YANGIANS AND CLASSICAL LIE ALGEBRAS

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0. Introduction

The term 'Yangian' was introduced by V. G. Drinfeld to specify quantum groups related to rational solutions of the classical Yang–Baxter equation; see Belavin–Drinfeld [BD1,BD2] for the description of these solutions. In Drinfeld [D1] for each simple finite-dimensional Lie algebra $\mathfrak{a}$, a certain Hopf algebra $\mathrm{Y}(\mathfrak{a})$ was constructed so that $\mathrm{Y}(\mathfrak{a})$ is a deformation of the universal enveloping algebra for the polynomial current Lie algebra $\mathfrak{a}[x]$. An alternative description of the algebra $\mathrm{Y}(\mathfrak{a})$ was given in Drinfeld [D3]; see Theorem 1 therein.

Prior to the intruduction of the Hopf algebra $\mathrm{Y}(\mathfrak{a})$ in Drinfeld [D1], the algebra which may be called the Yangian for the reductive Lie algebra $\mathfrak{gl}(N)$ and may be denoted by $\mathrm{Y}\big(\mathfrak{gl}(N)\big)$, was considered in the works of mathematical physicists from St.-Petersburg; see for instance Takhtajan–Faddeev [TF]. The latter algebra is a deformation of the universal enveloping algebra $\mathrm{U}\big(\mathfrak{gl}(N)[x]\big)$. Representations of the algebra $\mathrm{Y}\big(\mathfrak{gl}(N)\big)$ were studied in Kulish–Reshetikhin–Sklyanin [KRS] and in Tarasov [T1,T2].

For any $\mathfrak{a}$ the Yangian $\mathrm{Y}(\mathfrak{a})$ contains the universal enveloping algebra $\mathrm{U}(\mathfrak{a})$ as a subalgebra. However, the case $\mathfrak{a}=\mathfrak{sl}(N)$ seems to be exceptional since only for $\mathfrak{a}=\mathfrak{sl}(N)$ there exists a homomorphism $\mathrm{Y}(\mathfrak{a})\to\mathrm{U}(\mathfrak{a})$ identical on the subalgebra $\mathrm{U}(\mathfrak{a})$; see Drinfeld [D1], Theorem 9. In the present article we concentrate on this distinguished Yangian. For each of the remaining classical Lie algebras $\mathfrak{a}=\mathfrak{o}(2n+1)$, $\mathfrak{sp}(2n)$, $\mathfrak{o}(2n)$ we introduce instead of the Yangian $\mathrm{Y}(\mathfrak{a})$ a new algebra. This new algebra is a deformation of the universal enveloping algebra for a certain twisted polynomial current Lie algebra.

Let $\mathfrak{a}$ be one of the latter three classical Lie algebras. Consider $\mathfrak{a}$ as an involutive subalgebra in $\mathfrak{gl}(N)$ where $N=2n+1,\,2n,\,2n$ respectively. Let σ denote the corresponding involution of $\mathfrak{gl}(N)$. The subalgebra

$$\mathfrak{gl}(N)[x]^{\sigma}=\{\,A(x)\in\mathfrak{gl}(N)[x]\;:\;\sigma\big(A(x)\big)=A(-x)\,\}$$

in the Lie algebra $\mathfrak{gl}(N)[x]$ is called the twisted polynomial current Lie algebra related to the symmetric Lie algebra $(\mathfrak{gl}(N),\sigma)$; it is an involutive subalgebra in $\mathfrak{gl}(N)[x]$ also. In the present article we introduce an algebra $\mathrm{Y}\big(\mathfrak{gl}(N),\sigma\big)$ which is a deformation of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}(N)[x]^{\sigma})$. The algebra $\mathrm{Y}\big(\mathfrak{gl}(N),\sigma\big)$ is a subalgebra in $\mathrm{Y}\big(\mathfrak{gl}(N)\big)$ and we call it 'twisted Yangian'.

As well as the Yangian $\mathrm{Y}\big(\mathfrak{gl}(N)\big)$, the twisted Yangian $\mathrm{Y}\big(\mathfrak{gl}(N),\sigma\big)$ contains the universal enveloping algebra $\mathrm{U}(\mathfrak{a})$ as a subalgebra and admits a homomorphism $\mathrm{Y}\big(\mathfrak{gl}(N),\sigma\big)\to\mathrm{U}(\mathfrak{a})$ identical on $\mathrm{U}(\mathfrak{a})$. Contrary to the Yangian $\mathrm{Y}(\mathfrak{a})$ defined in Drinfeld [D1], the twisted Yangian $\mathrm{Y}\big(\mathfrak{gl}(N),\sigma\big)$ has no natural Hopf algebra structure. However, it turns out to be a one-sided coideal in the Hopf algebra $\mathrm{Y}\big(\mathfrak{gl}(N)\big)$. Let us now describe the algebras $\mathrm{Y}\big(\mathfrak{gl}(N)\big)$ and $\mathrm{Y}\big(\mathfrak{gl}(N),\sigma\big)$ more explicitly.

Let the indices i,j run through the set $\{-n,\dots,-1,0,1,\dots,n\}$ if $N=2n+1$ and through the set $\{-n,\dots,-1,1,\dots,n\}$ if $N=2n$. Let $E_{ij}\in\mathrm{End}(\mathbb{C}^N)$ be the standard matrix units. We will also regard them as generators of the algebra $\mathrm{U}\big(\mathfrak{gl}(N)\big)$. The algebra $\mathrm{Y}\big(\mathfrak{gl}(N)\big)$ is generated by the elements $t_{ij}^{(k)}$ where $k=1,2,\dots$, subject to the following relations. Introduce the formal power series in

u^{-1} with the coefficients in $\mathrm{Y}(\mathfrak{gl}(N)) \otimes \mathrm{End}(\mathbb{C}^N)$

$$T(u) = \sum_{i,j} t_{ij}(u) \otimes E_{ij}, \quad t_{ij}(u) = \delta_{ij} + \sum_k t_{ij}^{(k)} u^{-k}.$$

Introduce also the formal power series in u^{-1}, v^{-1} with the coefficients in $\mathrm{Y}(\mathfrak{gl}(N)) \otimes \mathrm{End}(\mathbb{C}^N) \otimes \mathrm{End}(\mathbb{C}^N)$

$$T_1 = \sum_{i,j} t_{ij}(u) \otimes E_{ij} \otimes 1, \quad T_2 = \sum_{i,j} t_{ij}(v) \otimes 1 \otimes E_{ij}$$

and put

$$R_{12} = 1 \otimes R(u,v), \quad R(u,v) = 1 - (u-v)^{-1} \cdot \sum_{i,j} E_{ij} \otimes E_{ji}\,.$$

Then the defining relations in $\mathrm{Y}(\mathfrak{gl}(N))$ can be written as the 'ternary relation'

$$R_{12}\, T_1\, T_2 = T_2\, T_1\, R_{12}. \tag{1}$$

The exact meaning of relation (1) will be thoroughly explained in Section 1.

The imbedding $\mathrm{U}(\mathfrak{gl}(N)) \hookrightarrow \mathrm{Y}(\mathfrak{gl}(N))$ and the homomorphism $\mathrm{Y}(\mathfrak{gl}(N)) \to \mathrm{U}(\mathfrak{gl}(N))$ identical on $\mathrm{U}(\mathfrak{gl}(N))$ are defined respectively by $E_{ij} \mapsto t_{ij}^{(1)}$ and

$$T(u) \mapsto E(u) = 1 + u^{-1} \cdot \sum_{i,j} E_{ij} \otimes E_{ij}. \tag{2}$$

Thus if we denote

$$E_1 = 1 + u^{-1} \cdot \sum_{i,j} E_{ij} \otimes E_{ij} \otimes 1, \quad E_2 = 1 + v^{-1} \cdot \sum_{i,j} E_{ij} \otimes 1 \otimes E_{ij}$$

then the defining relations in $\mathrm{U}(\mathfrak{gl}(N))$ can be rewritten as

$$R_{12}\, E_1\, E_2 = E_2\, E_1\, R_{12}.$$

It turns out that the defining relations for the generators $F_{ij} = E_{ij} + \sigma(E_{ij})$ of the subalgebra $\mathrm{U}(\mathfrak{a}) \subset \mathrm{U}(\mathfrak{gl}(N))$ can be rewritten in an analogous way.

