DETERMINANTS IN PLANE PARTITIONS ENUMERATION

BY

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ABSTRACT. In this paper we provide a new and concise evaluation of

$$\det\left(\delta_{ij}+\binom{x+i+j}{j}\right)_{0\leqq i,j\leqq n-1}.$$

This determinant arose in the enumeration of certain set of plane partitions. Its previous evaluation was lengthy and complicated.

Introduction.

Our central object in this paper is to prove the following result:

Theorem 1. [8] *Let* $\Delta_0(x)=2$, *and for* $j>0$

$$\Delta_{2j}(x)=\frac{(x+2j+2)_j(x/2+2j+3/2)_{j-1}}{(j)_j(x/2+j+3/2)_{j-1}}, \tag{1.1}$$

$$\Delta_{2j-1}(x)=\frac{(x+2j)_{j-1}(x/2+2j+1/2)_j}{(j)_j(x/2+j+1/2)_{j-1}}, \tag{1.2}$$

then

$$det\left(\delta_{ij}+\binom{x+i+j}{j}\right)_{0\leqq i,j\leqq n-1}=\prod_{k=0}^{n-1}\Delta_k(x), \tag{1.3}$$

where

$$(A)_j=A(A+1)\cdots(A+j-1).$$

Using the work of Mills, Robbins and Rumsey [8], it is possible to show that Theorem 1 follows from the next two theorems:

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Theorem 2. [8].

$$(1.4) \qquad det\left(\binom{x+i+j}{2i-j}\right)_{0\leqq i,j\leqq n-1} = 2^{-n}\prod_{k=0}^{n-1}\Delta_{2k}(2x).$$

Theorem 3. [3].

$$(1.5) \qquad det\left(\binom{i+j+x+1}{2i-j+1}+\binom{i+j+x}{2i-j}\right)_{0\leqq i,j\leqq n-1} = \prod_{k=1}^{n}\Delta_{2k-1}(2x-1).$$

In Section 2, we describe how Theorem 1 follows from Theorem 2 and 3.

The proofs of Theorems 2 and 3 rely primarily on two hypergeometric series identities respectively:

$$(1.6) \qquad {}_5F_4\left(\begin{matrix}-2i-1, x+2i+2, x-z+1/2, x+i+1, z+i+1; 1\\ 1+x/2, (1+x)/2, 2z+2i+2, 2x-2z+1\end{matrix}\right)=0,$$

and

(1.7)
$${}_6F_5\left(\begin{matrix}-2i-1, x+2i+1, x-z+1/2, x+i+1/2, z+i+1/2, 2(x+i+2)/3; 1\\ 1+x/2, (1+x)/2, 2z+2i+1, 2x-2z+1, (2x+2i+1)/3\end{matrix}\right)=0.$$

In [2], identity (1.6) was first proved using an elementary method due to Pfaff. Once (1.6) was proved, Section 7 of [2] was devoted to deriving Theorem 2 directly therefrom.

In [2], identity (1.7) was stated (originally as a conjecture), and it was pointed out there that the author was unable to prove it using Pfaff's method.

At the June, 1994 Ann Arbor Conference on Algebra and Combinatorics, the first author listed (1.7) as a conjecture. The second author saw that (1.7) is a special case of work he did jointly with I. Gessel [6], and he was able to see how the same methods imply (1.6). At the meeting Doron Zéilberger proved (1.7) (and subsequently (1.6)) using the WZ-method. Since these methods are radically distinct from Pfaff's method and much more concise, it seems appropriate to record them here in the context of providing a new proof of Theorem 1.

Section 2 will be devoted to showing how Theorem 1 follows from Theorems 2 and 3. In Section 3 we prove (1.6) and (1.7). Section 4 then contains a proof of Theorem 3 and exactly parallels Section 7 of [2]. In Section 5 we present variations and extensions of our work in Section 3. Finally in Section 6 we discuss results concerning possible q-analogs of Theorems 1, 2 and 3. We conclude with some general observations.

We should also point out that recently Wilf and Petkovsek have found an incredibly short proof of Theorem 1 in the spirit of the WZ-method.

2. Theorems 2 and 3 imply Theorem 1.

Nothing in this section is new, and we therefore present a telegraphed account. All the work in this section occurred in [3] and [8].

