ASYMPTOTICS OF JACK POLYNOMIALS AS THE NUMBER OF VARIABLES GOES TO INFINITY

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Abstract. In this paper we study the asymptotic behavior of the Jack rational functions $P_\lambda(z_1, \dots, z_n; \theta)$ as the number of variables n and the signature λ grow to infinity. Our results generalize the results of A. Vershik and S. Kerov [VK2] obtained in the Schur function case ($\theta = 1$). For $\theta = 1/2, 2$ our results describe approximation of the spherical functions of the infinite-dimensional symmetric spaces $U(\infty)/O(\infty)$ and $U(2\infty)/Sp(\infty)$ by the spherical functions of the corresponding finite-dimensional symmetric spaces.

Contents

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1. Introduction

1.1 Statement of the main result. Jack symmetric functions

$$P_\lambda(z_1,\dots,z_n;\theta)\in\mathbb{Q}(\theta)[z^{\pm1}]^{S(n)}$$

which are indexed by decreasing sequences of integers (called signatures)

$$\lambda=(\lambda_1\ge\dots\ge\lambda_n)\in\mathbb{Z}^n\,,$$

are eigenfunctions of the quantum Calogero-Sutherland Hamiltonian [C,Su]

$$H_{CS}=\frac{1}{2}\sum_i\left(z_i\frac{\partial}{\partial z_i}\right)^2+\theta\sum_{i\neq j}\frac{z_i^2}{z_i-z_j}\frac{\partial}{\partial z_i}\,. \tag{1.1}$$

The Hamiltonian H_{CS} describes n identical quantum particles living on the circle $\mathbb{T}=\{z\in\mathbb{C},|z|=1\}$ and interacting via an inverse-square potential. Here $\theta>0$ is a fixed positive parameter (a coupling constant). For 3 special values

$$\theta=1,\frac{1}{2},2$$

the functions $P_\lambda(z;\theta)$ reduce to Schur functions (which are spherical functions of the symmetric space $U(n)$) and spherical functions of Gelfand pairs

$$U(n)\supset O(n)\,,\quad U(2n)\supset Sp(n) \tag{1.2}$$

respectively. For these values of θ, the meaning of the parameter θ is half the multiplicity of a root in the restricted root system (of type A_n) associated to the corresponding symmetric space.

In this paper we study the behavior of the Jack polynomials as the number of variables grows to infinity in such a way that all but finitely many variables z_i are kept equal to 1. More precisely, set $\mathbb{T}^k=(\mathbb{T})^{\times k}$ and consider the direct limit

$$\mathbb{T}^\infty:=\varinjlim\mathbb{T}^k$$

with respect to embeddings $(z_1,\dots,z_k)\mapsto(z_1,\dots,z_k,1)$. Equip $\mathbb{T}^\infty$ with the direct limit topology.

Let $\lambda(n)=(\lambda_1(n)\ge\dots\ge\lambda_n(n))$ be a sequence of signatures. We call this sequence *regular* if the following sequence of *normalized Jack functions*

$$\Phi_{\lambda(n)}(z_1,\dots,z_n;\theta):=\frac{P_{\lambda(n)}(z_1,\dots,z_n;\theta)}{P_{\lambda(n)}(1,\dots,1;\theta)} \tag{1.3}$$

converges uniformly on the compact subsets of $\mathbb{T}^\infty$ (it suffices to consider the subsets $\mathbb{T}^k\subset\mathbb{T}^\infty$). Notice that the normalization in (1.3) is necessary to make these functions converge at least at the point $(1,1,\dots)\in\mathbb{T}^\infty$. Note also that the normalization (1.3) is precisely the normalization for the spherical function (for $\theta=1,\frac{1}{2},2$).

Our main result is an explicit description of all regular sequences and an explicit computation of the corresponding limits

$$\lim_{n\to\infty} \Phi_{\lambda(n)}(z;\theta)\,, \tag{1.4}$$

given in Theorem 1.1 below. In the Schur function case ($\theta = 1$) this program was carried out by A. Vershik and S. Kerov[1] (see [VK2]) in their proof of the classification theorem for characters of the group $U(\infty)$. In particular they have found (for $\theta = 1$) a beautiful description of the set of regular sequences. Their necessary and sufficient conditions, which we shall call the *Vershik-Kerov conditions*, are as follows.

First, suppose we have a sequence of partitions $\{\lambda(n)\}_{n=1,2,\dots}$. This sequence is said to be a *Vershik-Kerov sequence* (or VK for short) if the following limits exist:

$$\begin{aligned} \alpha_i &:= \lim \frac{\lambda(n)_i}{n} & < \infty\,, \\ \beta_i &:= \lim \frac{\lambda'(n)_i}{n} & < \infty\,, \\ \delta &:= \lim \frac{|\lambda(n)|}{n} & < \infty\,. \end{aligned}$$

Here λ' denotes the partition conjugated to λ; in other words, λ'_i is the length of the i-th column in the diagram of λ. It is easy to see that the number

$$\gamma := \delta - \sum (\alpha_i + \beta_i) \geq 0$$

is nonnegative. The numbers α_i, β_i, γ are called the *VK parameters* of the sequence $\lambda(n)$.

Now, with each signature λ one can associate two partitions λ^+ and λ^- as follows. Suppose $\lambda_p \geq 0 \geq \lambda_{p+1}$ for some $p = 1, \dots, n$. Then, by definition,

$$\begin{aligned} \lambda^+ &= (\lambda_1, \dots, \lambda_p)\,, \\ \lambda^- &= (-\lambda_n, \dots, -\lambda_{p+1})\,. \end{aligned}$$

A sequence of signatures

$$\lambda(n) = (\lambda_1(n) \geq \cdots \geq \lambda_n(n))$$

is called a *VK sequence of signatures* if the two sequences of partitions $\lambda^+(n)$ and $\lambda^-(n)$ are Vershik-Kerov sequences. The corresponding numbers $\alpha_i^\pm$, $\beta_i^\pm$, and $\gamma^\pm$ are called the VK parameters of the sequence $\lambda(n)$. A typical member of a typical VK sequence looks like this:

[1]we order the names of A. Vershik and S. Kerov in the Russian alphabetical order

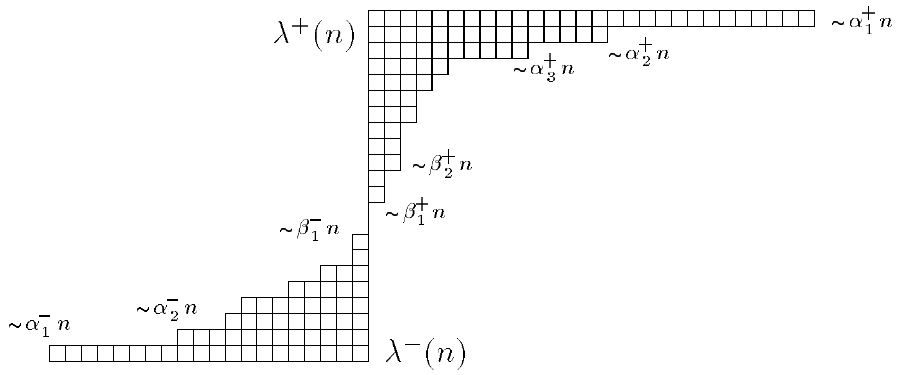

Fig. 1 A typical element of a Vershik-Kerov sequence

Throughout this paper we assume the following important notational convention. For a sequence $(\lambda(n))$ of signatures the letter n will always stand for the length of the signature $\lambda(n)$. The values of n may range, however, over some proper infinite subset of $\{1, 2, \dots\}$. In other words, we allow gaps in the values of n.

Our main result, which is a direct generalization of the result of Vershik and Kerov [VK2] is the following:

Theorem 1.1. *For any $\theta > 0$, a sequence $\lambda(n)$ of signatures is regular if and only if it is VK. If $\lambda(n)$ is VK with parameters $\alpha_i^\pm$, $\beta_i^\pm$, and $\gamma^\pm$ then we have*

$$\lim_{n\to\infty} \Phi_{\lambda(n)}(z;\theta) = \phi_{\alpha,\beta,\gamma}(z_1)\phi_{\alpha,\beta,\gamma}(z_2)\dots, \tag{1.5}$$

where the function $\phi_{\alpha,\beta,\gamma}(z)$ is the following product
(1.6)
$$\phi_{\alpha,\beta,\gamma}(z) := e^{\gamma^+(z-1)+\gamma^-(z^{-1}-1)} \prod_i \frac{(1+\beta_i^+(z-1))}{(1-\alpha_i^+(z-1)/\theta)^\theta} \frac{(1+\beta_i^-(z^{-1}-1))}{(1-\alpha_i^-(z^{-1}-1)/\theta)^\theta}.$$

The "if" part of this theorem will be established in Theorem 4.1 ; the "only if" part will follow from Theorem 5.1 in Section 5.1.

We shall denote by $\Phi_{\alpha,\beta,\gamma}(z)$ the function in the right-hand side of (1.5). The remarkable twofold factorization of the limit $\Phi_{\alpha,\beta,\gamma}$ has the following interpretation in the group-theoretic situation $\theta = 1, \frac{1}{2}, 2$ (see [Vo,Ol1]). The factorization (1.5), which states that all variables z_i become, in a sense, independent of each other, is the infinite-dimensional degeneration of the functional equation satisfied by the spherical functions. It reflects the fact that the convolution algebra of $K(n)$-biinvariant functions on $G(n)$ (here $G(n) \supset K(n)$ stands for a series of Gelfand pairs) becomes, in a certain precise sense, a semigroup algebra as $n \to \infty$.

The second factorization (1.6) asserts that all spherical representations of the corresponding Gelfand pairs are tensor products of elementary (anti)-bosonic, (anti)-fermionic, and certain intermediate representations.

In the context of the quantum Calogero-Sutherland model the parameters α and β are (in a somewhat different limit transition) associated with quasi-particles and quasi-holes, see [Ha,LPS,Ok11].

1.2 Regular and infinitesimally regular sequences.

Although there are several more or less explicit formulas for Jack polynomials (see, for example, (2.3) below), none of them seems to be suitable for direct investigation of the limit (1.4). The strategy we shall use in this paper (and which in an implicit form was present already in the work of Vershik and Kerov) is to replace the study of the uniform convergence of (1.3) by the investigation of convergence of the coefficients in the Taylor expansion of (1.3) about the point $z = (1, 1, \dots)$. The uniform convergence of (1.3) is, in fact, equivalent to weak convergence of certain measures (see below); the study of its Taylor series is then equivalent to the study of moments of those measures, which is a time-honored way of proving limit theorems. In the group-theoretic situation $\theta = 1, \frac{1}{2}, 2$, this means that one replaces the study of functions on a Lie group G by the study of the corresponding linear functionals on the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of G.

Let us call a sequence $\lambda(n)$ an *infinitesimally regular* sequence if the Taylor series of the functions (1.3) about the point $(1, 1, \dots)$ tend to a limit coefficient by coefficient. Since it is very often that quite complicated special functions have rather simple Taylor series, it is no surprise that this condition turns out to be easier to verify than the regularity condition. Of course, this condition is not *a priori* equivalent to the regularity condition. However we shall prove the following result (the last condition (iv) in that theorem will be explained below)

Theorem 1.2. *Given a sequence $\lambda(n)$, the following conditions are equivalent*

(i) *$\lambda(n)$ is regular,*
(ii) *$\lambda(n)$ is infinitesimally regular,*
(iii) *$\lambda(n)$ is a Vershik-Kerov sequence,*
(iv) *for any polynomial $f \in \Lambda^\theta$ the limit $\lim_{n\to\infty} \dfrac{f(\lambda(n))}{n^{\deg f}}$ exists,*

and the 4 above conditions are also equivalent to the hypothesis of Theorem 5.1.

The proof of this theorem will be completed in Section 5.1.