Let the superscript t denote the transposition in $\mathrm{End}(\mathbb{C}^N)$ corresponding to the bilinear (symmetric or alternating) form which is preserved by the subalgebra $\mathfrak{a} \subset \mathfrak{gl}(N)$; then $F_{ij} = E_{ij} - E_{ij}^t$ in $\mathrm{End}(\mathbb{C}^N)$. The algebra $\mathrm{Y}(\mathfrak{gl}(N), \sigma)$ is generated by the elements $s_{ij}^{(k)}$ where $k = 1, 2, \ldots$, subject to the following relations. As in the case of $\mathrm{Y}(\mathfrak{gl}(N))$, introduce the formal power series in u^{-1} with the coefficients in $\mathrm{Y}(\mathfrak{gl}(N), \sigma) \otimes \mathrm{End}(\mathbb{C}^N)$

$$S(u) = \sum_{i,j} s_{ij}(u) \otimes E_{ij}, \quad s_{ij}(u) = \delta_{ij} + \sum_k s_{ij}^{(k)} u^{-k}.$$

Introduce also the formal power series in u^{-1}, v^{-1} with the coefficients from $\mathrm{Y}\big(\mathfrak{gl}(N),\sigma\big)\otimes\mathrm{End}(\mathbb{C}^N)\otimes\mathrm{End}(\mathbb{C}^N)$

$$S_1=\sum_{i,j} s_{ij}(u)\otimes E_{ij}\otimes 1,\quad S_2=\sum_{i,j} s_{ij}(v)\otimes 1\otimes E_{ij}$$

and put

$$R'_{12}=1\otimes R'(-u,v),\quad R'(u,v)=1-(u-v)^{-1}\cdot\sum_{i,j} E^t_{ij}\otimes E_{ji}\,.$$

Then the defining relations in $\mathrm{Y}\big(\mathfrak{gl}(N),\sigma\big)$ can be written as the 'quaternary relation'

$$R_{12}\,S_1\,R'_{12}\,S_2=S_2\,R'_{12}\,S_1\,R_{12} \tag{3}$$

along with the 'symmetry relation'

$$S(u)-S^t(-u)=\mp\frac{1}{2u}\big(S(u)-S(-u)\big), \tag{4}$$

where

$$S^t(u)=\sum_{i,j} s_{ij}(u)\otimes E^t_{ij}\,.$$

Whenever the double sign $\pm$ or $\mp$ occurs, the upper sign corresponds to the case $\mathfrak{a}=\mathfrak{o}(N)$ while the lower sign corresponds to $\mathfrak{a}=\mathfrak{sp}(N)$.

The imbedding $\mathrm{U}(\mathfrak{a})\hookrightarrow\mathrm{Y}\big(\mathfrak{gl}(N),\sigma\big)$ and the homomorphism $\mathrm{Y}\big(\mathfrak{gl}(N),\sigma\big)\to\mathrm{U}(\mathfrak{a})$ identical on $\mathrm{U}(\mathfrak{a})$ are defined respectively by $F_{ij}\mapsto s^{(1)}_{ij}$ and

$$S(u)\mapsto F(u)=1+(u\pm\frac{1}{2})^{-1}\cdot\sum_{i,j} F_{ij}\otimes E_{ij}. \tag{5}$$

Thus if we denote

$$F_1=1+(u\pm\frac{1}{2})^{-1}\cdot\sum_{i,j} F_{ij}\otimes E_{ij}\otimes 1,\quad F_2=1+(v\pm\frac{1}{2})^{-1}\cdot\sum_{i,j} F_{ij}\otimes 1\otimes E_{ij}$$

then the defining relations in $\mathrm{U}(\mathfrak{a})$ can be rewritten as

$$R_{12}\,F_1\,R'_{12}\,F_2=F_2\,R'_{12}\,F_1\,R_{12},$$
$$F(u)-F^t(-u)=\mp\frac{1}{2u}\big(F(u)-F(-u)\big),$$

where

$$F^t(u)=1+(u\pm\frac{1}{2})^{-1}\sum_{i,j} F_{ij}\otimes E^t_{ij}\,.$$

The imbedding $\mathrm{Y}(\mathfrak{gl}(N),\sigma) \hookrightarrow \mathrm{Y}(\mathfrak{gl}(N))$ can be defined by

$$S(u) \mapsto T(u)\,T^t(-u), \quad \text{where} \quad T^t(u) = \sum_{i,j} t_{ij}(u) \otimes E^t_{ij}. \tag{6}$$

The ternary relation (1) has a rich and extensive background; see for instance Takhtajan–Faddeev [TF] and Drinfeld [D4]. This relation originates from the quantum Yang–Baxter equation (see Kulish–Sklyanin [KS1]), and the Yangians themselves were primarily regarded as a vehicle for producing new solutions of that equation; cf. Drinfeld [D1]. Conversely, the ternary relation (1) was used in Reshetikhin–Takhtajan–Faddeev [RTF] as a tool for studying quantum groups.

The quaternary relation (3) has its own history. Relations of the type (3) appeared for the first time in Cherednik [C1] and Sklyanin [S2], where integrable systems with boundary conditions were studied. Various versions of (3) were employed in Reshetikhin–Semenov [RS] to extend the approach of Reshetikhin–Takhtajan–Faddeev [RTF] from finite-dimensional to affine Lie algebras, and in Noumi [No] to obtain the q-analogues of spherical functions on the classical symmetric spaces. Algebraic structures related to (3) were discussed in Kulish–Sklyanin [KS3] and in Kulish–Sasaki–Schwiebert [KSS]. In these two papers a quaternary type relation was called the 'reflection equation'.

On the other hand, the Yangian $\mathrm{Y}(\mathfrak{gl}(N))$ has proved to be useful in the theory of finite-dimensional representations of the Lie algebra $\mathfrak{gl}(N)$. It gives rise to canonical generators of the center of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}(N))$ and to a variety of commutative subalgebras in $\mathrm{U}(\mathfrak{gl}(N))$; see Cherednik [C2] and Kirillov–Reshetikhin [KR]. Applications of the algebra $\mathrm{Y}(\mathfrak{gl}(N))$ to constructing the Gelfand–Zetlin bases for irreducible finite-dimensional representations of $\mathfrak{gl}(N)$ were considered in Cherednik [C2] and in Nazarov–Tarasov [NT], Molev [M2]. All of these applications are based on the existence of a homomorphism $\mathrm{Y}(\mathfrak{gl}(N)) \to \mathrm{U}(\mathfrak{gl}(N))$ identical on $\mathrm{U}(\mathfrak{gl}(N))$. We believe that the twisted Yangians will play the role of $\mathrm{Y}(\mathfrak{gl}(N))$ for the other classical Lie algebras $\mathfrak{a} = \mathfrak{o}(2n+1)$, $\mathfrak{sp}(2n)$, $\mathfrak{o}(2n)$. The results of Molev [M3] come in support of this claim. In forthcoming publications we will consider further applications of the twisted Yangians to the theory of finite-dimensional representations of the classical Lie algebras.