We shall use the notation of [1], [3] and [8] slightly modified:

$$Z_n(x) = \det\left(\delta_{ij} + \binom{i+j+x}{j}\right)_{0\leqq i,j\leqq n-1}, \tag{2.1}$$

$$T_n(x) = \det\left(\binom{i+j+x}{2j-i}\right)_{0\leqq i,j\leqq n-1} \tag{2.2}$$

$$R_n(x) = \det\left(\binom{i+j+x}{2i-j} + 2\binom{i+j+x+2}{2i-j+1}\right)_{0\leqq i,j\leqq n-1} \tag{2.3}$$

$$W_n(x) = \det\left(\binom{i+j+x+1}{2i-j+1} + \binom{i+j+x}{2i-j}\right)_{0\leqq i,j\leqq n-1}. \tag{2.4}$$

The first step in the proof of equivalence is an immediate consequence of Theorem 5 in [8]. Namely

$$Z_{2n}(2x) = T_n(x)R_n(x), \tag{2.5}$$

$$Z_{2n+1}(2x) = 2T_{n+1}(x)R_n(x), \tag{2.6}$$

Indeed in Theorem 5 of [8; with the x there set $=1$ and μ replaced by x] we have exactly these equations except the factor $R_n(x)$ is instead

$$\begin{aligned}
&\det\left[\sum_{t=0}^{2n-1}\left(\binom{i+x}{2i+1+x-t}+\binom{i+1+x}{2i+1+x-t}\right)\right.\\
&\quad\left.\times\left(\binom{j}{2j+1-t}+\binom{j+1}{2j+1-t}\right)\right]_{0\leqq i,j\leqq n-1}\\
&=\det\left(\binom{i+j+x}{2i-j+x}+\binom{i+j+x+1}{2i-j+x}\right.\\
&\left.+\binom{i+j+x+1}{2i-j+x+1}+\binom{i+j+x+2}{2i-j+x+1}\right)_{0\leqq i,j\leqq n-1}
\end{aligned}$$

(by the Chu-Vandermonde summation [5; p. 3])

$$=\det\left(\binom{i+j+x}{2i-j}+2\binom{i+j+x+2}{2i-j+1}\right)_{0\leq i,j\leq n-1}$$

(where the matrix has been transposed and the second and third binomial coefficients have been combined by the Pascal rule).

$$=R_n(x).$$

The $T_n(x)$ and $Z_n(x)$ factors are simplified similarly using the Chu-Vandermonde summation.

Next as shown in [3; p. 12] it is possible to show that

$$\mathcal{S}_n\cdot W_n(x)=R_n(x-1/2), \tag{2.7}$$

where

$$\mathcal{S}_n=\det\left(\frac{\left(-\frac{1}{2}\right)_{i-j}(-1)^{i-j}}{2^{2i-2j-1}(i-j)!}\right)_{0\leqq i,j\leqq n-1}. \tag{2.8}$$

In fact (2.7) follows immediately once the two determinants on the left are multiplied together and the Pfaff-Saalschutz summation is applied [5; p.9, Section 2.2, eq. (1)].

Also

$$\mathcal{S}_n=2^n, \tag{2.9}$$

because $\mathcal{S}_n$ is lower triangular with 2's on the main diagonal.

Hence with $\nu_1 = \lfloor \frac{n}{2} \rfloor$, $\nu_2 = \lfloor \frac{n+1}{2} \rfloor$ we may combine the above observations as follows

$$\begin{aligned} Z_n(x) &= 2^{\nu_2-\nu_1} T_{\nu_2}(x) R_{\nu_1}(x) \\ &\qquad \text{(by (2.6) and (2.7))} \\ &= 2^{\nu_2-\nu_1} T_{\nu_2}(x) 2^{\nu_1} W_{\nu_1}\left(x+\frac{1}{2}\right) \\ &\qquad \text{(by (2.7) and (2.9))} \\ &= 2^{\nu_2-\nu_1} 2^{-\nu_2} \left(\prod_{k=0}^{\nu_2-1} \Delta_{2k}(2x)\right) 2^{\nu_1} \prod_{k=1}^{\nu_1} \Delta_{2k-1}(2x) \\ &\qquad \text{(by Theorems 2 and 3)} \\ &= \prod_{k=0}^{n-1} \Delta_k(2x), \end{aligned} \tag{2.10}$$

which is Theorem 1.

3. Proofs of (1.6) and (1.7).

In [6; p. 296, eq. (1.8), first line], it was proved that for n odd

$$ {}_7F_6\left(\begin{matrix} a, b, a+1/2-b, 1+2a/3, 1-2d, 2a+2d+n, -n; 1 \\ 2a+1-2b, 2b, 2a/3, a+d+1/2, 1-d-n/2, 1+a+n/2 \end{matrix}\right) = 0. \tag{3.1}$$

If we set $n = 2i+1$, $a = x+i+1/2$, $b = z+i+1/2$, $d = -i-x/2$, we obtain (1.7).