The meaning of the condition (iv) is the following. The quantum Calogero-Sutherland Hamiltonian (1.1) is completely integrable: there are n independent commuting differential operators which commute with H_{CS}, or, in other words, there are n independent quantum integrals of motion, see e.g. [He]. The values of those conserving quantities in the quantum state described by $P_\lambda(z)$ are polynomials in $\lambda_1, \dots, \lambda_n$ symmetric in variables $\lambda_i - \theta\, i$. Conversely, each polynomial with such symmetry corresponds to some integral of motion. Thus, the subalgebra

$$\Lambda^\theta(n) \subset \mathbb{C}[\lambda_1, \dots, \lambda_n]$$

of polynomials symmetric in variables $\lambda_i - \theta i$ is a natural commutative algebra of observables in the model. For example, the polynomial

$$\lambda_1 + \cdots + \lambda_n \in \Lambda^\theta(n)$$

corresponds to the differential operator $\sum z_i \frac{\partial}{\partial z_i}$ which is the total momentum of the system. The polynomial

$$E(\lambda) = \frac{1}{2} \sum_i \lambda_i^2 + \theta \sum_i (n-i)\lambda_i$$

corresponds to the energy of the system, that is, to the operator H_{CS} itself.

We denote by

$$\Lambda^\theta := \varprojlim \Lambda^\theta(n),$$

the inverse limit of $\Lambda^\theta(n)$ as filtered (by degree of polynomials) algebras with respect to homomorphisms:

$$\begin{aligned} \Lambda^\theta(n+1) \quad &\to \quad \Lambda^\theta(n)\,, \\ f(\lambda_1,\dots,\lambda_{n+1}) &\mapsto f(\lambda_1,\dots,\lambda_n,0)\,. \end{aligned} \tag{1.7}$$

Examples of elements of Λ^θ are the functions $\sum \lambda_i$, $\sum \lambda_i^2 - 2\theta \sum i\lambda_i$, or, more generally, $\sum((\lambda_i-\theta i)^k - (-\theta i)^k)$, where $k = 0,1,2,\dots$.

The condition (iv) now means that all "stable" observables f, after proper normalization, have a limit value as $n \to \infty$. Observe that energy

$$E(\lambda) = \frac{1}{2}\left(\sum_i \lambda_i^2 - 2\theta \sum_i i\lambda_i\right) + n\theta \sum_i \lambda_i$$

is not a stable polynomial. Notice, however, that if the condition (iv) is satisfied then the limit

$$\lim_{n\to\infty} \frac{E(\lambda)}{n^2}$$

exists. More generally, one can consider the algebra $\Lambda^\theta[n]$ of polynomials in n with coefficients in stable polynomials in λ and introduce the notion of degree in this algebra by setting

$$\deg n = \deg \lambda_i = 1\,.$$

Then, for example, $E(\lambda)$ is an element of this algebra of degree 2. It is clear that (iv) is equivalent to the more general condition

$$\forall f \in \Lambda^\theta[n] \quad \exists \lim_{n\to\infty} \frac{f(\lambda,n)}{n^{\deg f}}\,.$$

Note finally that, although we shall make no use of those n commuting differential operators in the present paper, their role is crucial in the proof of the binomial theorem (see Section 2.4).

1.3 Extremality of the limit functions.

For $n = 1,2,\dots$ define the following convex subsets $\Upsilon_n^\theta \subset C(\mathbb{T}^n)$. The subset Υ_n^θ consists, by definition, of the functions of the form

$$\phi(z_1,\dots,z_n) = \sum_\lambda c_\lambda \Phi_\lambda(z;\theta)\,,$$

where λ ranges over signatures of length n and the coefficients c_λ are arbitrary satisfying

$$c_\lambda \ge 0\,, \quad \sum_\lambda c_\lambda = 1\,.$$

It is clear that Υ^θ_n is a simplex with countably many vertices. It follows from the positivity of the branching coefficients for Jack polynomials (see the formula (2.3) below) that the simplex Υ^θ_n is contained in (and for $n = 1$ coincides with) the simplex of characteristic functions of probability measures on $\mathbb{Z}^n$.

It also follows from the same positivity that the map

$$\phi(z_1, \dots, z_n) \mapsto \phi(z_1, \dots, z_{n-1}, 1)$$

defines an affine map

$$\Upsilon^\theta_n \to \Upsilon^\theta_{n-1}\,.$$

We set

$$\Upsilon^\theta := \varprojlim \Upsilon^\theta_n\,.$$

By definition, each element of Υ^θ is a function on $\mathbb{T}^\infty$. It is clear that the set Υ^θ is convex. Denote by $\operatorname{Ex}\Upsilon^\theta$ the set of its extreme points. Recall that $\Phi_{\alpha,\beta,\gamma}(z)$, $z \in \mathbb{T}^\infty$, denotes the RHS of the formula (1.5). In Section 6 we shall demonstrate the following

Theorem 1.3. *We have*

$$\operatorname{Ex}\Upsilon^\theta = \left\{\Phi_{\alpha,\beta,\gamma}\right\}_{\alpha^\pm,\beta^\pm,\gamma^\pm}\,,$$

where $\alpha^\pm, \beta^\pm, \gamma^\pm$ range over all possible values of VK parameters:

$$\begin{gathered}
\alpha_1^+ \geq \alpha_2^+ \geq \dots 0\,, \quad \alpha_1^- \geq \alpha_2^- \geq \dots 0\,,\\
\beta_1^+ \geq \beta_2^+ \geq \dots 0\,, \quad \beta_1^- \geq \beta_2^- \geq \dots 0\,,\\
\sum(\alpha_i^+ + \alpha_i^- + \beta_i^+ + \beta_i^-) < \infty\,,\\
\beta_1^+ + \beta_1^- \leq 1\,,\\
\gamma^\pm \geq 0\,.
\end{gathered}$$

Theorem 1.3 is a rather straightforward corollary of Theorem 1.1 and some general abstract theorems, see Section 6.

The statement of Theorem 1.3 was known for $\theta = 1, \frac{1}{2}, 2$ where the set $\operatorname{Ex}\Upsilon^\theta$ is the set of characters of

$$U(\infty) = \varinjlim U(n)$$

and the set of spherical functions of Gelfand pairs

$$U(\infty) \supset O(\infty)\,, \quad U(2\infty) \supset Sp(\infty) \tag{1.8}$$

respectively.

The description of Υ^1 is a famous theorem, various pieces of which were proved by A. Edrei, D. Voiculescu, R. Boyer, and A. Vershik and S. Kerov. Namely, D. Voiculescu constructed in [Vo] a large supply of characters of $U(\infty)$ and conjectured that his list is complete. Later R. Boyer [B1] and independently A. Vershik and S. Kerov [VK2] observed that that completeness follows from an old hard theorem of A. Edrei [Ed]. In the same note, A. Vershik and S. Kerov have also outlined an alternative plan of proof (which uses approximation of characters of $U(\infty)$ by characters of $U(n)$) and have formulated the $\theta = 1$ case of above Theorem 1.1. A complete proof of a particular case of that result of Vershik and Kerov (the case when $\lambda(n)$ is assumed to be a partition) was published later by R. Boyer in [B2]. In that proof he used techniques different from those announced by Vershik and Kerov (and different from those used in our paper).

The description of Υ^θ for $\theta = \frac{1}{2}, 2$ was obtained in the papers of G. Olshanski and D. Pickrell. Spherical (and more general admissible) representations of the infinite-dimensional Gelfand pairs corresponding to classical series of Riemannian symmetric spaces were constructed and studied by Olshanski in [Ol1]. The proof of the completeness of Olshanski's lists of spherical representations was completed by D. Pickrell in [P] by using embeddings of symmetric spaces into each other.

The new piece of information we obtain in the case $\theta = \frac{1}{2}, 2$ is how exactly the spherical functions of the Gelfand pairs (1.2) approximate the spherical functions of the corresponding infinite-dimensional Gelfand pairs (1.8).

1.4 Related results.

1.4.1 Shifted Schur functions and binomial formula.

Our research presented in this paper started in 1994 in the framework of the general classification problem of admissible representations of infinite-dimensional Gelfand pairs, see [Ol1-2,KO,OV,Ok1-2]. We began by analyzing the details of the proofs sketched in the seminal paper [VK2] of A. Vershik and S. Kerov. As we already mentioned above, that paper contained, among other things, the following important idea: one has to consider the convergence of Taylor series coefficients.

The Taylor expansion of the Schur functions about the point $(1, \dots, 1)$ are given by the following *binomial formula*

$$\frac{s_\lambda(1+x_1, \dots, 1+x_n)}{s_\lambda(1, \dots, 1)} = \sum_\mu \frac{s^*_\mu(\lambda)\, s_\mu(x)}{\prod_{(i,j)\in\mu}(n+j-i)}\,. \tag{1.9}$$

Here s^*_μ are certain factorial analogs of Schur functions which can be defined by the following determinant ratio formula

$$s^*_\mu(x_1, \dots, x_n) = \frac{\det\left[(x_i+n-i)\cdots(x_i-i+j-\mu_j+1)\right]_{1\le i,j\le n}}{\prod_{1\le i<j\le n}(x_i-x_j+j-i)}\,. \tag{1.10}$$

We call these functions the *shifted Schur functions*, see [OO]. They differ by a shift of variables only from the factorial Schur functions introduced in [BL] and studied further in [M2] and other papers cited in [OO].

A formula equivalent to the binomial formula (1.9) was established by A. Lascoux, see [Lasc] and also Example I.3.10 in [M2]. Its proof is elementary and based on the determinant ratio formula for the Schur functions

$$s_\mu(x_1, \dots, x_n) = \frac{\det\left[x_i^{\mu_j+n-j}\right]_{1\le i,j\le n}}{\prod_{1\le i<j\le n}(x_i-x_j)}\,. \tag{1.11}$$

The observation that the coefficients appearing in Lascoux's binomial formula are factorial analogs of ordinary Schur functions was made by Olshanski in [Ol3], see also [OO] and the next subsection.

From (1.9) it is clear that for $\theta = 1$ a sequence $\lambda(n)$ of signatures is infinitesimally regular if and only if the limit

$$\lim_{n\to\infty} \frac{s^*_\mu(\lambda(n))}{n^{|\mu|}}$$

exists for every partition μ.

1.4.2 Quantum immanants([OO,Ok6-7]**).**

For general Jack polynomials (or even for $\theta = 1/2, 2$), no formulas as simple as (1.11) are available. Therefore, in order to generalize (1.9), one needs some insight into the invariant meaning of the expansion (1.9).

Observe that in the binomial formula (1.9) there is a certain symmetry between λ and x. There exists, in fact, a very precise parallel between the functions s_μ and s^*_μ. In short, the relation between them is the same as the relation between the algebra

$$Z(G) = C(G)^G \tag{1.12}$$

of central continuous functions on a compact group G and the algebra of the *Laplace operators*

$$\mathfrak{Z}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})^G \tag{1.13}$$

on the group G. Here $\mathfrak{g}$ is the (complexified) Lie algebra of G and $\mathcal{U}(\mathfrak{g})$ stands for its universal enveloping algebra. (In our case, $G = U(n)$). Instead of the whole group G, one can look at its maximal torus and the radial parts of the elements of $Z(G)$ and $\mathfrak{Z}(\mathfrak{g})$.

The functions s_μ and s^*_μ correspond to distinguished linear bases of the spaces (1.12) and (1.13) respectively. In (1.12), the Haar measure defines a natural scalar product

$$(f_1, f_2) = \int_G f_1(g)\,\overline{f_2(g)}\,dg\,,$$

with respect to which the characters (i.e. Schur functions) form an orthogonal basis. Another way of expressing this orthogonality is to say that a character (as a function on G) vanishes in all but one irreducible representations.