Our definition of the twisted Yangian $\mathrm{Y}(\mathfrak{gl}(N),\sigma)$ is motivated by the results of Olshanskiĭ [O1] where a natural extension of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}(\infty))$ is constructed. In that paper the Yangian $\mathrm{Y}(\mathfrak{gl}(N))$ arises in the following way. For each $m = N+1, N+2, \ldots$ consider the subalgebra

$$\mathfrak{gl}(N) \oplus \mathfrak{gl}(m-N) \subset \mathfrak{gl}(m)$$

and denote by $\mathrm{A}_N(m)$ the centralizer of $\mathfrak{gl}(m-N)$ in the algebra $\mathrm{U}(\mathfrak{gl}(m))$. In particular, $\mathrm{A}_0(m)$ coincides with the center $\mathrm{Z}(\mathfrak{gl}(m))$ of $\mathrm{U}(\mathfrak{gl}(m))$. Then there is a canonical chain of homomorphisms

$$\mathrm{A}_N(N+1) \leftarrow \mathrm{A}_N(N+2) \leftarrow \ldots \leftarrow \mathrm{A}_N(m) \leftarrow \ldots .$$

In particular, for $N=0$ we obtain a canonical chain

$$\mathrm{Z}(\mathfrak{gl}(1)) \leftarrow \mathrm{Z}(\mathfrak{gl}(2)) \leftarrow \ldots \leftarrow \mathrm{Z}(\mathfrak{gl}(m)) \leftarrow \ldots .$$

Then Theorem 2.1.5 from Olshanskiĭ [O1] establishes an isomorphism

$$\lim_{m\to\infty} \text{proj } \mathrm{A}_N(m) \cong \mathrm{Y}\big(\mathfrak{gl}(N)\big) \otimes \lim_{m\to\infty} \text{proj } \mathrm{Z}\big(\mathfrak{gl}(m)\big).$$

It was shown in Olshanskiĭ [O2] that by applying an analogous construction to the Lie algebra $\mathfrak{a}$ instead of $\mathfrak{gl}(N)$, one obtains the twisted Yangian $\mathrm{Y}\big(\mathfrak{gl}(N),\sigma\big)$ in place of $\mathrm{Y}\big(\mathfrak{gl}(N)\big)$.

A systematic study of finite-dimensional representations of Yangians was commenced in Drinfeld [D3]. The case of the Yangian $\mathrm{Y}\big(\mathfrak{sl}(2)\big)$ was primarily investigated in Tarasov [T1,T2]; see also Chari–Pressley [CP1]. Finite-dimensional representations of the Yangian $\mathrm{Y}(\mathfrak{a})$ were studied in Reshetikhin [R], and the paper Chari–Pressley [CP2] is concerned with general Yangians. However, the distinguished case of the Yangian $\mathrm{Y}\big(\mathfrak{gl}(N)\big)$ is the most fully studied; for this case an analogue of the classical Schur–Weyl duality is established in Cherednik [C2] and Drinfeld [D2]. We are convinced that finite-dimensional representations of the twisted Yangians also deserve a thorough study.

In Molev [M1] a general theory of finite-dimensional representations of the twisted Yangian $\mathrm{Y}\big(\mathfrak{gl}(N),\sigma\big)$ was addressed. In that paper an analogue of the classification theorem from Drinfeld [D3] was obtained and the simplest cases $\mathfrak{a}=\mathfrak{sp}(2)$, $\mathfrak{o}(2)$ were thoroughly examined.

In the present paper we study in detail the algebraic structure of the Yangian $\mathrm{Y}\big(\mathfrak{gl}(N)\big)$ and that of the twisted Yangian $\mathrm{Y}\big(\mathfrak{gl}(N),\sigma\big)$. Most of the results about the structure of $\mathrm{Y}\big(\mathfrak{gl}(N)\big)$ are known but there is no exposition of them available; the results concerning $\mathrm{Y}\big(\mathfrak{gl}(N),\sigma\big)$ were announced in Olshanskiĭ [O2] without proofs. Let us now give an overview of the contents of the present paper.

In Section 1 we begin with the definition of the Yangian $\mathrm{Y}\big(\mathfrak{gl}(N)\big)$ and introduce certain useful automorphisms of this algebra. The main result of Section 1 is an analogue of the Poincaré–Birkhoff–Witt theorem (Theorem 1.22). We also introduce a filtration on the algebra $\mathrm{Y}\big(\mathfrak{gl}(N)\big)$ such that the corresponding graded algebra coincides with the universal enveloping algebra $\mathrm{U}\big(\mathfrak{gl}(N)[x]\big)$; see Theorem 1.26.

In Section 2 we give a complete description (Theorem 2.13) of the center of the algebra $\mathrm{Y}\big(\mathfrak{gl}(N)\big)$. Here we use an important notion of 'quantum determinant' from Kulish–Sklyanin [KS2]. For $N=2$ this notion appeared earlier in Izergin–Korepin [IK]. Consider the formal series in u^{-1} with the coefficients in $\mathrm{Y}\big(\mathfrak{gl}(N)\big)$

$$(7)\quad \sum_{p\in\mathfrak{S}_N} \operatorname{sgn}(p)\, t_{p(1),1}(u)\, t_{p(2),2}(u-1)\dots t_{p(N),N}(u-N+1) = 1+\sum_k d_k\, u^{-k},$$

where $\mathfrak{S}_N$ is the symmetric group. This series is called the quantum determinant of the $N\times N$-matrix formed by $t_{ij}(u)$. We prove that the coefficients $d_1, d_2, \dots$ generate the center of $\mathrm{Y}\big(\mathfrak{gl}(N)\big)$. The images of the elements $d_1, d_2, \dots, d_N$ under the homomorphism $\mathrm{Y}\big(\mathfrak{gl}(N)\big)\to\mathrm{U}\big(\mathfrak{gl}(N)\big)$ defined by (2) turn out to be the generators of the center $\mathrm{Z}\big(\mathfrak{gl}(N)\big)$, introduced in Capelli [Ca1,Ca2] and also considered in Carter–Lusztig [CL] and Howe [H]. Furthermore, the algebra $\mathrm{Y}\big(\mathfrak{sl}(N)\big)$ can be defined as the quotient of $\mathrm{Y}\big(\mathfrak{gl}(N)\big)$ by the relations $d_1=d_2=\ldots=0$ (Corollary 2.18).

In Section 3 we give two alternative definitions of the algebra $\mathrm{Y}(\mathfrak{gl}(N),\sigma)$ and establish their equivalence. By one of the definitions, $\mathrm{Y}(\mathfrak{gl}(N),\sigma)$ is the algebra with the generators $s_{ij}^{(k)}$ and relations (3,4). We prove (Theorem 3.5) that the mapping (6) extends to a homomorhism of algebras $\mathrm{Y}(\mathfrak{gl}(N),\sigma)\to\mathrm{Y}(\mathfrak{gl}(N))$ and that this homomoprhism is injective (Theorem 3.8). Thus $\mathrm{Y}(\mathfrak{gl}(N),\sigma)$ can be also defined as a certain subalgebra in $\mathrm{Y}(\mathfrak{gl}(N))$. Moreover, this subalgebra turns out to be a left coideal in the Hopf algebra $\mathrm{Y}(\mathfrak{gl}(N))$; see Theorem 4.17. We point out (Remark 3.14) an analogue of the Poincaré–Birkhoff–Witt theorem for the algebra $\mathrm{Y}(\mathfrak{gl}(N),\sigma)$. As well as in Section 1, we introduce a filtration on the algebra $\mathrm{Y}(\mathfrak{gl}(N),\sigma)$ such that the corresponding graded algebra coincides with $\mathrm{U}(\mathfrak{gl}(N)[x]^\sigma)$, see Theorem 3.15.

In Section 4 we construct generators $c_1,\,c_2,\,\ldots$ of the center of $\mathrm{Y}(\mathfrak{gl}(N),\sigma)$ analogous to $d_1,\,d_2,\,\ldots\in\mathrm{Y}(\mathfrak{gl}(N))$; see Proposition 4.4, Theorem 4.7 and Theorem 4.11. Here we generalize one particular construction from Sklyanin [S2]. Through the homomorphism $\mathrm{Y}(\mathfrak{gl}(N),\sigma)\to\mathrm{U}(\mathfrak{a})$ defined by (5), we then obtain generators of the center of $\mathrm{U}(\mathfrak{a})$ which seem to be new; cf. Howe–Umeda [HU], Appendix 2. The quotient of the algebra $\mathrm{Y}(\mathfrak{gl}(N),\sigma)$ by the relations $c_1=c_2=\cdots=0$ is a deformation of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}(N)[x]^\sigma$; see Corollary 4.16.