The proof of (1.6) requires two cubic transformations following the pattern of [6; §5]. First is one of Bailey's other cubic transformations [4; eq. (4.06)]

$$ {}_3F_2\left(\begin{matrix} a, b, a-b+1/2; 4x \\ 2b, 2a-2b+1 \end{matrix}\right) = (1-x)^{-a} \, {}_3F_2\left(\begin{matrix} a/3, (a+1)/3, (a+2)/3\ ;\ \frac{27x^2}{4(1-x)^3} \\ b+1/2, a-b+1 \end{matrix}\right). \tag{3.2}$$

If in (3.2), we set $a = D-1$, $b = D/2$

$$\begin{aligned} &(1-x)^{1-D} \, {}_3F_2\left(\begin{matrix} (D-1)/3, D/3, (D+1)/3; \frac{27x^2}{4(1-x)^3} \\ D/2, (D+1)/2 \end{matrix}\right) \\ &= {}_2F_1\left(\begin{matrix} D/2, (D-1)/2; 4x \\ D \end{matrix}\right) \\ &= (1-4x)^{1/2} \, {}_2F_1\left(\begin{matrix} D/2, (D+1)/2; 4x \\ D \end{matrix}\right) \\ &\qquad \text{(by [5; p. 2, eq. (2)]).} \end{aligned} \tag{3.3}$$

We rewrite (3.3) as

$$(3.4)\quad {}_2F_1\left(\begin{matrix}D/2,(D+1)/2;4x\\ D\end{matrix}\right) = (1-x)^{1-D}\,{}_3F_2\left(\begin{matrix}(D-1)/3,D/3,(D+1)/3;\frac{27x^2}{4(1-x)^3}\\ D/2,(D+1)/2\end{matrix}\right)(1-4x)^{-1/2}.$$

Now we multiply equations (3.2) and (3.4) together, and we find that the coefficient of x^n on the left-hand side is

$$(3.5)\quad \frac{(D/2)_n((D+1)/2)_n 4^n}{n!(D)_n}\,{}_5F_4\left(\begin{matrix}a,b,a-b+1/2,-n,1-n-D;1\\ 2b,2a-2b+1,1-n-D/2,1/2-n-D/2\end{matrix}\right).$$

On the other hand the coefficient of x^n on the right-hand side is

$$\begin{aligned}&\operatorname*{Res}_x\left\{(1-x)^{-a+1-D}\,{}_3F_2\left(\begin{matrix}a/3,(a+1)/3,(a+2)/3;\frac{27x^2}{4(1-x)^3}\\ b+1/2,a-b+1\end{matrix}\right)\right.\\ &\left.\times\,{}_3F_2\left(\begin{matrix}(D-1)/3,D/3,(D+1)/3;\frac{27x^2}{4(1-x)^3}\\ D/2,(D+1)/2\end{matrix}\right)\frac{(1-4x)^{-1/2}}{x^{n+1}}\right\}\\ (3.6)\qquad &=\operatorname*{Res}_y\left\{{}_3F_2\left(\begin{matrix}a/3,(a+1)/3,(a+2)/3;\frac{27y^2}{4}\\ b+1/2,a-b+1\end{matrix}\right)\right.\\ &\times\,{}_3F_2\left(\begin{matrix}(D-1)/3,D/3,(D+1)/3;\frac{27y^2}{4}\\ D/2,(D+1)/2\end{matrix}\right)y^{-n-1}\\ &\left.\times\frac{(1-4x)^{-1/2}}{1+x/2}(1-x)^{-a+1-D-3(n+1)/2+5/2}\right\}\end{aligned}$$

where we have made the substitution $y=x(1-x)^{-3/2}$ (so that $dx=(1-x)^{5/2}(1+x/2)^{-1}dy$).