In the algebra $\mathfrak{Z}(\mathfrak{g})$, there is no natural scalar product. However, there is a natural pairing between (1.12) and (1.13), namely

$$\langle D, f\rangle = \big(Df\big)(1,\dots,1)\,, \quad D \in \mathfrak{Z}(\mathfrak{g})\,,$$

where D is the radial part of a Laplace operator and f is the radial part of an element of $Z(G)$. In particular,

$$\langle D, s_\lambda\rangle = \operatorname{tr}\pi_\lambda(D)\,,$$

where π_λ stands for the irreducible representation with highest weight λ. There exists a linear basis $\{\mathbb{S}_\mu\} \subset \mathfrak{Z}(\mathfrak{g})$ indexed by partitions μ with at most n parts such that

(1) $\mathbb{S}_\mu$ is an operator of order $|\mu|$,
(2) $\pi_\lambda(\mathbb{S}_\mu) = 0$ for all partitions λ such that $\mu \not\subset \lambda$.

In words, $\mathbb{S}_\mu$ vanish in as many irreducible representations as possible (it is clearly impossible for an element of $\mathfrak{Z}(\mathfrak{g})$ to vanish in all but one irreducible representation). We call these elements of $\mathfrak{Z}(\mathfrak{g})$ the *quantum immanants*, see [OO,Ok6-7]. Their relation to the s^*-functions is the following:

$$\pi_\lambda(\mathbb{S}_\mu) = s^*_\mu(\lambda) \cdot \mathrm{Id}\,,$$

which follows easily from the definition of $\mathbb{S}_\mu$, the Harish-Chandra isomorphism theorem, and the following three direct consequences of (1.10):

(1.14) $\deg s^*_\mu = |\mu|\,,$

(1.15) the polynomial $s^*_\mu(\lambda)$ is symmetric in variables $\lambda_i - i$,

(1.16) $s^*_\mu(\lambda) = 0$ for all partitions λ such that $\mu \not\subset \lambda$.

The quantum immanants and shifted Schur functions enjoy quite a few remarkable properties, some of which do and some of which don't have analogs for the ordinary Schur functions. We studied them at length in our papers [OO,Ok6-7]. For example, the explicit formulas for quantum immanants proved in [Ok6] (see also [N2,Ok7]) generalize the classical Capelli identities (see, for example, the papers [HU,N1] and also [S1]).

It is amazing how powerful is the vanishing property (1.16). For example, it yields a short computation-less proof of the expansion (1.9) (see [OO], Section 5) and also other previously known properties of the s^*-functions.

To summarize, the role which the quantum immanants play behind the scene in the Vershik-Kerov theorem is the following. When one goes from the Schur functions to their Taylor expansions, one replaces the study of the functions

$$\frac{\operatorname{tr}\pi_\lambda(g)}{\dim \pi_\lambda}\,, \quad g \in U(n)\,, \qquad \pi_\lambda \in U(N)^\wedge\,, \quad n \le N\,,$$

by the study of the functions

$$\frac{\operatorname{tr}\pi_\lambda(X)}{\dim \pi_\lambda}\,, \quad X \in \mathfrak{Z}(\mathfrak{gl}(n))\,, \quad \pi_\lambda \in U(N)^\wedge\,, \quad n \le N\,. \tag{1.17}$$

The quantum immanants are the distinguished elements of $\mathfrak{Z}(\mathfrak{gl}(n))$ on that the functions (1.17) are easy to compute (see the formulas in [OO], Section 10, and also in [Ok6], Section 5).

1.4.3 The case of general $\theta > 0$.

For general $\theta > 0$, we have the algebra of quantum integrals of motion in the Calogero-Sutherland model instead of the algebra of the radial parts of Laplace operators. Instead of quantum immanants, we have differential operators that annihilate as many Jack polynomials $P_\lambda(x;\theta)$ as possible. From the discussion after Theorem 1.2 it is clear that the role of the s^*_μ functions is to be played by the polynomials $P^*_\mu(x;\theta)$ satisfying the following conditions (compare with (1.14-16)):

$$\deg P^*_\mu = |\mu|\,, \tag{1.18}$$

$$\text{the polynomial } P^*_\mu(\lambda;\theta) \text{ is symmetric in variables } \lambda_i - \theta i\,, \tag{1.19}$$

$$P^*_\mu(\lambda;\theta) = 0 \text{ for all partitions } \lambda \text{ such that } \mu \not\subset \lambda\,. \tag{1.20}$$

Note that the polynomials $P^*_\mu(x;\theta)$ are a particular case of the polynomials considered by S. Sahi in [S1]. We call these polynomials the *shifted Jack polynomials*; they shall play a central role in the present paper and will be discussed more formally in Section 2.

Modifying suitably (see [OO2]) the argument from [OO], Section 5 (2nd proof), one deduces from the definition (1.18-20) the binomial formula given in the Section 2.4 below. However, the abstract formula (2.6) alone is not sufficient for the purposes of the present paper. One needs some control of the polynomials $P^*_\mu(x;\theta)$, for example, one would like to know their highest degree term. We have conjectured the explicit formula (2.4) for the shifted Jack polynomials $P^*_\mu(x;\theta)$, which, in particular, implies that

$$P^*_\mu(x;\theta) = P_\mu(x;\theta) + \dots\,, \tag{1.21}$$

where dots stand for lower degree terms. However, we were only able to prove our conjecture for $\mu = (m)$ (using the argument reproduced in Section 7) and to check (1.21) in the group-theoretic situation $\theta = \frac{1}{2}, 2$.

First proof of (1.21) for general θ was given, together with some other fundamental results, by F. Knop and S. Sahi in [KS]; see also their papers [Kn,S]. They used certain difference equations for these polynomials.[2 3]

Later the formula (2.4) and analogs of other properties of shifted Schur functions were established for shifted Jack polynomials and also their q-analogs in [OO2,Ok8-9].

1.4.4 Log-concavity and the "only if" part of Theorem 1.1.

The most subtle point of the "only if" part of Theorem 1.1 seems to be to justify the following implication (see the statement of Theorem 1.2)

$$\text{(i)} \overset{?}{\Longrightarrow} \text{(ii)}\,. \tag{1.22}$$

This point is quite subtle even in the $\theta = 1$ case. (A. Vershik and S. Kerov did not discuss the proof of the "only if" part of their theorem in [VK2].)

[2]It seems to be an interesting and important problem to understand better the nature of the Knop-Sahi difference equations. For example, their analogs do not exists for other series of root systems, see [Ok10].

[3]Also note, that by virtue of the binomial formula (2.6), the relation (1.21) is equivalent to formula (4.2) in [OO2] for the Bessel functions. Apparently, that formula for the Bessel functions was also known to M. Lassalle, see the paper [BF], Sections 3 and 6, where the authors cite Lassalle's unpublished letters. Lassalle's argument is very different from proofs of (1.21) given in [KS] and [Ok8].

A proof and an explanation of (1.22) based on some general log-concavity results was proposed in the paper [Ok3]. (See also the subsequent papers [Gr,Ok4-5,Ka].) Similar log-concavity argument applies in the situation considered in [OV].

Unfortunately, that log-concavity fails for general $\theta > 0$, even for the limit functions (1.5). Indeed, the coefficients in the expansion

$$\frac{1}{(1-z)^\theta} = 1 + \theta\, z + \frac{\theta(\theta+1)}{2} z^2 + \dots$$

do not form a log-concave sequence if $\theta < 1$. In this paper (in the proof of the "only if" part of Theorem 1.1) we shall use a different approach, where the main role is played by the simple yet effective Lemma 5.2.

1.4.5 $S(\infty)$ and Kerov's conjecture ([VK1,K,KOO]).

The work of A. Vershik and S. Kerov on the characters of $U(\infty)$ had very much in common with their earlier work [VK1] on the characters of $S(\infty)$. The problems considered in this paper do have their "symmetric group counterparts", too. Those are very similar, but less technically involved. For example, the analog of the problem, addressed in Theorem 1.3, is is the following problem: describe all homomorphisms

$$\sigma : \Lambda \to \mathbb{C}$$

such that

$$\forall \lambda \quad \sigma(P_\lambda(x;\theta)) \geq 0\,. \tag{1.23}$$

Here Λ denotes the $\mathbb{C}$-algebra of symmetric functions in infinitely many variables. In [K], S. Kerov conjectured an explicit description of all homomorphisms $\Lambda \to \mathbb{C}$ that are positive on the Macdonald polynomials. In particular, it implies that all homomorphism σ satisfying (1.23) are given by extended symmetric functions (see Section 2.7)

$$\Lambda \ni f \xrightarrow{\ \sigma\ } f(\alpha;\beta;\gamma;\theta) \in \mathbb{C},$$

where $\alpha = (\alpha_1 \geq \alpha_2 \geq \dots \geq 0)$, $\beta = (\beta_1 \geq \beta_2 \geq \dots \geq 0)$ and $\gamma \geq 0$ are arbitrary. This particular case of Kerov's conjecture was proved in [KOO] using the approximation techniques proposed in [V,VK1] and the P^*-functions machinery. That paper contains also a discussion of other equivalent forms of this problem and a symmetric group analog of Theorem 1.1. In the present paper we shall use the estimate established in Theorem 7.1 of [KOO].

1.4.6 Other series of root systems.

In our next paper we shall prove analogs of the results obtained here for other classical series of root systems. The necessary P^*-functions techniques were developed in [Ok10], see also our paper [OO3]. Note also that the analogs of the Vershik-Kerov theorem for the groups $O(\infty)$ and $Sp(\infty)$ were obtained by R. Boyer in [B2].

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2. Jack polynomials and shifted Jack polynomials

In this section we fix notation and gather some fundamental results. We shall mostly follow the standard Macdonald's notation, with a few exceptions. Most importantly, as in [OO2], our fixed positive parameter $\theta > 0$ is inverse to the Macdonald's parameter $\alpha = 1/\theta$.

2.1 Orthogonality ([M1,St]). We denote by Λ the $\mathbb{C}$-algebra of symmetric functions (in infinitely many variables). The Jack polynomials $P_\lambda(x;\theta)$ form an orthogonal basis of Λ with respect to the following inner product:

$$(p_\lambda, p_\mu) := \delta_{\lambda,\mu} z_\lambda \, \theta^{-\ell(\lambda)}\,, \tag{2.1}$$

where λ and μ are two partitions, p_λ is the following symmetric function

$$p_\lambda := \prod_{k \le \ell(\lambda)} \left(\sum_i x_i^{\lambda_k} \right) ,$$

$\ell(\lambda)$ stands for the number of parts of λ, and the factor z_λ is defined in [M1], Section I.2. Let

$$Q_\lambda(x;\theta) := \frac{1}{(P_\lambda, P_\lambda)} P_\lambda(x;\theta)$$

be the dual (with respect to the inner product (2.1)) basis of Λ. The orthogonality relation $(P_\lambda, Q_\mu) = \delta_{\lambda,\mu}$ is equivalent to the following Cauchy-type identity:

$$\begin{aligned} \prod_{i,j} (1 - x_i y_j)^{-\theta} &= \sum_\lambda P_\lambda(x)\, Q_\lambda(y)\,, \\ &= \sum_\lambda Q_\lambda(x)\, P_\lambda(y)\,, \end{aligned} \tag{2.2}$$

2.2 Interpolation ([S1,OO,Ok6,KS,Kn,S2,Ok8]).

We denote by $\Lambda^\theta(n) \subset \mathbb{C}[x_1, \dots, x_n]$ the subalgebra of polynomials, symmetric in variables $x_i - \theta i$, and denote by

$$\Lambda^\theta := \varprojlim \Lambda^\theta(n),$$

the inverse limit of $\Lambda^\theta(n)$ as filtered (by degree of polynomials) algebras with respect to homomorphisms:

$$\begin{aligned} \Lambda^\theta(n+1) \quad &\to \quad \Lambda^\theta(n)\,, \\ f(x_1,\dots,x_{n+1}) &\mapsto f(x_1,\dots,x_n,0)\,. \end{aligned}$$