In Section 5 we construct generators $z_1,\,z_2,\,\ldots$ of the center of the algebra $\mathrm{Y}(\mathfrak{gl}(N))$ different from those considered in Section 2, cf. [N1]. The images of these generators under the homomorphism $\mathrm{Y}(\mathfrak{gl}(N))\to\mathrm{U}(\mathfrak{gl}(N))$ defined by (2) essentially coincide with the elements of $\mathrm{Z}(\mathfrak{gl}(N))$ found in Perelomov–Popov [PP]. We describe explicitly (Theorem 5.11) the automorphism S^2 of the algebra $\mathrm{Y}(\mathfrak{gl}(N))$, where S stands for the antipode; this description also involves the elements $z_1,\,z_2,\,\ldots$. We provide a formula which links these elements with $d_1,\,d_2,\,\ldots$ (Theorem 5.7).

In Section 6 we construct generators of the center of the algebra $\mathrm{Y}(\mathfrak{gl}(N),\sigma)$ analogous to $z_1,\,z_2,\,\ldots\in\mathrm{Y}(\mathfrak{gl}(N))$. Their images with respect to the homomorphism $\mathrm{Y}(\mathfrak{gl}(N),\sigma)\to\mathrm{U}(\mathfrak{a})$ defined by (5) again essentially coincide with the elements $\mathrm{Z}(\mathfrak{a})$ from Perelomov–Popov [PP]. The results of this section also allow us to reformulate the symmetry relation (4) in a rather elegant way (Theorem 6.4).

Finally, in Section 7 we resume consideration of the quantum determinant (7). In that section we provide a quantum analogue (Theorem 7.3) of the well-known expansion of the determinant of a block matrix

$$\det\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\det A\cdot\det(D-C\,A^{-1}B)$$

where the block A is assumed to be invertible. We also provide an analogue of the last expansion for the algebra $\mathrm{Y}(\mathfrak{gl}(N),\sigma)$; see Theorem 7.6.

A word of explanation is necessary in regard to the scheme of referring to formulae adopted in the present paper. We number the formulae in every subsection independently. When we refer back to these formulae in later subsections, triple numbering is employed. For example, formula (1) in Subsection 3.2 is referred to as (1) in that subsection, and as (3.2.1) later on.

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1. The Yangian $\mathrm{Y}(N)$

In this section, we keep $N \in \{1, 2, \dots\}$ fixed. The Yangian $\mathrm{Y}(N)$ is introduced as an associative algebra with the defining relations (1.1.1). Then we describe the basic tool to work with $\mathrm{Y}(N)$ — the R-matrix formalism. Further we define some automorphisms of the algebra $\mathrm{Y}(N)$ which will be extensively used later. In Subsections 1.16 – 1.19 we discuss fundamental links between $\mathrm{Y}(N)$ and the universal enveloping algebra of $\mathfrak{gl}(N)$. Then we define two filtrations in $\mathrm{Y}(N)$ and prove the main result of this section — the Poincaré–Birkhoff–Witt theorem (Theorem 1.22 and Corollary 1.23). We also show that $\mathrm{Y}(N)$ is a flat deformation of the universal enveloping algebra of the polynomial current Lie algebra $\mathfrak{gl}(N)\otimes\mathbb{C}[x]$. Finally we discuss the Hopf algebra structure of $\mathrm{Y}(N)$.

1.1. Definition. The *Yangian* $\mathrm{Y}(N) = \mathrm{Y}(\mathfrak{gl}(N))$ is defined as the complex associative unital algebra with countably many generators $t_{ij}^{(1)}, t_{ij}^{(2)}, \dots$ where $1 \le i, j \le N$, and defining relations

$$[t_{ij}^{(M+1)}, t_{kl}^{(L)}] - [t_{ij}^{(M)}, t_{kl}^{(L+1)}] = t_{kj}^{(M)} t_{il}^{(L)} - t_{kj}^{(L)} t_{il}^{(M)}, \tag{1}$$

where $M, L = 0, 1, 2, \dots$ and $t_{ij}^{(0)} := \delta_{ij} \cdot 1$.

1.2. Proposition. *The system* (1.1.1) *is equivalent to the system of commutation relations:*

$$[t_{ij}^{(M)}, t_{kl}^{(L)}] = \sum_{r=0}^{\min(M,L)-1} (t_{kj}^{(r)} t_{il}^{(M+L-1-r)} - t_{kj}^{(M+L-1-r)} t_{il}^{(r)}), \tag{1}$$

where $M, L = 1, 2, \dots$ *and* $1 \le i, j, k, l \le N$.

Proof. To simplify the notation, let us denote by $\mathrm{Left}(M, L)$ and $\mathrm{Right}(M, L)$ the left and right hand sides of (1.1.1), respectively. The system of relations (1.1.1) is clearly equivalent to the following system of relations:

$$\begin{aligned}\mathrm{Left}(M, L) &+ \mathrm{Left}(M-1, L+1) + \ldots + \mathrm{Left}(0, M+L)\\ &= \mathrm{Right}(M, L) + \mathrm{Right}(M-1, L+1) + \ldots + \mathrm{Right}(0, M+L),\end{aligned} \tag{2}$$

where $M, L = 0, 1, 2, \dots$. Observe that

$$\mathrm{Left}(0, M+L) = [t_{ij}^{(1)}, t_{kl}^{(M+L)}] - [t_{ij}^{(0)}, t_{kl}^{(M+L+1)}] = [t_{ij}^{(1)}, t_{kl}^{(M+L)}],$$

since $t_{ij}^{(0)} = \delta_{ij}$. Thus (2) may be rewritten as

$$[t_{ij}^{(M+1)}, t_{kl}^{(L)}] = \sum_{r=0}^{M} \mathrm{Right}(r, M+L-r). \tag{3}$$

Furthermore, observe that

$$\mathrm{Right}(r, s) = -\mathrm{Right}(s, r) \qquad \text{for} \quad r, s = 0, 1, \dots , \tag{4}$$

so that

$$\sum_{r=L}^{M} \mathrm{Right}(r, M+L-r) = 0 \qquad \text{if} \quad M \geq L,$$

and (3) becomes

$$[t_{ij}^{(M+1)}, t_{kl}^{(L)}] = \sum_{r=0}^{\min(M,L-1)} \mathrm{Right}(r, M+L-r).$$

Now by replacing M by $M-1$ we obtain the relation (1).

1.3. The next few subsections contain preliminaries on the R-matrix formalism. In this formalism, one deals with the multiple tensor products $\mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N$ and operators therein. Let us set $\mathcal{E} = \mathbb{C}^N$. For an operator $A \in \mathrm{End}\,\mathcal{E}$ and a number $m = 1, 2, \ldots$ we set

$$(1) \qquad A_k := 1^{\otimes(k-1)} \otimes A \otimes 1^{\otimes(m-k)} \in \mathrm{End}\,\mathcal{E}^{\otimes m}, \quad 1 \leq k \leq m.$$

If $A \in \mathrm{End}\,\mathcal{E}^{\otimes 2}$ then for any k, l such that $1 \leq k, l \leq m$ and $k \neq l$, we denote by A_{kl} the operator in $\mathcal{E}^{\otimes m}$ which acts as A in the product of k-th and l-th copies and as 1 in all other copies. That is,

$$(2) \quad A = \sum_{r,s,t,u} a_{rstu} E_{rs} \otimes E_{tu}, \quad a_{rstu} \in \mathbb{C} \qquad \Rightarrow \qquad A_{kl} = \sum_{r,s,t,u} a_{rstu} (E_{rs})_k \, (E_{tu})_l$$

where, in accordance with (1),

$$(3) \qquad (E_{rs})_k = 1^{\otimes(k-1)} \otimes E_{rs} \otimes 1^{\otimes(m-k)}.$$

1.4. We will denote by P the permutation operator in $\mathcal{E} \otimes \mathcal{E}$:

$$(1) \qquad P := \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji}.$$

Set

$$(2) \qquad R(u) := 1 - \frac{P}{u} \in \mathrm{End}\,(\mathcal{E} \otimes \mathcal{E}) \otimes \mathbb{C}(u),$$

where u is a formal variable; this object is called the *Yang R-matrix*.