Now clearly each of these ${}_3F_2$'s has no odd powers of y. Hence identifying (3.5) and (3.6) and setting $n=2i+1$, $D=-a-3i-1$, we find

$$\begin{aligned}&\frac{((-a-3i-1)/2)_{2i+1}((-a-3i)/2)_{2i+1}4^{2i+1}}{(2i+1)!(-a-3i-1)_{2i+1}}\\ &{}_5F_4\left(\begin{matrix}a,b,a-b+1/2,-2i-1,a+i+1;1\\ 2b,2a-2b+1,(a-i+1)/2,(a-i)/2\end{matrix}\right)\\ &=\operatorname*{Res}_y\left\{{}_3F_2\left(\begin{matrix}a/3,(a+1)/3,(a+2)/3;\frac{27y^2}{4}\\ b+1/2,a-b+1\end{matrix}\right)\right.\\ (3.7)\qquad &\times\,{}_3F_2\left(\begin{matrix}(-a-3i-2)/3,(-a-3i-1)/3,(-a-3i)/3;\frac{27y^2}{4}\\ (-a-3i-1)/2,(-a-3i)/2\end{matrix}\right)\\ &\left.\times\left(1-\frac{27}{4}y^2\right)^{-1/2}y^{-2i-2}\right\}\\ &=0\end{aligned}$$

because the penultimate expression is clearly an even function of y.

Equation (3.7) reduces to (1.6) by setting $a = x + i + 1$, $b = z + i + 1$.

4. Proof of Theorems 2 and 3.

As noted in the introduction, Section 7 of [2] shows how Theorem 2 follows from (1.6). The proof proceeds by proving the identity

$$M_n(x)E_n(x) = L_n(x), \tag{4.1}$$

where

$$M_n(x) = \left(\binom{x+i+j}{2i-j}\right)_{0\leqq i,j\leqq n-1} \tag{4.2}$$

(note $M_n(x)$ is the transpose of $T_n(x)$ given in (2.2)),

$$E_n(x) = (e_{i,j}(x))_{0\leqq i,j\leqq n-1}, \tag{4.3}$$

$$e_{i,j}(x) = \begin{cases} 0 & \text{if } i > j \\ \frac{(-1)^{j-i}(i)_{2j-2i}(2x+2j+i+2)_{j-i}}{4^{j-i}(j-i)!(x+i+1)_{j-i}(x+j+i+1/2)_{j-i}} & \text{otherwise.} \end{cases} \tag{4.4}$$

and where the matrix $L_n(x)$ is lower triangular with $\frac{1}{2}\Delta_2(2x)$, $\frac{1}{2}\Delta_4(2x), \dots, \frac{1}{2}\Delta_{2n-2}(2x)$ on the main diagonal. Theorem 2 is immediate when one takes determinants in (4.1). Identity (1.6) is the key to evaluating the diagonal of $L_n(x)$ and to showing that it is lower triangular.

Precisely the same sort of argument applies for Theorem 3. In this case, we display the details. Here we prove

$$W_n(x)G_n(x) = K_n(x), \tag{4.5}$$

where

$$\begin{aligned} W_n(x) &= \left(\binom{i+j+x+1}{2i-j+1} + \binom{i+j+x}{2i-j}\right)_{0\leqq i,j\leqq n-1} \\ &= \left(\binom{i+j+x+1}{2i-j+1}\frac{x+3i+2}{i+j+x+1}\right)_{0\leqq i,j\leqq n-1}, \end{aligned} \tag{4.6}$$

$$G_n(x) = (g_{ij}(x))_{0\leqq i,j\leqq n-1} \tag{4.7}$$

$$g_{ij}(x) = \begin{cases} 0 & \text{if } i > j \\ \frac{(-1)^{j-i}(i+1)_{2j-2i}(2x+i+2j+2)_{j-i}(2x+3i+1)}{4^{j-i}(j-i)!(x+i+j+1)_{j-i}(x+i+1/2)_{j-i}(2x+3j+1)}, & \text{otherwise}, \end{cases} \tag{4.8}$$

and where the matrix $K_n(x)$ is lower triangular with $\Delta_1(2x-1), \Delta_3(2x-1), \ldots, \Delta_{2n-1}(2x-1)$ on the main diagonal. Theorem 3 is immediate from (4.5) merely by taking determinants.

So what we must prove is that $K_n(x) = (k_{ij}(x))_{0\leqq i,j\leqq n-1}$ is indeed lower triangular with the stated main diagonal.