Note that, in particular, the value $f(\lambda)$ is well defined for any partition λ and any $f \in \Lambda^\theta$. The *shifted* (or *interpolation*) Jack polynomials $P^*_\mu(x;\theta)$ is the unique polynomial satisfying the following Newton interpolation conditions:

(1) $P^*_\mu(x;\theta) \in \Lambda^\theta$,
(2) $\deg P^*_\mu(x;\theta) = |\mu|$,
(3) $P^*_\mu(\lambda;\theta) = 0$ unless $\mu \subset \lambda$,
(4) $P^*_\mu(\mu;\theta) = H(\mu)$,

where $H(\mu)$ is a normalization constant defined as follows. Recall that for a square $s=(i,j)\in\mu$ in the diagram of a partition μ the numbers

$$\begin{aligned} a(s) &= \mu_i - j, \qquad & a'(s) &= j-1, \\ l(s) &= \mu'_j - i, \qquad & l'(s) &= i-1, \end{aligned}$$

are called arm-length, arm-colength, leg-length, and leg-colength, respectively. We set

$$H(\mu) := \prod_{s\in\mu} (a(s) + \theta\, l(s) + 1)\,.$$

Set also

$$H'(\mu) := \prod_{s\in\mu} (a(s) + \theta\, l(s) + \theta)\,.$$

Then we have

$$(P_\lambda, P_\lambda) = H(\lambda)/H'(\lambda)\,.$$

2.3 Branching rules. One has the following branching rule for the Jack polynomials (see [St] and also [M1])

$$P_\lambda(x_1,\dots,x_n;\theta) = \sum_{\mu\prec\lambda} \psi_{\lambda/\mu}\, x_1^{|\lambda/\mu|}\, P_\mu(x_2,\dots,x_n;\theta)\,, \tag{2.3}$$

where $\mu \prec \lambda$ stands for the inequalities of interlacing

$$\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \ge \dots \ge \mu_{n-1} \ge \lambda_n\,,$$

and (we use the standard notation $(t)_m = t(t+1)\cdots(t+m-1)$)

$$\psi_{\lambda/\mu} := \prod_{1\le i\le j<n} \frac{(\mu_i - \mu_j + \theta(j-i) + \theta)_{\mu_j-\lambda_{j+1}}}{(\mu_i - \mu_j + \theta(j-i) + 1)_{\mu_j-\lambda_{j+1}}} \frac{(\lambda_i - \mu_j + \theta(j-i) + 1)_{\mu_j-\lambda_{j+1}}}{(\lambda_i - \mu_j + \theta(j-i) + \theta)_{\mu_j-\lambda_{j+1}}}$$

is a positive coefficient

$$\psi_{\lambda/\mu} > 0\,, \quad \mu\prec\lambda\,,$$

which is rational in θ and equals 1 for $\theta = 1$ (the Schur functions case).

The analog of (2.3) for the interpolation Jack polynomials is the following formula (see [Ok8]):

$$P^*_\lambda(x_1, x_2, \dots; \theta) = \sum_{\mu \prec \lambda} \psi_{\lambda/\mu} \left(\prod_{s \in \lambda/\mu} (x_1 - a'(s) + \theta\, l'(s)) \right) P^*_\mu(x_2, \dots; \theta). \tag{2.4}$$

Here λ is a partition whereas in (2.3) it can be a signature (see section 2.6). In particular, one has (see [KS] and also [Ok8]):

$$P^*_\lambda(x; \theta) = P_\lambda(x; \theta) + \dots, \tag{2.5}$$

where dots stand for the lower degree terms.

Such a close relationship between the orthogonal polynomials P_μ and the interpolation polynomials P^*_μ seems to be a rather remarkable phenomenon (notice that their definitions have nothing in common). The interplay between P_μ and P^*_μ will play the central role in this paper.

2.4 Binomial formula ([OO2]). Given a partition μ and a number t set

$$(t)_\mu = \prod_{s \in \mu} (t + a'(s) - \theta\, l'(s)).$$

If $\mu = (m)$ then $(t)_\mu = (t)_m$ is the standard shifted factorial. We have the following equivalent formulas

$$\begin{aligned} \frac{P_\lambda(1 + x_1, \dots, 1 + x_n; \theta)}{P_\lambda(1, \dots, 1; \theta)} &= \sum_\mu \frac{P^*_\mu(\lambda; \theta)\, Q_\mu(x_1, \dots, x_n; \theta)}{(n\theta)_\mu} \\ &= \sum_\mu \frac{Q^*_\mu(\lambda; \theta)\, P_\mu(x_1, \dots, x_n; \theta)}{(n\theta)_\mu}, \end{aligned} \tag{2.6}$$

where $Q^*_\mu(x) := P^*_\mu(x; \theta)/(P_\mu, P_\mu)$. Here λ can be a signature but μ is a partition. Recall that by Φ_λ we denote the normalized Jack rational function

$$\Phi_\lambda(z_1, \dots, z_n; \theta) := \frac{P_\lambda(z_1, \dots, z_n; \theta)}{P_\lambda(1, \dots, 1; \theta)}.$$

It follows that

$$\Phi_\lambda(z_1, \dots, z_k, \overbrace{1, \dots, 1}^{n-k \text{ times}}; \theta) = \sum_{\ell(\mu) \le k} \frac{Q^*_\mu(\lambda; \theta)\, P_\mu(z_1 - 1, \dots, z_k - 1; \theta)}{(n\theta)_\mu}. \tag{2.7}$$

This expansion (2.7) will play a fundamental role in this paper.

2.5 Generating functions. By definition, set

$$g_k(x;\theta) := Q_{(k)}(x;\theta)\,, \quad g^*_k(x;\theta) := Q^*_{(k)}(x;\theta)\,, \quad k = 0,1,2,\dots\,.$$

From the general formulas (2.3-4) we have

$$\begin{aligned} g_k(x;\theta) &= \sum_{1\le i_1\le\dots\le i_k} \frac{(\theta)_{m_1}(\theta)_{m_2}\cdots}{m_1!\,m_2!\,\cdots}\, x_{i_1}x_{i_2}\cdots x_{i_k}\,, \\ g^*_k(x;\theta) &= \sum_{1\le i_1\le\dots\le i_k} \frac{(\theta)_{m_1}(\theta)_{m_2}\cdots}{m_1!\,m_2!\,\cdots}\,(x_{i_1}-k+1)\cdots(x_{i_{k-1}}-1)x_{i_k}\,, \end{aligned} \tag{2.8}$$

where $m_l := \#\{r\,|\,i_r = l\}$ stand for the multiplicities with which the numbers $l = 1,2\dots$ occur in $i_1,\dots,i_k$.

Consider the following generating functions:

$$G(x;t) := \sum_{k\ge 0} g_k(x)\,t^k\,, \quad G^*(x;u) := \sum_{k\ge 0} \frac{g^*_k(x)}{u(u-1)\cdots(u-k+1)}\,.$$

Observe that the last series is a well defined element of $\Lambda^\theta[[u^{-1}]]$ and

$$G(x;t) = \lim_{a\to\infty} G^*(ax;a/t)\,.$$

It follows from the Cauchy identity (2.2) that

$$G(x;t) = \prod_i (1-tx_i)^{-\theta}\,. \tag{2.9}$$

The interpolation analog of (2.9) is the following evaluation

$$G^*(x;u) = \prod_i \frac{\Gamma(x_i-u-\theta\, i)}{\Gamma(x_i-u-\theta\, i+\theta)}\,\frac{\Gamma(-u-\theta\, i+\theta)}{\Gamma(-u-\theta\, i)}\,. \tag{2.10}$$

Originally, we obtained the formula (2.10) as an auxiliary statement in the proof of the formula (2.8) for the polynomials $g^*_k(x;\theta)$ (unpublished). That proof of (2.10) was based on iterated use of the Gauss summation formula; it is reproduced in the Appendix to this paper (see Section 7). Alternatively, the formula (2.10) can be obtained by taking the limit $q\to 1$ in its q-analog proved in [Ok9] or by using the very same argument as used in the proof of the generating functions (12.3) in [OO] and (2.9) in [Ok9]. Also, it can be regarded as a particular case of the binomial formula for P^*-functions, see [Ok9].

2.6 Partitions and signatures. We call any non-increasing sequence of integers

$$\lambda = (\lambda_1 \ge \dots \ge \lambda_n)\,, \quad n = 1,2,\dots$$

a *signature*. The number n is called the *length* of the signature λ. Using the relation

$$P_{(\lambda_1+1,\dots,\lambda_n+1)}(x_1,\dots,x_n;\theta)=\left(\prod_{i=1}^n x_i\right)P_\lambda(x;\theta)$$

one can define rational Jack functions $P_\lambda(x;\theta)$ for any signature λ. These functions enjoy the same branching rules and binomial expansions as the Jack polynomials. It is important to notice that the definition of the rational Jack polynomials makes sense only for finitely many variables (namely, for n variables where n is the length of the signature λ).

To each signature λ one can associate two partitions λ^+ and λ^- as follows. Suppose $\lambda_p\ge 0\ge\lambda_{p+1}$ for some $p=1,\dots,n$. Then, by definition,

$$\begin{aligned}\lambda^+&=(\lambda_1,\dots,\lambda_p)\,,\\ \lambda^-&=(-\lambda_n,\dots,-\lambda_{p+1})\,.\end{aligned}$$

This decomposition has the following property

$$G^*(\lambda;u)=G^*(\lambda^+;u)\,G^*(\lambda^-;-u-\theta\,n-1)\,, \tag{2.11}$$

which follows from (2.10) and the following elementary identity

$$\begin{aligned}&\frac{\Gamma(x-u-\theta)}{\Gamma(x-u)}\,\frac{\Gamma(-u)}{\Gamma(-u-\theta)}=\frac{(x-u)\cdots(-1-u)}{(x-u-\theta)\cdots(-1-u-\theta)}=\\ &=\frac{(u+1)\cdots(u-x)}{(u+\theta+1)\cdots(u+\theta-x)}=\frac{\Gamma(-x+u+1)}{\Gamma(-x+u+\theta+1)}\,\frac{\Gamma(u+\theta+1)}{\Gamma(u+1)}\,,\qquad x\in\mathbb{Z}_{\le 0}\,.\end{aligned}$$

A similar argument works for $x\in\mathbb{Z}_{\ge0}$. Observe, however, that the above identity is false if $x\notin\mathbb{Z}$.

2.7 Extended symmetric functions. Let f be an element of Λ. Recall that we have fixed a positive real parameter $\theta>0$.

Given two sequence of variables $\alpha=(\alpha_1,\alpha_2,\dots)$, $\beta=(\beta_1,\beta_2,\dots)$ and one extra variable γ we define (cf. [K]) the *extended* symmetric function $f(\alpha;\beta;\gamma;\theta)$ as the result of specialization

$$\Lambda\to\mathbb{C}[\alpha,\beta,\gamma]$$

given by

$$\begin{aligned}p_1&\mapsto\sum\alpha_i+\sum\beta_i+\gamma\,,\\ p_k&\mapsto\sum\alpha_i^k+(-\theta)^{k-1}\sum\beta_i^k\,,\qquad k\ge2\,.\end{aligned}$$

Equivalently, this specialization can be described by its action on the generating series $G(t)$:

$$G(t)\mapsto e^{\gamma t\theta}\prod_i\frac{(1+t\theta\,\beta_i)}{(1-t\alpha_i)^\theta}\,.$$

We shall use this definition in the situation when $\alpha_i, \beta_i, \gamma \in \mathbb{R}$ and the series

$$\sum \alpha_i, \ \sum \beta_i < \infty$$

converge absolutely. In the sequel we write simply $f(\alpha; \beta; \gamma)$ instead of $f(\alpha; \beta; \gamma; \theta)$

More generally, given 4 sequences $\alpha^+, \alpha^-, \beta^+, \beta^-$ of variables and two extra variables γ^+, γ^- we define the *doubly extended* symmetric function

$$f\begin{pmatrix} \alpha^+; \beta^+; \gamma^+ \\ \alpha^-; \beta^-; \gamma^- \end{pmatrix}$$

as the result of specialization defined by

$$G(t) \mapsto e^{\gamma^+ \theta t + \gamma^- \theta t'} \prod_i \frac{(1 + t\theta\beta_i^+)}{(1 - t\alpha_i^+)^\theta} \frac{(1 + t'\theta\beta_i^-)}{(1 - t'\alpha_i^-)^\theta}, \tag{2.12}$$

where $t' := -t/(1 + \theta t)$, or, equivalently

$$(1 + \theta t)(1 + \theta t') = 1 .$$

3. Asymptotic properties of Vershik-Kerov sequences of signatures

In this section we prove the following

Theorem 3.1. *Suppose $\lambda(n)$ is a VK sequence of signatures with parameters $\alpha_i^\pm$, $\beta_i^\pm$, and $\gamma^\pm$. Suppose $f^* \in \Lambda^\theta$ and let $f \in \Lambda$ be the highest degree term of f^*. Then*

$$\lim_{n\to\infty} \frac{f^*(\lambda(n))}{n^{\deg f^*}} = f\begin{pmatrix} \alpha^+; \beta^+; \gamma^+ \\ \alpha^-; \beta^-; \gamma^- \end{pmatrix} . \tag{3.1}$$

In particular, the limit depends on the highest degree term f of f^* only.