Let $m = 2, 3, \ldots$ and $u_1, \ldots, u_m$ be formal variables. For $1 \leq k, l \leq m$, $k \neq l$ consider the operator P_{kl} obtained from P by using the general rule (1.3.2); this is simply the permutation of k-th and l-th factors in $\mathcal{E}^{\otimes m}$. Now set

$$(3) \qquad R_{kl}(u_k - u_l) := 1 - \frac{P_{kl}}{u_k - u_l} \in \mathrm{End}\,\mathcal{E}^{\otimes m} \otimes \mathbb{C}(u_1, \ldots, u_m).$$

When $m = 2$, we will write $R(u - v)$ instead of $R_{12}(u_1 - u_2)$.

1.5. Proposition. *If i, j, k are pairwise distinct, then the following identity holds:*

$$R_{ij}(u)R_{ik}(u+v)R_{jk}(v) = R_{jk}(v)R_{ik}(u+v)R_{ij}(u). \tag{1}$$

This identity is called the *Yang–Baxter equation.* Sometimes it is convenient to write (1) in a slightly different form

$$R_{12}(u_1-u_2)R_{13}(u_1-u_3)R_{23}(u_2-u_3) = R_{23}(u_2-u_3)R_{13}(u_1-u_3)R_{12}(u_1-u_2). \tag{2}$$

Proof. One may assume that $i=1$, $j=2$, $k=3$. Using (1.4.2) and multiplying (1) by $uv(u+v)$, we obtain, after obvious transformations,

$$P_{12}P_{13}v + P_{12}P_{23}(u+v) + P_{13}P_{23}u - P_{12}P_{13}P_{23} =$$

$$P_{13}P_{12}v + P_{23}P_{12}(u+v) + P_{23}P_{13}u - P_{23}P_{13}P_{12}.$$

Since $P_{12}P_{13}P_{23} = P_{23}P_{13}P_{12}$ (this is essentially an identity in the symmetric group $\mathfrak{S}_3$), we have to verify the following two identities:

$$\begin{aligned} P_{12}P_{23} + P_{13}P_{23} &= P_{23}P_{12} + P_{23}P_{13} \\ P_{12}P_{13} + P_{12}P_{23} &= P_{13}P_{12} + P_{23}P_{12}. \end{aligned}$$

However, these identities are indeed true since

$$P_{12}P_{23} = P_{23}P_{13} = P_{13}P_{12}, \qquad P_{13}P_{23} = P_{23}P_{12} = P_{12}P_{13}.$$

1.6. Now we introduce the so-called T-matrix which is a matrix-valued formal generating series for the generators $t_{ij}^{(M)}$ of the Yangian $\mathrm{Y}(N)$. For certain reasons (see, e.g. Remark 2.2), it is convenient to deal with series in the *negative* powers of a formal variable.

Firstly, for any $i, j = 1, \dots, N$ define the generating series for the sequence $t_{ij}^{(M)}, M = 1, 2, \dots$ as follows:

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)}u^{-1} + t_{ij}^{(2)}u^{-2} + \cdots \in \mathrm{Y}(N)[[u^{-1}]]. \tag{1}$$

Then combine all these series into a single "T-matrix":

$$T(u) := \sum_{i,j=1}^{N} t_{ij}(u) \otimes E_{ij} \in \mathrm{Y}(N)[[u^{-1}]] \otimes \operatorname{End}\mathcal{E}. \tag{2}$$

More generally, given a number $m = 2, 3\dots$ and formal variables $u_1, \dots, u_m$, we set for any $k = 1, \dots, m$

$$T_k(u_k) := \sum_{i,j=1}^{N} t_{ij}(u_k) \otimes (E_{ij})_k \in \mathrm{Y}(N)[[u_1^{-1}, \dots, u_m^{-1}]] \otimes \operatorname{End}\mathcal{E}^{\otimes m}. \tag{3}$$

If $m=2$, we will prefer to write u,v instead of u_1,u_2.

1.7. We will often have to deal with the operators $R_{kl}(u_k-u_l)$ and $T_k(u_k)$ simultaneously. Then the algebra $\mathrm{Y}(N)[[u_1^{-1},\dots,u_m^{-1}]]$ should be replaced by an appropriate extension $\mathrm{Y}(N)[[u_1^{-1},\dots,u_m^{-1}]]_{ext}$ containing the elements $(u_k-u_l)^{-1}$. It is easy to construct such an extension. For example, we write

$$(u_k-u_l)^{-1}=-\frac{u_k^{-1}u_l^{-1}}{u_k^{-1}-u_l^{-1}}$$

and then localize $\mathrm{Y}(N)[[u_1^{-1},\dots,u_m^{-1}]]$ with respect to the multiplicative family generated by the elements $u_k^{-1}-u_l^{-1}$, $k\neq l$. The localization is well-defined since the algebra $\mathrm{Y}(N)[[u_1^{-1},\dots,u_m^{-1}]]$ has no divisors of zero. This is the minimal possible extension, and sometimes we will need a larger one (see, e.g., Subsection 1.10).

1.8. Proposition. *The system* (1.1.1) *of the defining relations of* $\mathrm{Y}(N)$ *is equivalent to the following single relation on the T-matrix:*

$$R(u-v)T_1(u)T_2(v)=T_2(v)T_1(u)R(u-v). \tag{1}$$

We will refer to (1) as the *ternary relation.*

Proof. It is easily seen that, in terms of the generating series (1.6.1), the initial system (1.1.1) may be rewritten as follows:

$$[t_{ij}(u),t_{kl}(v)]=\frac{1}{u-v}(t_{kj}(u)t_{il}(v)-t_{kj}(v)t_{il}(u)) \tag{2}$$

where $1\le i,j,k,l\le N$. Indeed, if we multiply both sides of (2) by $u-v$ and compare the terms having the same degrees in u and v, then we will return to (1.1.1).

On the other hand, by definitions (1.3.1) and (1.4.2), formula (1) reads as follows:

$$\begin{aligned}(1-\frac{P}{u-v})\sum_{i,j,k,l}t_{ij}(u)t_{kl}(v)\,(E_{ij}\otimes E_{kl})=\\ =\sum_{i,j,k,l}t_{kl}(v)t_{ij}(u)\,(E_{ij}\otimes E_{kl})\,(1-\frac{P}{u-v}).\end{aligned} \tag{3}$$

This may be rewritten as

$$\begin{aligned}&\sum_{i,j,k,l}[t_{ij}(u),t_{kl}(v)]\,(E_{ij}\otimes E_{kl})=\\ &=\frac{1}{u-v}\sum_{i,j,k,l}t_{ij}(u)t_{kl}(v)\,P\,(E_{ij}\otimes E_{kl})-\frac{1}{u-v}\sum_{i,j,k,l}t_{kl}(v)t_{ij}(u)\,(E_{ij}\otimes E_{kl})\,P.\end{aligned} \tag{4}$$

Observe now that, by definition of P,

$$P\,(E_{ij}\otimes E_{kl})=E_{kj}\otimes E_{il},\qquad (E_{ij}\otimes E_{kl})\,P=E_{il}\otimes E_{kj}.$$

Substituting this in (4) and changing the notation of the indices in an obvious manner one finally obtains that

$$\sum_{i,j,k,l}[t_{ij}(u),t_{kl}(v)]\,(E_{ij}\otimes E_{kl})=$$
$$=\frac{1}{u-v}\sum_{i,j,k,l}(t_{kj}(u)t_{il}(v)-t_{kj}(v)t_{il}(u))\,(E_{ij}\otimes E_{kl}),$$

which is equivalent to the system (2).