Now for $0 \leqq i \leqq j$, we have

$$\begin{aligned} k_{ij}(x) &= \sum_{h=0}^{j}\left(\binom{i+h+x+1}{2i-h+1} + \binom{i+h+x}{2i-h}\right) g_{hj}(x) \\ &= \sum_{h=0}^{j} \frac{(i+h+x)!(x+3i+2)}{(2i-h+1)!(2h-i+x)!} \times \frac{(-1)^j(2j)!(2x+2j+2)_{j-1}}{4^j j!(x+j+1)_j(x+1/2)_j}. \\ &\quad \times \frac{(-1)^h(-j)_h(x+j+1)_h(x+1/2)_h(2x+3h+1)}{4^{-h}(-2j)_h h!(2x+2j+2)_h} \\ &= \frac{(i+x)!(x+3i+2)(2x+1)(-1)^j(2j)!(2x+2j+2)_{j-1}}{(2i+1)!(x-i)!4^j j!(x+j+1)_j(x+1/2)_j} \\ &\quad {}_6F_5\left(\begin{matrix} -2i-1, i+x+1, -j, x+j+1, x+1/2, (2x+4)/3; 1 \\ (x-i+1)/2, (x-i+2)/2, -2j, 2x+2j+2, (2x+1)/3 \end{matrix}\right) \\ &\quad \text{(valid for } i<j, \text{ for } i=j \text{ see next paragraph)} \\ &= 0 \end{aligned} \tag{4.9}$$

for $i < j$ by (1.7) wherein we have replaced x by $x-i$ and z by $-i-j-1/2$.

If $i = j$, then the $-2j$ among the lower parameters of the ${}_6F_5$ means that a non-zero term in the above sum is ignored in the passage from $\sum_{h=0}^{j}$ to $\sum_{h=0}^{2i+1}$.

Consequently

$$
\begin{aligned}
k_{jj}(x) &= \sum_{h=0}^{j} \left(\binom{j+h+x+1}{2j-h+1} + \binom{j+h+x}{2j-h} \right) g_{hj}(x) \\
&= \frac{(j+x)!(x+3j+2)(2x+1)(-1)^j(2j)!(2x+2j+2)_{j-1}}{(2j+1)!(x-j)!4^j j!(x+j+1)_j(x+1/2)_j} \\
&\times \left\{ {}_6F_5\left(\begin{matrix} -2i-1, i+x+1, -j, x+j+1, x+1/2, (2x+4)/3; 1 \\ (x-i+1)/2, (x-i+2)/2, -2j, 2x+2j+2, (2x+1)/3 \end{matrix} \right) \right. \\
&- \frac{(-2j-1)_{2j+1}(j+x+1)_{2j+1}^2(2x+6j+4)(x+1/2)_{2j+1}}{(2j+1)!\, 4^{-2j-1}(x-j+1)_{4j+2}(2x+2j+2)_{2j+1}(2x+1)} \\
&\left. \times \lim_{y\to j} \frac{(-y)_{2j+1}}{(-2y)_{2j+1}} \right\} \\
&= \frac{(j+x)!(x+3j+2)^2(-1)^j(2x+2j+2)_{j-1}}{(2j+1)(x-j)!\, 4^{-j-1} j!(x+j+1)_j(x+1/2)_j} \\
&\times \frac{(j+x+1)_{2j+1}^2(x+1/2)_{2j+1}}{(x-j+1)_{4j+2}(2x+2j+2)_{2j+1}} \quad \frac{(-1)^j j!^2}{(2j)!} \\
&= \frac{(2x+2j+1)_j\ (x+2j+2)_{j+1}}{(j+1)_{j+1}(x+j+1)_j} \\
&= \Delta_{2j+1}(2x-1).
\end{aligned} \tag{4.10}
$$

Thus $K_n(x)$ has the desired properties and Theorem 3 is proved.

5. Further Hypergeometric Identities.

The results in Section 3 can be deduced from substantially more general hypergeometric identities.

Theorem 4. *If p is a nonnegative integer,*

$$
\begin{aligned}
&(a)_p\ {}_5F_4\left(\begin{matrix} x+1, z+1, x-z+1/2, -p, 1-a; 1 \\ 2z+2, 2x-2z+1, (1-a-p)/2, (2-a-p)/2 \end{matrix} \right) \\
&= (a+x+1)_p\ {}_5F_4\left(\begin{matrix} x+1, x-2z, 1-x+2z, -p/2, (1-p)/2; 1 \\ z+3/2, x-z+1, -a-p-x, x+a+1 \end{matrix} \right)
\end{aligned} \tag{5.1}
$$