Proof. In the particular case when $\lambda(n)$ is, in fact, a sequence of partitions this theorem was established in [KOO]. Namely, it follows from the estimate stated in Theorem 7.1 of [KOO] and Lemma 5.2 of [KOO]. Notice that in [KOO] a different convention about the letter n was used: n stood there for the number of squares $|\lambda(n)|$ in the partition $\lambda(n)$. Here we have weaker assumptions, namely

$$\ell(\lambda(n)) \le n , \quad \text{and} \quad \exists \lim_{n\to\infty} \frac{|\lambda(n)|}{n} .$$

It is easy to see that this difference in assumptions is unessential.

We shall deduce the general statement (3.1) from that particular case. Observe that it suffices to prove (3.1) for

$$f^* = g_k^* , \qquad f = g_k , \quad k = 1, 2, \dots ,$$

because the functions $\{g_k\}$ are homogeneous generators of the algebra Λ. To simplify notation, set

$$g(k, n) = g_k^*(\lambda(n)) , \quad g^\pm(k, n) = g_k^*(\lambda^\pm(n)) .$$

Since $\lambda^{\pm}(n)$ are two VK sequences of partitions, we conclude that the following limits exist:

$$g^{+}(k) := \lim_{n\to\infty} \frac{g^{+}(k,n)}{n^k} = g_k(\alpha^{+};\beta^{+};\gamma^{+}),$$
$$g^{-}(k) := \lim_{n\to\infty} \frac{g^{-}(k,n)}{n^k} = g_k(\alpha^{-};\beta^{-};\gamma^{-})$$

and satisfy the equality

$$\sum_k g^{\pm}(k)\, t^k = e^{\gamma^{\pm}\theta t} \prod_i \frac{(1+t\theta\beta_i^{\pm})}{(1-t\alpha_i^{\pm})^{\theta}}\,. \tag{3.2}$$

The identity (2.11) can be rewritten as

$$\sum_{k\geq 0} \frac{g(k,n)}{u(u-1)\cdots(u-k+1)} = \left(\sum_{p\geq 0} \frac{g^{+}(p,n)}{u(u-1)\cdots(u-p+1)}\right)\left(\sum_{q\geq 0} \frac{g^{-}(q,n)}{u'(u'-1)\cdots(u'-q+1)}\right), \tag{3.3}$$

where

$$u' := -u - \theta\, n - 1\,.$$

We introduce a new variable v

$$u = nv\,.$$

Then (3.3) becomes

$$\sum_{k\geq 0} \frac{g(k,n)}{n^k} \frac{1}{v(v-\frac{1}{n})\cdots(v-\frac{k-1}{n})} = \left(\sum_{p\geq 0} \frac{g^{+}(p,n)}{n^p} \frac{1}{v(v-\frac{1}{n})\cdots(v-\frac{p-1}{n})}\right)\left(\sum_{q\geq 0} \frac{g^{-}(q,n)}{n^q} \frac{1}{(v'-\frac{1}{n})\cdots(v'-\frac{q}{n})}\right), \tag{3.4}$$

where

$$v' := -v - \theta\,.$$

Recall that we consider (3.4) as an identity in the formal power series algebra $\mathbb{C}[[1/v]]$. We equip $\mathbb{C}[[1/v]]$ with the topology of the coefficient-wise convergence. Clearly, in this topology

$$\frac{1}{v-a_n} \to \frac{1}{v-a_\infty}, \quad n\to\infty\,,$$

provided $a_n \to \alpha_\infty$. Therefore the RHS of (3.4) converges to the following element of $\mathbb{C}[[1/v]]$

$$\left(\sum_{p\geq 0} \frac{g^{+}(p)}{v^p}\right)\left(\sum_{q\geq 0} \frac{g^{-}(q)}{(-v-\theta)^q}\right).$$

Comparing the like powers of $1/v$ in (3.4) we conclude by induction on k that the following limits exist

$$g(k) := \lim_{n\to\infty} \frac{g(k,n)}{n^k}$$

and satisfy the equality

$$\sum_k \frac{g(k)}{v^k} = \left(\sum_{p\geq 0} \frac{g^+(p)}{v^p}\right)\left(\sum_{q\geq 0} \frac{g^-(q)}{(-v-\theta)^q}\right).$$

Set $t = 1/v$. Then we obtain from (3.2)

$$\sum_k g(k)\, t^k = e^{\gamma^+\theta t + \gamma^-\theta t'} \prod_i \frac{(1+t\theta\beta_i^+)}{(1-t\alpha_i^+)^\theta} \frac{(1+t'\theta\beta_i^-)}{(1-t'\alpha_i^-)^\theta},$$

where

$$t' = -\frac{1}{v+\theta} = -\frac{t}{1+\theta t}.$$

This concludes the proof of the theorem. $\square$

4. Sufficient conditions of regularity

In this section we prove the following

Theorem 4.1. *Suppose $\lambda(n)$ is a VK sequence of signatures with parameters $\alpha_i^\pm$, $\beta_i^\pm$, and $\gamma^\pm$. Then $\lambda(n)$ is regular and infinitesimally regular. Set*

$$\Phi_{\alpha,\beta,\gamma}(z_1, z_2, \dots) := \phi_{\alpha,\beta,\gamma}(z_1)\phi_{\alpha,\beta,\gamma}(z_2)\dots,$$

where the function $\phi_{\alpha,\beta,\gamma}(z)$ is the following product

$$\phi_{\alpha,\beta,\gamma}(z) := e^{\gamma^+(z-1)+\gamma^-(z^{-1}-1)} \prod_i \frac{(1+\beta_i^+(z-1))}{(1-\alpha_i^+(z-1)/\theta)^\theta} \frac{(1+\beta_i^-(z^{-1}-1))}{(1-\alpha_i^-(z^{-1}-1)/\theta)^\theta}.$$

Then

$$\Phi_{\lambda(n)}(z_1, \dots, z_k, 1, \dots; \theta) \to \Phi_{\alpha,\beta,\gamma}(z_1, \dots, z_k, 1, \dots)$$

uniformly on each torus $\mathbb{T}^k$, $k = 1, 2, \dots$.

We shall deduce the above theorem from Theorem 3.1 of the previous section and the following

Lemma 4.2. *Fix some $k = 1, 2, \dots$ and suppose that $\psi_1, \psi_2, \dots, \psi_\infty$ are smooth functions on the torus $\mathbb{T}^k$ such that*

(a) *$\psi_n(1, \dots, 1) = 1$, $\quad n = 1, 2, \dots, \infty$;*
(b) *the functions ψ_n are positive definite;*
(c) *ψ_∞ is a real-analytic function on $\mathbb{T}^k$;*
(d) *the Taylor expansions of ψ_n about the point $(1, \dots, 1) \in \mathbb{T}^k$ converge coefficient-wise to the Taylor expansion of ψ_∞;*

then $\psi_n \to \psi_\infty$ *uniformly on* $\mathbb{T}^k$.

Proof of Theorem 4.1. We have to verify that the functions

$$\psi_n(z_1,\dots,z_k) := \Phi_{\lambda(n)}(z_1,\dots,z_k,\underbrace{1,\dots,1}_{n-k \text{ times}};\theta),$$
$$\psi_\infty(z_1,\dots,z_k) := \Phi_{\alpha,\beta,\gamma}(z_1,\dots,z_k,1,\dots)$$

satisfy all conditions of Lemma 4.2. Obviously, ψ_n are smooth and $\psi_n(1,\dots,1)=1$ for all n.

Since the coefficients $\psi_{\lambda/\mu}$ in the branching rules (2.3) are nonnegative, it follows that ψ_n are polynomials in $z_i^{\pm}$ with nonnegative coefficients and hence positive definite functions.

Since the functions ψ_n are symmetric and the Jack polynomials form a linear basis of Λ, we can rewrite the Taylor expansion of ψ_n as an expansion in polynomials

$$P_\mu(t_1,\dots,t_k;\theta), \quad t_i = z_i - 1, \quad \ell(\mu)\le k.$$

This expansion is given by the formula (2.7), which in our current notation reads

$$\psi_n(z_1,\dots,z_k) = \sum_{\ell(\mu)\le k} \frac{Q^*_\mu(\lambda(n))}{(\theta\, n)_\mu} P_\mu(t_1,\dots,t_k;\theta). \tag{4.1}$$

It is clear that

$$\frac{Q^*_\mu(\lambda(n))}{(\theta\, n)_\mu} \sim \theta^{-|\mu|}\,\frac{Q^*_\mu(\lambda(n))}{n^{|\mu|}}, \quad n\to\infty.$$

By assumption the sequence $\lambda(n)$ is VK; therefore, by (2.5) and Theorem 3.1 we have

$$\frac{Q^*_\mu(\lambda(n))}{(\theta\, n)_\mu} \to \theta^{-|\mu|}\, Q_\mu\begin{pmatrix}\alpha^+;\beta^+;\gamma^+\\ \alpha^-;\beta^-;\gamma^-\end{pmatrix}, \quad n\to\infty.$$

Now it suffices to verify that

$$\psi_\infty(z_1,\dots,z_k) = \sum_{\ell(\mu)\le k} \theta^{-|\mu|}\, Q_\mu\begin{pmatrix}\alpha^+;\beta^+;\gamma^+\\ \alpha^-;\beta^-;\gamma^-\end{pmatrix} P_\mu(t_1,\dots,t_k;\theta).$$

But this follows from the Cauchy identity (2.2) which implies

$$\sum_{\ell(\mu)\le k} \theta^{-|\mu|}\, Q_\mu(x_1,x_2,\dots)P_\mu(t_1,\dots,t_k;\theta) =$$
$$= \prod_{i=1}^{\infty}\prod_{j=1}^{k} \frac{1}{(1-x_i t_j/\theta)^\theta} = \prod_{j=1}^{k} G(x_1,x_2,\dots;t_j/\theta)$$

and from the definition (2.12) of the specialization

$$Q_\mu\begin{pmatrix}\alpha^+;\beta^+;\gamma^+\\ \alpha^-;\beta^-;\gamma^-\end{pmatrix}.$$

This concludes the proof of the theorem. □

We have the following corollary of the above proof:

Corollary 4.3. *We have the following equivalence between two conditions listed in Theorem 1.2*

$$\text{(ii)} \iff \text{(iv)}\,.$$

Proof. Follows from (4.1) and the fact that the polynomials Q_μ form a homogeneous linear basis of the algebra Λ. □

Now it remains to establish Lemma 4.2. The proof of the lemma is quite standard. To avoid multiindices we assume $k=1$; the argument in the general case is the same.