1.9. Remark. One could propose the following informal interpretation of identity (1.8.1). Let us suppose that the generators $t_{ij}^{(1)},t_{ij}^{(2)},\dots$ operate in a certain vector space W (the nature of W is irrelevant; e.g., one may take the left regular representation of the algebra $\mathrm{Y}(N)$). Then $T(u)$ may be regarded as an operator in $W\otimes\mathcal{E}$ depending on a (formal) parameter u, so that (1.8.1) may be regarded as a relation in $\operatorname{End}W\otimes\mathcal{E}\otimes\mathcal{E}$.

1.10. Remark. The commutation relations (1.2.1) can be also written as follows:

$$(1)\qquad [T_1^{(M)},T_2^{(L)}]=\sum_{r=0}^{\min(M,L)-1}(PT_1^{(r)}T_2^{(M+L-1-r)}-T_2^{(M+L-1-r)}T_1^{(r)}P)$$

where

$$T^{(M)}:=\sum_{i,j=1}^{N}t_{ij}^{(M)}\otimes E_{ij}\in\mathrm{Y}(N)\otimes\operatorname{End}\mathcal{E}$$

and $T_1^{(M)}$ and $T_2^{(M)}$ are built following the general prescription (1.3.1).

Let us show how (1) can be derived from the ternary relation (1.8.1). First, we write (1.8.1) as

$$(2)\qquad [T_1(u),T_2(v)]=\frac{1}{u-v}(PT_1(u)T_2(v)-T_2(v)T_1(u)P).$$

Next we write

$$(3)\qquad \frac{1}{u-v}=\frac{u^{-1}}{1-vu^{-1}}=\sum_{s=0}^{\infty}v^su^{-s-1}$$

and regard both sides of (2) as the elements of the extended algebra

$$(4)\qquad \mathrm{Y}(N)((v^{-1}))[[u^{-1}]]\otimes\operatorname{End}(\mathcal{E}\otimes\mathcal{E})$$

(note that (3) does belong to this algebra). Then (2) is rewritten as the system of the following relations for $M, L = 0, 1, 2, \ldots$

$$(5) \qquad [T_1^{(M)}, T_2^{(L)}] = \sum_{s=0}^{\infty} (PT_1^{(M-s-1)} T_2^{(L+s)} - T_2^{(L+s)} T_1^{(M-s-1)} P).$$

The sum in the right hand side of (5) is actually taken over $s = 0, 1, \ldots, M-1$ so that we may replace s by $r := M-1-s$. Then (5) takes the form

$$(6) \qquad [T_1^{(M)}, T_2^{(L)}] = \sum_{r=0}^{M-1} (PT_1^{(r)} T_2^{(M+L-1-r)} - T_2^{(M+L-1-r)} T_1^{(r)} P)$$

which is simply another form of (1.2.3).

It remains to note that for any r, s the conjugation by P sends $T_1^{(r)}$ into $T_2^{(r)}$ and $T_2^{(s)}$ into $T_1^{(s)}$, so that the expression

$$PT_1^{(r)} T_2^{(s)} - T_2^{(s)} T_1^{(r)} P$$

is antisymmetric in (r, s) (compare with (1.2.4)). This shows that the summation in (6) can be actually made over $r = 0, 1, \ldots, \min(M, L) - 1$.

1.11. Proposition. *There exists an involutive antiautomorphism of the algebra* $\mathrm{Y}(N)$ *defined by*

$$(1) \qquad \mathrm{sign} : T(u) \mapsto T(-u).$$

Proof. This is almost trivial. We have to check that

$$(2) \qquad R(u-v) T_2(-v) T_1(-u) = T_1(-u) T_2(-v) R(u-v)$$

but this follows from the ternary relation (1.8.1), if we conjugate both of its sides by P and then replace (u, v) by $(-v, -u)$.

1.12. Proposition. *The following mappings define automorphisms of the algebra* $\mathrm{Y}(N)$. (i) *The shift in u:*

$$(1) \qquad \sigma_a : T(u) \mapsto T(u+a), \quad a \in \mathbb{C}.$$

(ii) *The multiplication by a formal power series:*

$$(2) \qquad \mu_f : T(u) \mapsto f(u) T(u)$$

where

$$(3) \qquad f(u) := 1 + f_1 u^{-1} + f_2 u^{-2} + \ldots \in \mathbb{C}[[u^{-1}]]$$

or, more explicitly,

$$t_{ij}^{(1)} \mapsto t_{ij}^{(1)} + f_1\delta_{ij}, \quad t_{ij}^{(2)} \mapsto t_{ij}^{(2)} + f_1 t_{ij}^{(1)} + f_2\delta_{ij}, \quad \textit{etc.} \tag{4}$$

(iii) *Inversion:*

$$\mathrm{inv} : T(u) \mapsto T^{-1}(-u). \tag{5}$$

(iv) *Transposition:*

$$T(u) \mapsto T^t(-u) \tag{6}$$

where $t : \operatorname{End}\mathcal{E} \to \operatorname{End}\mathcal{E}$ *is an arbitrary antiautomorphism of the algebra* $\operatorname{End}\mathcal{E}$ *(e.g.,* $E_{ij} \mapsto E_{ji}$*) and*

$$T^t(u) := \sum t_{ij}(u) \otimes (E_{ij})^t. \tag{7}$$

Proof. We will verify first that each of the mappings (i) – (iv) preserves the defining relations (1.8.1) of $\mathrm{Y}(N)$.

(i) This seems to be trivial since the ternary relation is clearly invariant under the shift of the parameter u. There is, however, an important detail: it should be stressed that a shift of the (formal) parameter u is a well defined operation in $\mathrm{Y}(N)[[u^{-1}]]$. Note that in the case of $\mathrm{Y}(N)[[u]]$ this is no longer true.

(ii) It suffices to multiply both sides of the ternary relation by $f(u)f(v)$.

(iii) We have to verify the relation

$$R(u-v)T_1^{-1}(-u)T_2^{-1}(-v) = T_2^{-1}(-v)T_1^{-1}(-u)R(u-v). \tag{8}$$

This can be done as follows. First, one multiplies both sides of the ternary relation by $T_1^{-1}(u)T_2^{-1}(v)$ on the left and by $T_2^{-1}(v)T_1^{-1}(u)$ on the right; the result looks like

$$T_1^{-1}(u)T_2^{-1}(v)R(u-v) = R(u-v)T_2^{-1}(v)T_1^{-1}(u). \tag{9}$$

Next one interchanges both sides of (9) and conjugates them by the permutation operator P; the result then looks like

$$R(u-v)T_1^{-1}(v)T_2^{-1}(u) = T_2^{-1}(u)T_1^{-1}(v)R(u-v). \tag{10}$$

Finally one replaces (u,v) by $(-v,-u)$; this clearly transforms (10) to (8).

(iv) First of all, observe that any antiautomorphism of $\operatorname{End}\mathcal{E}$ can be written as the composition of the “standard” transposition $E_{ij} \mapsto E_{ji}$ and an interior automorphism (i.e., conjugation by an invertible operator). It implies that $P \in \operatorname{End}(\mathcal{E}\otimes\mathcal{E})$ is invariant with respect to $t\otimes t$, so that $R(u-v)$ is invariant too.

Introduce the partial transpositions

$$t_1 := t\otimes 1, \qquad t_2 := 1\otimes t. \tag{11}$$

Our claim is equivalent to the validity of the relation

$$R(u-v)T_1^{t_1}(-u)T_2^{t_2}(-v) = T_2^{t_2}(-v)T_1^{t_1}(-u)R(u-v). \tag{12}$$

We will deduce (12) from the ternary relation by means of the following transformations.