Proof. W. N. Bailey [4; eqs. (4.05) and (4.06)] has proved that

$$
\begin{aligned}
&(1-w)^{x+1}\ {}_3F_2\left(\begin{matrix} x+1, x-2z, 1-x+2z; w/4 \\ z+3/2, x-z+1 \end{matrix} \right) \\
&= {}_3F_2\left(\begin{matrix} \frac{x+1}{3}, \frac{x+2}{3}, \frac{x+3}{3}; \frac{-27w}{4(1-w)^3} \\ z+3/2, x-z+1 \end{matrix} \right)
\end{aligned} \tag{5.2}
$$

$$(5.3)\quad \begin{aligned}&(1-v)^{x+1}\,{}_3F_2\left(\begin{matrix}x+1, z+1, x-z+1/2; 4v\\ 2z+2, 2x-2z+1\end{matrix}\right)\\ &= {}_3F_2\left(\begin{matrix}\frac{x+1}{3}, \frac{x+2}{3}, \frac{x+3}{3}; \frac{-27v^2}{4(1-v)^3}\\ z+3/2, x-z+1\end{matrix}\right).\end{aligned}$$

If $w = -y^2/(1-y)$ and $v = y(1-y)$, then the right sides of (5.2) and (5.3) are identical; so

$$(5.4)\quad \begin{aligned}&(1-y)^{-x-1}\,{}_3F_2\left(\begin{matrix}x+1, x-2z, 1-x+2z; -\frac{-y^2}{4(1-y)}\\ z+3/2, x-z+1\end{matrix}\right)\\ &= {}_3F_2\left(\begin{matrix}x+1, z+1, x-z+1/2; 4y(1-y)\\ 2z+2, 2x-2z+1\end{matrix}\right)\end{aligned}$$

If we multiply (5.4) by $(1-y)^{-a}$ and equate coefficients of y^p, we obtain (5.1).

□

The choices $w = -y^2/(1-y)$ and $v = y/(1+y)^2$ also make the right sides of (5.2) and (5.3) identical; so

$$(5.5)\quad \begin{aligned}&(1+y)^{x+1}\,{}_3F_2\left(\begin{matrix}x+1, x-2z, 1-x+2z; \frac{-y^2}{4(1+y)}\\ z+3/2, x-z+1\end{matrix}\right)\\ &= {}_3F_2\left(\begin{matrix}x+1, z+1, x-z+1/2; \frac{4y}{(1+y)^2}\\ 2z+2, 2x-2z+1\end{matrix}\right),\end{aligned}$$

and a result equivalent to (5.1) may be obtained form (5.5).

Corollary 1. *Identity (1.6) is valid.*

Proof. Replace x and z by $x+i$ and $z+i$ respectively in Theorem 4. Then set $p = 2i+1$ and $a = -1-x-2i$. □

If we replace y by $-y$ in (5.5) and compare with (5.4), we obtain the following result.

Theorem 5.

$$(5.6)\quad \begin{aligned}&(1-y)^{x+1}\,{}_3F_2\left(\begin{matrix}x+1, z+1, x-z+1/2; 4y(1-y)\\ 2z+2, 2x-2z+1\end{matrix}\right)\\ &= (1-y)^{-x-1}\,{}_3F_2\left(\begin{matrix}x+1, z+1, x-z+1/2; \frac{-4y}{(1-y)^2}\\ 2z+2, 2x-2z+1\end{matrix}\right).\end{aligned}$$

If we now multiply both sides of (5.6) by $(1-y)^{-a}$ and equate coefficients of y^p we find (in analogy with Theorem 4):

(5.7)
$$(a+x+1)_p\ {}_5F_4\left(\begin{matrix} x+1, z+1, x-z+1/2, -p, a+x+1+p; 1 \\ 2z+2, 2x-2z+1, (a+x+1)/2, (a+x+2)/2 \end{matrix}\right)$$
$$= (a-x-1)_p\ {}_5F_4\left(\begin{matrix} x+1, z+1, x-z+1/2, -p, z-a+x; 1 \\ 2z+2, 2x-2z+1, (2-a+x-p)/2, (3-a+x-p)/2 \end{matrix}\right).$$

As with Theorem 4, equation (5.7) also implies Corollary 1.

We may also treat ${}_6F_5$ hypergeometric series by using ${}_4F_3$ cubic transformations [6; eqs. (5.4), (5.7)]. The analogs of (5.4) and (5.5) are

(5.8)
$$(1-y)^{-x-1}(1-y/2)\,{}_4F_3\left(\begin{matrix} x+1, x-2z, 1-x+2z, (x+4)/3; \frac{-y^2}{4(1-y)} \\ z+3/2, x-z+1, (x+1)/3 \end{matrix}\right)$$
$$= (1-2y)\ {}_4F_3\left(\begin{matrix} x+1, z+1, x-z+1/2, (2x+5)/3; 4y(1-y) \\ 2z+2, 2x-2z+1, (2x+2)/3 \end{matrix}\right),$$

and

(5.9)
$$(1+y)^{x+1}(1+y/2)\ {}_4F_3\left(\begin{matrix} x+1, x-2z, 1-x+2z, (x+4)/3; \frac{-y^2}{4(1+y)} \\ z+3/2, x-z+1, (x+1)/3 \end{matrix}\right)$$
$$= (1-y)\ {}_4F_3\left(\begin{matrix} x+1, z+1, x-z+1/2, (2x+5)/3; \frac{4y}{(1+y)^2} \\ 2z+2, 2x-2z+1, (2x+2)/3 \end{matrix}\right).$$