Proof of Lemma 4.2. By (a) and (b) we have[4]

$$\psi_n(e^{2\pi i s}) = \int e^{2\pi i s \xi}\, M_n(d\xi)\,, \quad s \in \mathbb{R}/\mathbb{Z}\,, \quad \xi \in \mathbb{Z}\,,$$

where M_n is a certain probability measure supported on $\mathbb{Z} = \mathbb{T}^\wedge$. By assumption, all functions ψ_n are smooth and hence all moments of all measures M_n are finite. By (d) the limits

$$m_l := \lim \int \xi^l\, M_n(d\xi)\,, \quad n \to \infty\,,$$

exist. In particular, we have certain upper bounds

$$\int \xi^l\, M_n(d\xi) \le c(l)$$

uniformly in n. The following Chebyshev-type uniform tail estimates

$$\int_{|\xi|>a} \xi^{2l}\, M_n(d\xi) \le \int_{|\xi|>a} \frac{\xi^{2l+2}}{a^2}\, M_n(d\xi) \le \frac{c(2l+2)}{a^2}\,,$$

where a is arbitrary positive, imply that the family $\{M_n\}$ is tight (and hence relatively compact in the weak convergence topology) and, moreover,

$$\int \xi^l\, M_\infty(d\xi) = m_l$$

for any limit point M_∞ of the set $\{M_n\}$. Note that M_∞ is a probability measure.

By (c) the series

$$\psi_\infty(e^{2\pi i s}) = \sum_{l\ge 0} m_l \frac{(2\pi i s)^l}{l!}$$

converges in some neighborhood of $s=0$. Hence (see e.g. Theorem II.12.7 in [Sh]) the moments $\{m_l\}$ determine the measure M_∞ uniquely and (see the proof of the

[4] Here and below it will be convenient for us to write sums as integrals with respect to a discrete measure. Most of our computations (such as Lemma 5.2 below) apply to arbitrary measures on $\mathbb{R}$.

aforementioned theorem) the Fourier transform of M_∞ is real-analytic and hence equals ψ_∞ everywhere.

It follows that the set $\{M_n\}$ has a unique limit point M_∞, whence

$$M_n \xrightarrow{\text{weak}} M_\infty\,, \quad n \to \infty$$

and hence

$$\psi_n \to \int e^{2\pi i s\xi}\, M_\infty(d\xi) = \psi_\infty$$

uniformly on compact sets. Since $\mathbb{T}$ is compact, the lemma follows. □

5. Necessary conditions of regularity

5.1 The "only if" part of Theorem 1.1.

Given a vector $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbb{R}^n$ set, by definition,

$$\mathfrak{N}(\lambda)^2 := \sum \lambda_i^2 + \theta \left(\sum \lambda_i\right)^2 + \theta(\lambda, 2\rho)\,,$$

where $2\rho := (n-1, n-3, \ldots, 3-n, 1-n)$ and hence

$$(\lambda, 2\rho) = \sum_{i<j} (\lambda_i - \lambda_j) \geq 0\,.$$

The function $\mathfrak{N}(\lambda)$ will play the role of a "norm" of λ.

The main theorem which we shall prove in this section is the following

Theorem 5.1. *If $\lambda(n)$ is a sequence of signatures and the following limit*

$$\lim_{n\to\infty} \Phi_{\lambda(n)}(z, \underbrace{1, \ldots, 1}_{n-1 \text{ times}})\,, \quad z \in \mathbb{T}\,, \tag{5.1}$$

exists point-wise and is continuous at $z = 1$ then the sequence $\lambda(n)$ is a Vershik-Kerov sequence.

In particular, the hypothesis of Theorem 5.1 is clearly satisfied if the sequence $\lambda(n)$ is regular. Therefore, the combination of this theorem with Theorem 4.1 implies Theorem 1.1.

We shall prove Theorem 5.1 in three steps: we shall prove that its hypothesis implies that

(1) $\mathfrak{N}(\lambda(n)) = O(n)\,, \quad n \to \infty$,
(2) $|\lambda^\pm(n)| = O(n)\,, \quad n \to \infty$, and, finally,
(3) $\lambda(n)$ is a Vershik-Kerov sequence.

To justify the first step we shall need the following

Lemma 5.2. *Let $\{M_n\}$ be a tight family of probability measures on $\mathbb{R}$ with finite 4-th moments*

$$\int x^4\, M_n(dx) < \infty\,,$$

and suppose that

$$\int x^2\, M_n(dx) \to \infty\,.$$

Then the following ratio grows to infinity

$$\int x^4\, M_n(dx) \Big/ \left(\int x^2\, M_n(dx)\right)^2 \to \infty\,.$$

Proof. Fix some $\varepsilon > 0$ and show that

$$4\varepsilon \int x^4\, M_n(dx) \geq \left(\int x^2\, M_n(dx)\right)^2\,, \tag{5.2}$$

provided n is sufficiently large. Since $\{M_n\}$ is tight we can find (and fix) some $a > 0$ such that

$$\int_{|x|\geq a} M_n(dx) \leq \varepsilon$$

for all n. By Cauchy inequality we have

$$\left(\int_{|x|\geq a} x^2\, M_n(dx)\right)^2 \leq \varepsilon \int_{|x|\geq a} x^4\, M_n(dx) \leq \varepsilon \int x^4\, M_n(dx)\,. \tag{5.3}$$

On the other hand

$$\int x^2\, M_n(dx) \leq a^2 + \int_{|x|\geq a} x^2\, M_n(dx)$$

and since $\int x^2\, M_n(dx) \to \infty$ we have

$$\int_{|x|\geq a} x^2\, M_n(dx) \geq \frac{1}{2} \int x^2\, M_n(dx) \tag{5.4}$$

for all sufficiently large n. Substituting (5.4) into (5.3) we obtain (5.2). $\square$

Now we begin with the proof of the Theorem 5.1.

Proof of Theorem 5.1.
Step 1. Our goal now is to prove that

$$\mathfrak{N}(\lambda(n)) = O(n)\,, \quad n \to \infty\,. \tag{5.5}$$

We set

$$\phi_n(z) := \Phi_{\lambda(n)}(z, \underbrace{1, \dots, 1}_{n-1 \text{ times}})$$

and denote by ϕ_∞ the limit function (5.1).

As in the previous section we shall represent the function $\phi_n(z)$ as the Fourier transform of some probability measure M_n on $\mathbb{Z}$

$$\phi_n(z) = \int z^\xi M_n(d\xi)\,, \quad z \in \mathbb{T}^1\,, \quad \xi \in \mathbb{Z}\,.$$

By the continuity property of characteristic functions (see, for example, [Sh], Theorem III.3.1) we have

$$M_n \xrightarrow{\text{weak}} M_\infty\,,$$

where

$$\phi_\infty(z) = \int z^\xi M_\infty(d\xi)\,.$$

To simplify notation we shall write λ in place of $\lambda(n)$. The binomial expansion

$$\phi_n(z) = \sum_{k\ge 0} \frac{g_k^*(\lambda)}{(n\theta)_k}\,(z-1)^k$$

implies that

$$\int \xi(\xi-1)\dots(\xi-k+1)\,M_n(d\xi) = k!\,\frac{g_k^*(\lambda)}{(n\theta)_k}\,,$$

whence the k-th moment of M_n is a linear combination of the numbers

$$\frac{g_1^*(\lambda)}{(n\theta)_1}, \quad \dots \quad , \frac{g_k^*(\lambda)}{(n\theta)_k}\,.$$

For example, we have

$$\int \xi^2\, M_n(d\xi) = \frac{g_1^*(\lambda)}{n\theta} + 2\frac{g_2^*(\lambda)}{n\theta(n\theta+1)}\,.$$

The formula (2.8) specializes to

$$\begin{aligned} g_1^*(\lambda) &= \lambda_1 + \dots + \lambda_n\,, \\ g_2^*(\lambda) &= \theta^2 \sum_{i<j} (\lambda_i - 1)\lambda_j + \frac{\theta(1+\theta)}{2} \sum_i (\lambda_i - 1)\lambda_i\,. \end{aligned}$$

After some simple algebra, we obtain

$$\int \xi^2\, M_n(d\xi) = \frac{\mathfrak{N}(\lambda)^2}{n(n\theta+1)}\,.$$

Therefore, to complete the first step of the proof we have to show that the 2-nd moments of M_n remain bounded as $n \to \infty$

$$\int \xi^2\, M_n(d\xi) = O(1)\,, \quad n \to \infty\,. \tag{5.6}$$

Below in Theorem 5.4 we shall prove a general growth estimate on functions $|f(\lambda)|$, where $f \in \Lambda^\theta$ is arbitrary. It implies, in particular, that

$$\frac{g_k^*(\lambda)}{n^k} = O\left(1 + \frac{\mathfrak{N}(\lambda)^k}{n^k}\right).$$

This yields that

$$\int \xi^k \, M_n(d\xi) = O\left(1 + \frac{\mathfrak{N}(\lambda)^k}{n^k}\right),$$

which together with Lemma 5.2 implies (5.6) and, hence, (5.5).

Step 2. Our goal now is to prove that

$$\big(\mathfrak{N}(\lambda(n)) = O(n)\big) \Longrightarrow \big(|\lambda^{\pm}(n)| = O(n)\big).$$

Again, to simplify notation we shall write simply λ in place of $\lambda(n)$. By definition of $\mathfrak{N}(\lambda)$ we have

$$\sum \lambda_i = O(n), \tag{5.7}$$

$$\sum_{i<j} (\lambda_i - \lambda_j) = O(n^2). \tag{5.8}$$

We can find two integers $p = p(n)$ and $q = q(n)$ such that $p + q = n$, $\ell(\lambda^+) \le p$, and $\ell(\lambda^-) \le q$. Then we have

$$\begin{aligned} \sum_{i<j} (\lambda_i - \lambda_j) \ge \sum_{i \le p < j} (\lambda_i - \lambda_j) &= \\ = q|\lambda^+| + p|\lambda^-| &= \frac{n}{2}(|\lambda^+| + |\lambda^-|) + \frac{q-p}{2} \sum \lambda_i, \end{aligned} \tag{5.9}$$

where we used the equalities $\sum \lambda_i = |\lambda^+| - |\lambda^-|$ and $p + q = n$. Since $q - p = O(n)$ we obtain from (5.9) and (5.7-8)

$$n(|\lambda^+| + |\lambda^-|) = O(n^2),$$

which is equivalent to $|\lambda^{\pm}(n)| = O(n)$.