Firstly, apply t_1 to both sides of the ternary relation. It is easy to see that the result is

$$(R(u-v)T_1(u))^{t_1}T_2(v) = T_2(v)(T_1(u)R(u-v))^{t_1}. \tag{13}$$

Secondly, observe that one may regard $R(u-v)$ and $T_1(u)$ as $N\times N$ matrices, say A and B, such that each coefficient of A commutes with any coefficient of B. (In fact, the coefficients of A are essentially operators in the second copy of $\mathcal{E}$ while the coefficients of B are essentially elements of the Yangian.) In such a situation, we have $(AB)^t = B^tA^t$, therefore we may write

$$\begin{aligned}(R(u-v)T_1(u))^{t_1} &= T_1^{t_1}(u)R^{t_1}(u-v),\\ (T_1(u)R(u-v))^{t_1} &= R^{t_1}(u-v)T_1^{t_1}(u).\end{aligned}$$

Substituting this in (13), we obtain

$$T_1^{t_1}(u)R^{t_1}(u-v)T_2(v) = T_2(v)R^{t_1}(u-v)T_1^{t_1}(u). \tag{14}$$

Thirdly, applying the partial transposition t_2 to (14) and using the invariance of the R-matrix under $t_2\circ t_1 = t\otimes t$, we obtain that

$$T_1^{t_1}(u)T_2^{t_2}(v)R(u-v) = R(u-v)T_2^{t_2}(v)T_1^{t_1}(u). \tag{15}$$

Finally, conjugating both sides of (15) by P, then interchanging them and replacing (u,v) by $(-v,-u)$, we arrive at (12).

Thus each of the mappings (i) – (iv) preserves the defining relations of $\mathrm{Y}(N)$. To prove Proposition 1.12 it remains to verify that all these mappings are invertible. In cases (i), (ii) and (iv) this is clear. For (iii) this needs a bit of work. We start with the equality

$$(\operatorname{inv} T(u))T(-u) = 1$$

and apply inv to both sides. Then we get

$$\begin{gathered}(\operatorname{inv}\circ\operatorname{inv})(T(u))\operatorname{inv}(T(-u)) = 1,\\ (\operatorname{inv}\circ\operatorname{inv})(T(u))(T^{-1}(u)) = 1,\\ (\operatorname{inv}\circ\operatorname{inv})(T(u)) = T(u)\end{gathered}$$

so that $\operatorname{inv}\circ\operatorname{inv} = \mathrm{id}$.

1.13. Corollary to Propositions 1.11 and 1.12. *The mappings*

$$\mathrm{S} := \mathrm{inv} \circ \mathrm{sign} : T(u) \mapsto T^{-1}(u), \tag{1}$$

$$t \circ \mathrm{sign} : T(u) \mapsto T^{t}(u) \tag{2}$$

define antiautomorphisms of the algebra $\mathrm{Y}(N)$.

1.14. Remark. Note that the antiautomorphisms inv and sign do not commute, so that $\mathrm{S}^{-1} = \mathrm{sign} \circ \mathrm{inv} \neq \mathrm{S}$ and the automorphism S is not involutive! In fact,

$$(\mathrm{inv} \circ \mathrm{sign})(t_{ij}^{(M)}) = \mathrm{inv}(t_{ij}^{(M)})(-1)^{M}.$$

On the other hand, $\mathrm{inv}(t_{ij}^{(M)})$ is a rather complicated expression on the generators $t_{kl}^{(L)}$, e.g.,

$$\begin{aligned}
&\mathrm{inv}(t_{ij}^{(1)}) = t_{ij}^{(1)},\\
&\mathrm{inv}(t_{ij}^{(2)}) = -t_{ij}^{(2)} + \sum_{a=1}^{N} t_{ia}^{(1)} t_{aj}^{(1)},\\
&\mathrm{inv}(t_{ij}^{(3)}) = t_{ij}^{(3)} - \sum_{a=1}^{N} (t_{ia}^{(1)} t_{aj}^{(2)} + t_{ia}^{(2)} t_{aj}^{(1)}) + \sum_{a,b=1}^{N} t_{ia}^{(1)} t_{ab}^{(1)} t_{bj}^{(1)},
\end{aligned}$$

so that its behavior under the antiautomorphism which sends $t_{kl}^{(L)}$ into $t_{kl}^{(L)}(-1)^{L}$, is not easy to describe. We will calculate the square of S later on, see Subsection 5.11.

1.15. Remark. In what follows, we shall often use an assertion which is a natural generalization of that used in the proof of Proposition 1.12: if A and B are matrices whose coefficients belong to an associative algebra and each coefficient of A commutes with any coefficient of B, then $(AB)^{t} = B^{t} A^{t}$ for any antiautomorphism t of the matrix algebra.

Now we need some preparations to prove an important result, Theorem 1.22.

1.16. Let E_{ij} be the natural basis of the Lie algebra $\mathfrak{gl}(N)$ formed by matrix units.

Proposition. *The mapping*

$$\xi : t_{ij}(u) \mapsto \delta_{ij} + E_{ij} u^{-1} \tag{1}$$

defines the homomorphism of algebras $\xi : \mathrm{Y}(N) \to \mathrm{U}(\mathfrak{gl}(N))$.

Proof. By using the automorphism (1.12.6) of $\mathrm{Y}(N)$ we obtain that the required property of ξ is equivalent to that of the mapping

$$\xi' : t_{ij}(u) \mapsto \delta_{ji} - E_{ji} u^{-1}.$$

To prove that ξ' also defines an algebra homomorphism $\xi' : \mathrm{Y}(N) \to \mathrm{U}(\mathfrak{gl}(N))$, we have to verify that the ternary relation holds for

$$T(u) = 1 - u^{-1} \sum_{i,j} E_{ij} \otimes E_{ji}. \tag{2}$$

However, due to (1.4.1 and 1.4.2), the expression (2) coincides with $R(u)$. Hence the required statement is equivalent to the formula

$$R_{23}(u-v)R_{12}(u)R_{13}(v) = R_{13}(v)R_{12}(u)R_{23}(u-v).$$

After conjugation by P_{23} it turns into the Yang-Baxter equation (1.5.2) written in a slightly different form.

1.17. Proposition. *The mapping $\eta : E_{ij} \mapsto t_{ij}^{(1)}$ defines the inclusion of the algebra $\mathrm{U}(\mathfrak{gl}(N))$ into $\mathrm{Y}(N)$.*

Proof. It follows from the commutation relations (1.2.1) that η is extended to an algebra homomorphism. It is clear that $\xi \circ \eta = \mathrm{id}$, so the kernel of η is trivial.

1.18. Remark. Denote by E the $N \times N$-matrix whose entries are E_{ij}, i.e.,

$$E := \sum_{i,j} E_{ij} \otimes E_{ij} \in \mathrm{End}(\mathcal{E} \otimes \mathcal{E})$$

and set $\mathcal{T}(u) := 1 + Eu^{-1}$. We can summarize the previous results as follows: the fact that $\mathcal{T}(u)$ satisfies the ternary relation is equivalent to the fact that the basis elements E_{ij} satisfy the commutation relations

$$[E_{ij}, E_{kl}] = \delta_{kj}E_{il} - \delta_{il}E_{kj}.$$

1.19. We shall also need the composition $\xi \circ \mathrm{inv} : \mathrm{Y}(N) \to \mathrm{U}(\mathfrak{gl}(N))$ of the homomorphism ξ and the inversion (1.12.5):

$$\xi \circ \mathrm{inv} : T(u) \mapsto \mathcal{T}(-u)^{-1},$$

that is

$$\xi \circ \mathrm{inv}(t_{ij}^{(M)}) = \sum_{a_1,\cdots,a_{M-1}=1}^{N} E_{ia_1}E_{a_1a_2}\ldots E_{a_{M-1}j}. \tag{1}$$

1.20. Definition. The algebra $\mathrm{Y}(N)$ is equipped with two different ascending filtrations which are obtained by defining the degree of a generator in two different ways:

$$\deg_1(t_{ij}^{(M)}) = M \qquad \text{and} \qquad \deg_2(t_{ij}^{(M)}) = M - 1,$$

respectively. Let $\mathrm{gr}_1\mathrm{Y}(N)$ and $\mathrm{gr}_2\mathrm{Y}(N)$ denote the corresponding graded algebras.

1.21. Corollary. *The algebra* $\mathrm{gr}_1\mathrm{Y}(N)$ *is commutative.*

Proof. This follows directly from (1.2.1) since the degree $\deg_1(\cdot)$ of each term in the right hand side of (1.2.1) is less than that of the left hand side.