As (5.1) followed from (5.4) and (5.5), so does the following result follow from (5.8) and (5.9):

Theorem 6. *If p is a non-negative integer,*

(5.10)
$$(a)_{p-1}(a+2p-1)\,{}_6F_5\left(\begin{matrix} x+1, z+1, x-z+1/2, (2x+5)/3, -p, a+p-1; 1 \\ 2z+2, 2x-2z+1/2, (2x+2)/3, a/2, (1+a)/2 \end{matrix}\right)$$
$$= (a-x-1)_{p-1}(a-x-2+p/2)\,{}_6F_5\left(\begin{matrix} x+1, x-2z, 1-x+2z, (x+4)/3, -p/2, (1-p)/2; 1 \\ z+3/2, x-z+1, (x+1)/3, a-x-1, 3-a+x-p \end{matrix}\right).$$

Corollary 2. *Identity (1.7) is valid.*

Proof. In Theorem 6, replace x and z by $x+i-1/2$ and $z+i-1/2$ respectively. Then set $p=2i+1$, $a=x+1$. □

The ${}_6F_5$ hypergeometric analogs of Theorem 5 and equation (5.7) are

$$(5.11)\quad \begin{aligned}&(1-2y)(1-y)^{x+1}\ {}_4F_3\left(\begin{matrix}x+1,z+1,x-z+1/2,(2x+5)/3;4y(1-y)\\ 2z+2,2x-2z+1,(2x+2)/3\end{matrix}\right)\\ &=(1+y)(1-y)^{-x-1}\ {}_4F_3\left(\begin{matrix}x+1,z+1,x-z+1/2,(2x+5)/3;\frac{-4y}{(1-y)^2}\\ 2z+2,2x-2z+1,(2x+2)/3\end{matrix}\right)\end{aligned}$$

and

(5.12)

$$(a+x+1)_{p-1}(a+x+2p)_6F_5\left(\begin{matrix}x+1,z+1,x-z+1/2,(2x+5)/3,-p,a+x+p;1\\ 2z+2,2x-2z+1,(2x+2)/3,\frac{a+x+1}{2},\frac{a+x+2}{2}\end{matrix}\right)$$

$$=(a-x-1)_{p-1}(a-x-2-p)_6F_5\left(\begin{matrix}x+1,z+1,x-z+1/2,(2x+5)/3,-p,2-a+x;1\\ 2z+2,2x-2z+1,(2x+2)/3,\frac{3-a+x-p}{2},\frac{4-a+x-p}{2}\end{matrix}\right).$$

6. Generalized Determinants and q-Analogs.

The methods developed in the foregoing sections may be extended using the more general results of [6] and [7].

For example, if instead of (1.7) we base our work on [6; p. 296, eq. (1.8)]:

$$(6.1)\qquad {}_7F_6\left[\begin{matrix}a,b,a-b+1/2,1+2a/3,1-2d,2a+2d+n,-n;1\\ 2a+1-2b,2b,2a/3,a+d+1/2,1-d-n/2,1+a+n/2\end{matrix}\right]=0$$

in full generality, we may in the manner of Section 4 prove:

Theorem 7. *If*

$$(6.2)\qquad W_{ij}=\frac{(E-x/2-i/2)_{i-j}(1/2-x/2-E-i/2)_{i-j}(-1-x-2i)_{i-j}}{(2i+1-j)!(-x-2E-2i)_{i-j}(-1-x+2E-2i)_{i-j}}$$

then

$$(6.3)\quad det(W_{ij})_{0\leqq i,j\leqq n-1}=\prod_{j=0}^{n-1}\frac{(2x+2j+1)_j j!\,2^{2j}((2+x+2E+j)/2)_j((3+x-2E+j)/2)_j}{(x+j+1)_j(2j+1)!(1+x+2E+j)_j(2+x-2E+j)_j}.$$

Now instead of using (6.1), we begin with the q-analog of (6.1), namely [7; eq. (6.14)]

(6.4)

$$\sum_{k=0}^{n}\frac{(q^{-n/2};q^{1/2})_k(qA/F;q^{1/2})_k(Fq^{(n-1)/2};q^{1/2})_k(A;q)_k(qA/B;q)_k(Bq^{-1/2};q)_k}{(Aq^{1+n/2};q)_k(F;q)_k(Aq^{(3-n)/2}/F;q)_k(q^{1/2};q^{1/2})_k(Bq^{-1/2};q^{1/2})_k(qA/B;q^{1/2})_k}\frac{(1-Aq^{3k/2})}{1-A}q^{k/2}=0$$

provided n is a positive, odd integer.