Step 3. Now we finish the proof of the theorem. The standard compactness argument yields that any sequence $\{\lambda(n)\}$ of signatures such that $|\lambda^{\pm}(n)| = O(n)$ contains a VK subsequence. Therefore, in order to show that our sequence $\{\lambda(n)\}$ is VK it suffices to show that if

$$\{\dot\lambda(n)\}, \{\ddot\lambda(n)\} \subset \{\lambda(n)\} \tag{5.10}$$

are two VK subsequences then the corresponding VK parameters are equal:

$$(\dot\alpha^+, \dot\alpha^-, \dot\beta^+, \dot\beta^-, \dot\gamma^+, \dot\gamma^-) = (\ddot\alpha^+, \ddot\alpha^-, \ddot\beta^+, \ddot\beta^-, \ddot\gamma^+, \ddot\gamma^-). \tag{5.11}$$

Take two arbitrary VK subsequences (5.10). By Theorem 4.1 the corresponding subsequences $\{\dot\phi_n\}$ and $\{\ddot\phi_n\}$ converge uniformly and to the same limit:

$$e^{\dot\gamma^+(z-1)+\dot\gamma^-(z^{-1}-1)}\prod_i \frac{(1+\dot\beta_i^+(z-1))}{(1-\dot\alpha_i^+(z-1)/\theta)^\theta}\frac{(1+\dot\beta_i^-(z^{-1}-1))}{(1-\dot\alpha_i^-(z^{-1}-1)/\theta)^\theta} = \tag{5.12}$$
$$= e^{\ddot\gamma^+(z-1)+\ddot\gamma^-(z^{-1}-1)}\prod_i \frac{(1+\ddot\beta_i^+(z-1))}{(1-\ddot\alpha_i^+(z-1)/\theta)^\theta}\frac{(1+\ddot\beta_i^-(z^{-1}-1))}{(1-\ddot\alpha_i^-(z^{-1}-1)/\theta)^\theta}\,.$$

Consider the singularities of the two equal analytic functions in (5.12). The LHS is regular in the annulus

$$\frac{\dot\alpha_1^-}{\theta+\dot\alpha_1^-}<|z|<\frac{\theta+\dot\alpha_1^+}{\dot\alpha_1^+}$$

and becomes singular as $z\to\frac{\dot\alpha_1^-}{\theta+\dot\alpha_1^-},\frac{\theta+\dot\alpha_1^+}{\dot\alpha_1^+}$. It follows that

$$\dot\alpha_1^+=\ddot\alpha_1^+\,,\quad \dot\alpha_1^-=\ddot\alpha_1^-\,.$$

Thus, both sides of (5.12) have common factors which can be cancelled. Iterating the argument we obtain

$$\dot\alpha^+=\ddot\alpha^+\,,\quad \dot\alpha^-=\ddot\alpha^-\,.$$

Next, set

$$f(x):=1-\frac{1}{x}\,.$$

This function is increasing for $x>0$. Consider the zeros of (5.12). This yields the equality of the the following multisets (recall that $\beta_1^++\beta_1^-\le 1$)

$$\Big\{\dots\ge f(1-\dot\beta_2^-)\ge f(1-\dot\beta_1^-)\ge f(\dot\beta_1^+)\ge f(\dot\beta_2^+)\ge\dots\Big\}_{\dot\beta_i^\pm\ne 0}=$$
$$\Big\{\dots\ge f(1-\ddot\beta_2^-)\ge f(1-\ddot\beta_1^-)\ge f(\ddot\beta_1^+)\ge f(\ddot\beta_2^+)\ge\dots\Big\}_{\ddot\beta_i^\pm\ne 0}\,,$$

whence

$$\Big\{\dots\ge 1-\dot\beta_2^-\ge 1-\dot\beta_1^-\ge\dot\beta_1^+\ge\dot\beta_2^+\ge\dots\Big\}= \tag{5.13}$$
$$\Big\{\dots\ge 1-\ddot\beta_2^-\ge 1-\ddot\beta_1^-\ge\ddot\beta_1^+\ge\ddot\beta_2^+\ge\dots\Big\}\,,$$

or, in words, the two sequences in (5.13) coincide up to a possible shift of indices. Let $r\in\mathbb{Z}$ be the shift of indices in (5.13). Then

$$e^{\dot\gamma^+(z-1)+\dot\gamma^-(z^{-1}-1)}=z^r e^{\ddot\gamma^+(z-1)+\ddot\gamma^-(z^{-1}-1)}\,.$$

It follows that $\dot\gamma^\pm=\ddot\gamma^\pm$ and $r=0$. Hence

$$\dot\beta^+=\ddot\beta^+\,,\quad \dot\beta^-=\ddot\beta^-\,.$$

This establishes (5.11) and, thus, concludes the proof of the theorem (assuming the truth of Theorem 5.4 to be established in the next section). $\square$

We have the following corollary of the above proof:

Corollary 5.3. *We have the following implication between two conditions listed in Theorem 1.2*

$$(\mathrm{ii}) \Longrightarrow (\mathrm{iii})\,.$$

Proof. Argue as in the above theorem but without using Lemma 5.2 and Theorem 5.4. □

Now we have accumulated enough knowledge to prove Theorems 1.1 and 1.2

Proof of Theorem 1.1. Follows immediately from Theorems 4.1 and 5.1. □

Proof of Theorem 1.2. We merely list all already established implications

$$\begin{aligned} (\mathrm{iii}) &\Longrightarrow (\mathrm{i}), (\mathrm{ii})\,, && \text{by Theorem 4.1}\,,\\ (\mathrm{ii}) &\Longleftrightarrow (\mathrm{iv})\,, && \text{by Corollary 4.3}\,,\\ (\mathrm{i}) &\Longrightarrow (\mathrm{iii})\,, && \text{by Theorem 5.1}\,,\\ (\mathrm{ii}) &\Longrightarrow (\mathrm{iii})\,, && \text{by Corollary 5.3}\,, \end{aligned}$$

which prove that

$$(\mathrm{i}) \Longleftrightarrow (\mathrm{ii}) \Longleftrightarrow (\mathrm{iii}) \Longleftrightarrow (\mathrm{iv})\,.$$

Now the hypothesis of Theorem 5.1 is clearly weaker then (i). But by Theorems 5.1 and 4.1 it also implies (i). This concludes the proof. □

5.2 A growth estimate for $|f(\lambda)|$, $f \in \Lambda^\theta$.

In this subsection we shall prove the following estimate which we used in the proof of Theorem 5.1 (namely, in Step 2)

Theorem 5.4. *For any $f \in \Lambda^\theta$ there exists a constant C_f such that*

$$|f(\lambda)| \le C_f \max(\mathfrak{N}(\lambda), n)^{\deg f} \tag{5.14}$$

for all $n = 1, 2, \dots$ and all $\lambda = (\lambda_1 \ge \dots \ge \lambda_n) \in \mathbb{R}^n$.

Although rough, this estimate is sufficient for our purposes. First, we establish the following

Lemma 5.5. *Suppose $\lambda = (\lambda_1 \ge \dots \ge \lambda_n) \in \mathbb{R}^n$ then*

$$\left| \sum_{i=1}^{n} \lambda_i (i-1)^m \right| \le \left| \sum \lambda_i \right| n^m + 2(\lambda, 2\rho)\, n^{m-1} \tag{5.15}$$

for all $m = 0, 1, 2, \dots$.

Proof of Lemma 5.5. Denote the linear function to be estimated by

$$l_m(\lambda) := \sum \lambda_i (i-1)^m\,.$$

It is easy to see that using affine transformations

$$\lambda_i \mapsto a\lambda_i + b\,, \quad i = 1, \dots, n\,,$$

where $a, b \in \mathbb{R}$ and $a \neq 0$, one reduces the estimate (5.15) to the following estimate

$$\begin{pmatrix} \lambda_1 \geq \cdots \geq \lambda_n\,, \\ \sum \lambda_i = 0\,, \\ (\lambda, 2\rho) = n \end{pmatrix} \Longrightarrow |l_m(\lambda)| \leq 2n^m\,, \tag{5.16}$$

which shall now be established. The vectors λ satisfying the inequalities in the LHS of (5.16) form an $(n-2)$-dimensional simplex with extreme points

$$\lambda^{(k)} = \Big(\underbrace{\frac{1}{k}, \dots, \frac{1}{k}}_{k \text{ times}}, \underbrace{-\frac{1}{n-k}, \dots, -\frac{1}{n-k}}_{n-k \text{ times}}\Big)\,, \quad k = 1, \dots, n-1\,.$$

We estimate the values $l_m(\lambda^{(k)})$ by comparing sums to integrals and obtain

$$\begin{aligned} \left| l_m(\lambda^{(k)}) \right| &\leq \frac{1}{k} \int_0^k x^m\, dx + \frac{1}{n-k} \int_k^n x^m\, dx = \\ &= \frac{1}{m+1} \left(k^m + \frac{n^{m+1} - k^{m+1}}{n-k} \right) \leq 2n^m \end{aligned}$$

This concludes the proof. $\square$

Now we proceed with the proof of the theorem

Proof of Theorem 5.4. It is clear that it suffices to prove the estimate (5.15) for any set of functions $f_1, f_2, \dots \in \Lambda^\theta$ whose highest degree terms generate the algebra Λ. We take

$$f_m(\lambda) = \sum_i \left[(\lambda_i - \theta(i-1))^m - (-\theta(i-1))^m \right]\,, \quad m = 1, 2, \dots\,.$$

After we expand the binomials $(\lambda_i - \theta(i-1))^m$ we shall have to estimate the sums

$$\left| \sum_{i=1}^n \lambda_i^r (i-1)^{m-r} \right|\,, \quad r = 1, \dots, m\,.$$

For $r = 1$ this was done in Lemma 5.5 which implies

$$\left| \sum_{i=1}^n \lambda_i (i-1)^{m-1} \right| \leq n^{m-1}\, \mathfrak{N}(\lambda) + \frac{2}{\sqrt{\theta}}\, n^{m-2}\, \mathfrak{N}(\lambda)^2\,.$$

For $r \geq 2$ we have

$$\left| \sum_{i=1}^n \lambda_i^r (i-1)^{m-r} \right| \leq n^{m-r} \sum (\lambda_i^2)^{r/2} \leq n^{m-r} \left(\sum \lambda_i^2 \right)^{r/2} \leq n^{m-r}\, \mathfrak{N}(\lambda)^r\,.$$

This concludes the proof. $\square$

6. Proof of Theorem 1.3

In this section we shall give a proof of a general abstract approximation theorem which will imply that

$$\operatorname{Ex}\Upsilon^\theta \subset \left\{\Phi_{\alpha,\beta,\gamma}\right\}_{\alpha^\pm,\beta^\pm,\gamma^\pm}. \tag{6.1}$$

Similar approximation theorems are well-known in the literature (see e.g. [V], [VK1], and [Ol1], Section 22).

The inverse inclusion

$$\operatorname{Ex}\Upsilon^\theta \supset \left\{\Phi_{\alpha,\beta,\gamma}\right\}_{\alpha^\pm,\beta^\pm,\gamma^\pm}$$

follows immediately from the fact that the functions $\Phi_{\alpha,\beta,\gamma}$, being products of the form (1.5), are extreme in a larger convex set, namely, in the set of characteristic functions of $S(\infty)$-invariant measures on

$$\mathbb{Z}^\infty := \varprojlim \mathbb{Z}^n. \tag{6.2}$$

Recall that the description of ergodic $S(\infty)$-invariant measures on infinite Cartesian products like (6.2) is given by the De Finetti theorem, see e.g. [Al].

Now we formulate the abstract setup of the approximation theorem. Let a set A be presented as a disjoint union

$$A = \bigsqcup_{n\ge 1} A_n$$

of countable sets $A_1, A_2, \dots$. This presentation can be also encoded as a function

$$[\,\cdot\,] : A \to \mathbb{Z}_{>0}\,,$$
$$[a] = n \Leftrightarrow a \in A_n\,.$$

Suppose also that we are given a function

$$\omega : \bigsqcup_{n\ge 1} (A_n \times A_{n+1}) \to [0,1]$$

satisfying

$$\sum_{a\in A_n} \omega(a,a') = 1\,, \quad \forall n, \quad \forall a' \in A_{n+1}\,. \tag{6.3}$$

To this data one associates a convex set $\mathfrak{A}$ as follows. Let

$$\mathbb{R}^{A_n} \cong \mathbb{R}\times\mathbb{R}\times\dots$$

denote the vector space of real-valued functions

$$f : A_n \to \mathbb{R}\,.$$

This is a locally convex metrizable vector space with respect to the product topology. The subset

$$\mathbb{R}^{A_n} \supset \mathfrak{A}_n := \{f : A_n \to [0,1],\ \sum_{a\in A_n} f(a) = 1\}$$

is a simplex whose vertices are the delta-functions

$$\delta_a(a') = \begin{cases} 1\,, & a' = a\,, \\ 0\,, & a' \neq a\,, \end{cases} \qquad a, a' \in A_n\,.$$

The function ω defines a projection

$$\mathrm{Proj}_n^{n+1} : \mathfrak{A}_{n+1} \to \mathfrak{A}_n$$

by the following formula

$$\left(\mathrm{Proj}_n^{n+1} f\right)(a) := \sum_{a'\in A_{n+1}} \omega(a,a')f(a')\,, \quad a \in A_n, \quad f \in \mathfrak{A}_{n+1}\,. \tag{6.4}$$

For any $n > k \geq 1$ we denote by Proj_k^n the composition of the projections (6.4). Now we set, by definition

$$\mathfrak{A} := \varprojlim \mathfrak{A}_n$$

with respect to these projections. This is a convex set; any element of $\mathfrak{A}$ is by construction a function on A. Denote by $\mathrm{Ex}\,\mathfrak{A}$ the set of extreme points of $\mathfrak{A}$. We shall now establish the following approximation theorem for functions in $\mathrm{Ex}\,\mathfrak{A}$.

