Sundaram, *The homology representations of the symmetric group on Cohen-Macaulay subposets of the partition lattice*, Adv. in Math. **104 (2)** (1994), 225-296.

[SWa] S. Sundaram and M.L. Wachs, *The homology representations of the k-equal partition lattice*, preprint, 1994.

[SWe1] S. Sundaram and V. Welker, *Group actions on linear subspace arrangements and applications to configuration spaces*, preprint, 1994.

[SWe2] ———, *Group representations on the homology of products of posets*, in preparation (1994).

[Wa] M.L. Wachs, *A basis for the homology of the d-divisible lattice*, preprint, 1992, Adv. in Math. (to appear).

[Wal] J.W. Walker, *Canonical homeomorphisms of posets*, Europ. J. Comb. **9** (1988), 97-107.

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