From (6.4), we may deduce

Theorem 8. *If*

$$V_{ij} = \frac{(q^{2E-x-i};q^2)_{i-j}(q^{1-x-2E-i};q^2)_{i-j}(q^{-2-2x-4i};q^2)_{i-j}}{(q;q)_{2i+1-j}(q^{-x-2E-2i};q)_{i-j}(q^{-1-x+2E-2i};q)_{i-j}}, \tag{6.5}$$

then

$$\begin{aligned} &det(V_{ij})_{0\leqq i,j\leq n-1} \\ &\quad = \prod_{j=0}^{n-1} \frac{(q^{2x+2j+1};q)_j(q^{3+x-2E+j};q^2)_j(q^{2+x+2E+j};q^2)_j}{(q^{2x+2j+2};q^2)_j(q;q^2)_{j+1}(q^{1+x+2E+j};q)_j(q^{2+x-2E+j};q)_j}. \end{aligned} \tag{6.6}$$

where we are using the standard notation

$$(A;q)_n = \prod_{j=0}^{n-1}(1-Aq^j). \tag{6.7}$$

Corollary 3. *If*

$$U_{ij} = \begin{bmatrix} x+i+j+1 \\ 2i-j+1 \end{bmatrix} \frac{q^{-2i(1+j+x)}}{(1-q^{x+i+j+1})(-q^{x+2};q)_{i+j}}, \tag{6.8}$$

then

$$det(U_{ij})_{0\leqq i,j\leqq n-1} = \prod_{j=0}^{n-1} \frac{(q^{2x+2j+1};q)_j(q^{x+2j+2};q)_j q^{-2j(1+j+x)}}{(q^{2x+2j+2};q^2)_j(q;q^2)_{j+1}(-q^{x+2};q)_{2j}} \tag{6.9}$$

where

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{(q;q)_A}{(q;q)_B(q;q)_{A-B}}. \tag{6.10}$$

7. Conclusion

There are a number of mysteries left unresolved by our analysis. Perhaps the greatest surprise is the fact that we have no q-analog of Theorem 2 (and consequently none of Theorem 1). This was particularly noteworthy because Corollary 3 of Theorem 8 is, in fact, a q-analog of Theorem 3. Indeed this discrepancy arises because of the fact that we do know (6.4), the q-analog of (1.7), while we most emphatically do not know a q-analog of (1.6).

Since Theorems 1 and 2 have arisen in significant plane partition problems we would expect that Theorem 3 might as well. If such connections for Theorem 3 are found, then Corollary 3 of Theorem 8 might well provide a further tie.

Also attention should again be drawn to the marvelous proof of Theorem 1 by Wilf and Petkovsek which draws some of its inspiration from the WZ-method. Presumably their methods will apply to other determinant problems and may well help to answer some of the questions raised above.

References

1. G. E. Andrews, *Plane partitions III: the weak Macdonald conjecture*, Invent. Math. **53** (1979), 193–225.
2. G. E. Andrews, *Pfaff's method (I): the Mills-Robbins-Rumsey determinant*, (to appear).
3. G. E. Andrews and W. H. Burge, *Determinant identities*, Pac. J. Math. **158** (1993), 1–14.
4. W. N. Bailey, *Products of generalized hypergeometric series*, Proc. London Math. Soc. **Ser. 2, 28** (1928), 242–254.
5. W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge University Press, Cambridge, 1935 (Reprinted: Hafner, New York, 1964).
6. I. Gessel and D. Stanton, *Strange evaluations of hypergeometric series*, S.I.A.M. J. Math. Anal. **13** (1982), 295–308.
7. I. Gessel and D. Stanton, *Applications of q-Lagrange inversion to basic hypergeometric series*, Trans. Amer. Math. Soc. **277** (1983), 173–201.
8. W. H. Mills, D. P. Robbins and H. Rumsey, *Enumeration of a symmetry class of plane partitions*, Discrete Math. **67** (1987), 43–55.

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