Theorem 6.1. *For any function $f \in \mathrm{Ex}\,\mathfrak{A}$ there exists a sequence*

$$(a_n) \subset A\,, \quad [a_n] \to \infty\,,$$

of elements of the set A such that

$$\lim_{n\to\infty} \mathrm{Proj}_k^{[a_n]} \delta_{a_n} = f\big|_{A_k}\,, \quad \forall k = 1, 2, \dots\,.$$

Remark. In fact, a simple argument shows that we can always choose the sequence $(a_n) \subset A$ in such a way that $[a_n] = n$.

Proof of Theorem 6.1. We shall follow the argument used in proof of Theorem 22.9 in [Ol1] and shall consider certain compactifications of the non-compact (for infinite A_n) sets $\mathfrak{A}_n$. Namely, set

$$\overline{\mathfrak{A}}_n := \{f : A_n \to [0,1],\ \sum_{a\in A_n} f(a) \leq 1\} \subset \mathbb{R}^{A_n}\,.$$

It is clear that $\overline{\mathfrak{A}}_n \supset \mathfrak{A}_n$ and $\overline{\mathfrak{A}}_n$ is convex and compact. The same formula (6.4) gives a map

$$\overline{\mathrm{Proj}}_n^{n+1} : \overline{\mathfrak{A}}_{n+1} \to \overline{\mathfrak{A}}_n\,.$$

Observe that this map is not, in general, continuous. However, it is always lower semicontinuous

$$\overline{\mathrm{Proj}}_n^{n+1}(f_0) \le \varliminf_{f \to f_0} \overline{\mathrm{Proj}}_n^{n+1}(f)\,, \quad f, f_0 \in \overline{\mathfrak{A}}_{n+1}\,. \tag{6.5}$$

Here and below an inequality between two functions means inequality at every point. The semicontinuity (6.5) follows from the fact that $\omega(a,a') \in [0,1]$ for any a and a'. By virtue of (6.5) the convex set

$$\{(f,f'), f \ge \overline{\mathrm{Proj}}_n^{n+1} f'\} \subset \overline{\mathfrak{A}}_n \times \overline{\mathfrak{A}}_{n+1}$$

is compact. Similarly, the convex sets $\mathfrak{B}_n$

$$\mathfrak{B}_n := \{(f_1,\dots,f_n), f_i \ge \overline{\mathrm{Proj}}_i^{i+1} f_{i+1}, 1 \le i < n\} \subset \prod_1^n \overline{\mathfrak{A}}_i$$

are compact and so are the sets

$$\mathfrak{C}_n := \mathfrak{B}_n \times \prod_{n+1}^{\infty} \overline{\mathfrak{A}}_i \subset \prod_1^{\infty} \overline{\mathfrak{A}}_i \subset \mathbb{R}^A .$$

Clearly, we have

$$\mathfrak{C}_1 \supset \mathfrak{C}_2 \supset \dots .$$

Set

$$\mathfrak{C} := \bigcap_{n \ge 1} \mathfrak{C}_n\,.$$

Since all sets $\mathfrak{C}_n$ are compact convex subsets of a locally convex metrizable vector space $\mathbb{R}^A$ of real-valued functions on A it is well known (see e.g. Lemma 22.13 in [Ol1]) that for any $f \in \mathrm{Ex}\,\mathfrak{C}$ there exists a sequence

$$(f_n \in \mathrm{Ex}\,\mathfrak{C}_n)_{n=1,2,\dots}$$

such that

$$f_n \to f\,, \quad n \to \infty\,.$$

One easily obtains (see Sections 22.20–22.23 in [Ol1]) the following properties of the sets $\mathrm{Ex}\,\mathfrak{C}_n$ and $\mathrm{Ex}\,\mathfrak{C}$. We have:

(1) for any point $f \in \mathrm{Ex}\,\mathfrak{C}_n$ its projection onto $\mathfrak{B}_n$ (given by restriction to $\cup_{i \le n} A_i$) lies in $\mathrm{Ex}\,\mathfrak{B}_n$;
(2) a point in $\mathrm{Ex}\,\mathfrak{B}_n$ is either zero or of the form

$$\Delta_a(a') := \begin{cases} \left(\mathrm{Proj}_{[a']}^{[a]} \delta_a\right)(a')\,, & [a'] \le [a]\,, \\ 0\,, & [a'] > [a]\,, \end{cases}$$

where $a \in \mathfrak{A}$ and $[a] \le n$;
(3) the inclusion $\mathfrak{A} \subset \mathfrak{C}$ induces the inclusion

$$\mathrm{Ex}\,\mathfrak{A} \subset \mathrm{Ex}\,\mathfrak{C}\,.$$

Now we can finish the proof of the theorem. For any $f \in \operatorname{Ex}\mathfrak{A} \subset \operatorname{Ex}\mathfrak{C}$ there exists a sequence $(f_n \in \operatorname{Ex}\mathfrak{C}_n)_{n=1,2,\dots}$ such that $f_n \to f$. Consider the projections of f_n onto $\mathfrak{B}_n$. Clearly, only finitely many of them can be zero. The non-zero ones produce a sequence $(a_n) \subset A$ such that

$$\lim_{n\to\infty} \Delta_{a_n} = f\,. \tag{6.6}$$

It is clear that (6.6) implies $[a_n] \to \infty$ and that (a_n) is the desired sequence. This concludes the proof. $\square$

Now we shall specialize this general theorem to our particular situation. We set

$$A_n := \{\text{ signatures } \lambda \text{ of length } \ell(\lambda) = n\ \}\,.$$

If μ and λ are two signatures of length n and $n+1$ respectively we set

$$\begin{aligned}\omega(\mu,\lambda) := \ &\text{coefficient of } \Phi_\mu(z_1,\dots,z_n;\theta)\\ &\qquad\qquad\text{in the expansion of } \Phi_\lambda(z_1,\dots,z_n,1;\theta)\,.\end{aligned}$$

The normalization

$$\Phi_\nu(1,\dots,1;\theta) = 1\,,\quad \forall\nu\,,$$

implies the property (6.3). Now by the definition of $\mathfrak{A}$ and the definitions of Section 1.3 we have

$$\mathfrak{A}_n \cong \Upsilon_n^\theta\,,\quad \text{and}\quad \mathfrak{A} \cong \Upsilon^\theta\,.$$

Moreover, the convergence in $\mathfrak{A}_n$ is equivalent to the uniform convergence of functions in Υ_n^θ. Now Theorems 6.1 and 1.1 imply (6.1) and this concludes the proof of Theorem 1.3.

7. Appendix. A direct proof of the formula (2.10) for generating functions

In this section we give a direct derivation of the formula (2.10) for the generating functions and show how the formula (2.8) follows from this computation. Recall that the formula (2.8) reads

(7.1)

$$g_k^*(x_1,\dots,x_n;\theta) := \sum_{1\le i_1\le\dots\le i_k\le n} \frac{(\theta)_{m_1}\cdots(\theta)_{m_n}}{m_1!\cdots m_n!}(x_{i_1}-k+1)\cdots(x_{i_{k-1}}-1)x_{i_k}\,,$$

where $m_l := \#\{r \mid i_r = l\}$ are the multiplicities with which the numbers $l = 1,\dots,n$ occur in $i_1,\dots,i_k$. Set

$$G_n^*(x_1,\dots,x_n;u) := \sum_{k\ge 0} \frac{g_k^*(x_1,\dots,x_n)}{u(u-1)\cdots(u-k+1)}\,.$$

We shall establish the following

Proposition 7.1. *For fixed $x_1, \dots, x_n$ the series converges provided $\Re u \ll 0$ and*

$$G_n^*(x_1, \dots, x_n; u) = \prod_{i=1}^{n} \frac{\Gamma(x_i - u - \theta\, i)}{\Gamma(x_i - u - \theta\, i + \theta)} \frac{\Gamma(-u - \theta\, i)}{\Gamma(-u - \theta\, i + \theta)}\,. \tag{7.2}$$

Proof. Induct on n. For $n = 1$ we have the Gauss summation (see [WW])

$$\sum_{k \geq 0} \frac{(\theta)_k}{k!} \frac{x(x-1)\cdot(x-k+1)}{u(u-1)\cdot(u-k+1)} = {}_2F_1(-x, \theta; -u; 1) = \frac{\Gamma(x-u-\theta)}{\Gamma(x-u)} \frac{\Gamma(-u)}{\Gamma(-u-\theta)}\,, \tag{7.3}$$

provided $\Re u \ll 0$. Now suppose $n \geq 2$. By (7.1) we have

$$g_k^*(x_1, \dots, x_n) = \sum_{p+q=k} g_p^*(x_1 - q, \dots, x_{n-1} - q)\, g_q^*(x_n)\,.$$

This implies

$$G_n^*(x_1, \dots, x_n; u) = \sum_{q \geq 0} G_{n-1}^*(x_1 - q, \dots, x_{n-1} - q; u - q) \frac{g_q^*(x_n)}{u(u-1)\cdots(u-q+1)}\,. \tag{7.4}$$

Since we assume the truth of (7.2) for G_{n-1}^* we conclude that

$$\begin{aligned} \frac{G_{n-1}^*(x_1 - q, \dots, x_{n-1} - q; u - q)}{G_{n-1}^*(x_1, \dots, x_{n-1}; u)} &= \prod_{i=1}^{n-1} \frac{\Gamma(q - u - \theta\, i + \theta)}{\Gamma(q - u - \theta\, i)} \frac{\Gamma(-u - \theta\, i)}{\Gamma(-u - \theta\, i + \theta)} \\ &= \prod_{i=1}^{n-1} \frac{(-u - \theta\, i + \theta)_q}{(-u - \theta\, i)_q} \\ &= \frac{u \cdots (u - q + 1)}{(u + \theta(n-1)) \cdots (u + \theta(n-1) - q + 1)}\,. \end{aligned} \tag{7.5}$$

Substituting (7.5) into (7.4) and then using the Gauss summation we obtain

$$G_n^*(x_1, \dots, x_n; u) = G_{n-1}^*(x_1, \dots, x_{n-1}; u) \frac{\Gamma(x - u - \theta\, n)}{\Gamma(x - u - \theta(n-1))} \frac{\Gamma(-u - \theta(n-1))}{\Gamma(-u - \theta\, n)}\,,$$

which proves (7.2). □

Now we claim that this computation proves, in fact, the formula (7.1). Indeed, in the above prove we have used only the explicit formula appearing on the RHS of (7.1). The result of the computation shows that the polynomial in the RHS of (7.1) is an element of $\Lambda^\theta(n)$. One easily checks the degree and the vanishing conditions and concludes that (7.1) is true.

The above argument has a q-analog (where q is an extra parameter and not the index used above). It uses Heine's q-analog (see [GR]) of the Gauss summation (7.3) which is equivalent to the following summation (compare to (2.10) in [Ok9])

$$\sum_{k \geq 0} t^{-k} \frac{(t;q)_k}{(q;q)_k} \frac{(x-1)\cdots(x-q^{k-1})}{(u-1)\cdots(u-q^{k-1})} = \frac{(x/u\ ;q)_\infty}{(x/ut;q)_\infty} \frac{(1/ut;q)_\infty}{(1/u\ ;q)_\infty}\,.$$

Here $(a;q)_k = (1-a)\cdots(1-q^{k-1}a)$.

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