1.22. Theorem. *Let* $\bar{t}_{ij}^{(M)}$ *stand for the image of the generator* $t_{ij}^{(M)}$ *in the* M*-th component of* $\mathrm{gr}_1\mathrm{Y}(N)$. *The elements* $\bar{t}_{ij}^{(M)}$ *are algebraically independent, so that* $\mathrm{gr}_1\mathrm{Y}(N)$ *is the algebra of polynomials in countably many variables* $\bar{t}_{ij}^{(M)}$.

Proof. It follows from the defining relations (1.1.1) that for any $N' \geq N$ there is a natural homomorphism $\iota : \mathrm{Y}(N) \to \mathrm{Y}(N')$. Taking the composition of ι and the homomorphism $\xi \circ \mathrm{inv} : \mathrm{Y}(N') \to \mathrm{U}(\mathfrak{gl}(N'))$, we get another homomorphism $\zeta := \xi \circ \mathrm{inv} \circ \iota$,

$$\zeta : \mathrm{Y}(N) \to \mathrm{U}(\mathfrak{gl}(N'))$$

such that

$$\zeta(t_{ij}^{(M)}) = \sum_{a_1,\cdots,a_{M-1}=1}^{N'} E_{ia_1}E_{a_1a_2}\dots E_{a_{M-1}j}.$$

It prerserves the filtration of $\mathrm{Y}(N)$ defined by $\deg_1$ and the canonical filtration of $\mathrm{U}(\mathfrak{gl}(N'))$. Therefore it determines a homomorphism of graded algebras

$$\bar{\zeta} : \mathrm{gr}_1\mathrm{Y}(N) \to \mathrm{S}(\mathfrak{gl}(N')).$$

We shall consider elements of the symmetric algebra $\mathrm{S}(\mathfrak{gl}(N'))$ as polynomial functions on $\mathfrak{gl}(N')$. Thus the image of $\bar{t}_{ij}^{(M)}$ under $\bar{\zeta}$ is the polynomial $p_{ij}^{(M)}$ such that

$$p_{ij}^{(M)}(x) = (x^M)_{ij}, \qquad x \in \mathfrak{gl}(N').$$

Now it suffices to prove that for each fixed positive integer M' all the polynomials $p_{ij}^{(M)}$, $1 \leq i,j \leq N$, $1 \leq M \leq M'$, are algebraically independent for sufficiently large N'.

For any triple (i,j,M) satisfying the conditions $1 \leq i,j \leq N$, $1 \leq M \leq M'$ we can choose a subset

$$\Omega_{ij}^{(M)} \subset \{N+1, N+2, \dots\}$$

of cardinality $M-1$ in such a way that all these subsets are disjoint. Let N' be so large that all of them belong to $\{N+1, N+2, \dots, N'\}$. Let $y_{ij}^{(M)}$ be complex parameters. Define a linear operator $x_{ij}^{(M)}$ in $\mathbb{C}^{N'}$ depending on $y_{ij}^{(M)}$ as follows. Let $e_1, \dots, e_{N'}$ be the canonical basis in $\mathbb{C}^{N'}$ and $a_1 < \dots < a_{M-1}$ be all the elements of $\Omega_{ij}^{(M)}$. Then put

$$x_{ij}^{(M)} : e_j \mapsto y_{ij}^{(M)} e_{a_{M-1}}, \quad e_{a_{M-1}} \mapsto e_{a_{M-2}}, \; \dots, e_{a_1} \mapsto e_i,$$
$$x_{ij}^{(M)} : e_k \mapsto 0 \quad \text{for} \quad k \notin \{j\} \cup \Omega_{ij}^{(M)}$$

and set

$$x = \sum_{i,j,M} x_{ij}^{(M)}. \tag{1}$$

Then for any matrix x of the form (1) we have

$$p_{ij}^{(M)}(x) = y_{ij}^{(M)} + \psi$$

for certain polynomial ψ in $y_{kl}^{(L)}$ where $L < M$. Thus the polynomials $p_{ij}^{(M)}$ are algebraically independent even if they are restricted to the affine subspace of matrices of the form (1). Theorem 1.22 is proved.

1.23. Corollary. *Given an arbitrary linear order on the set of the generators $t_{ij}^{(M)}$, any element of the algebra $\mathrm{Y}(N)$ is uniquely written as a linear combination of ordered monomials in the generators.*

1.24. Remark. Theorem 1.22 (or the equivalent statement given in Corollary 1.23) is a fundamental fact which may be called *the Poincaré–Birkhoff–Witt theorem for the Yangian* $\mathrm{Y}(N)$.

1.25. Remark. Theorem 1.22 implies that $\mathrm{Y}(N)$ can be viewed as a flat deformation of the algebra of polynomials in countably many variables. To see this, for each $h \in \mathbb{C} \setminus \{0\}$ consider the algebra $\mathrm{Y}(N,h)$ with the generators $t_{ij}^{(M)}$ and the relations obtained from (1.2.1) by multiplying the right hand side by h:

$$(1) \qquad [\, t_{ij}^{(M)}, t_{kl}^{(L)} \,] = h \cdot \sum_{r=0}^{\min(M,L)-1} (\, t_{kj}^{(r)}\, t_{il}^{(M+L-1-r)} - t_{kj}^{(M+L-1-r)}\, t_{il}^{(r)} \,).$$

The algebras $\mathrm{Y}(N,h)$ are all isomorphic to each other; an isomorphism $\mathrm{Y}(N,h) \to \mathrm{Y}(N)$ can be defined by $t_{ij}^{(M)} \mapsto t_{ij}^{(M)} h^M$. On the other hand, in the limit $h \to 0$ we obtain from $\mathrm{Y}(N,h)$ the algebra of polynomials in the generators $t_{ij}^{(M)}$.

1.26. Now let us turn to the second filtration. Consider the *polynomial current Lie algebra*

$$(1) \qquad \mathfrak{gl}(N)[x] := \mathfrak{gl}(N) \otimes_{\mathbb{C}} \mathbb{C}[x]$$

and its universal enveloping algebra $\mathrm{U}(\mathfrak{gl}(N)[x])$. There is a natural basis in $\mathfrak{gl}(N)[x]$ formed by the elements $E_{ij}x^{M-1}$ where $1 \le i,j \le N$ and $M = 1,2,\dots$.

Theorem. *The algebra* $\mathrm{gr}_2\mathrm{Y}(N)$ *is isomorphic to the algebra* $\mathrm{U}(\mathfrak{gl}(N)[x])$.

Proof. Let us examine the degree $\deg_2(\cdot)$ of the different terms in (1.2.1). The degree of the left hand side equals $M+L-2$. The degree of each of the terms in the right hand side equals $M+L-3$ except the term with $r=0$. The latter one may be written as

$$(2) \qquad t_{kj}^{(0)} t_{il}^{(M+L-1)} - t_{kj}^{(M+L-1)} t_{il}^{(0)} = \delta_{kj} t_{il}^{(M+L-1)} - \delta_{il} t_{kj}^{(M+L-1)}$$

and has the same degree $M+L-2$ as the left hand side. This implies that in the algebra $\mathrm{gr}_2\mathrm{Y}(N)$, the commutation relations take the form

$$(3) \qquad [\tilde t_{ij}^{(M)}, \tilde t_{kl}^{(L)}] = \delta_{kj} \tilde t_{il}^{(M+L-1)} - \delta_{il} \tilde t_{kj}^{(M+L-1)},$$

where $\tilde t_{ij}^{(M)}$ stands for the image of $t_{ij}^{(M)}$ in the $(M-1)$-th component of $\mathrm{gr}_2\mathrm{Y}(N)$. Observe now that relations (3) are exactly the commutation relations of the Lie algebra $\mathfrak{gl}(N)[x]$ in its basis $\{E_{ij}x^{M-1}\}$. Thus there exists an algebra morphism $E_{ij}x^{M-1} \mapsto \tilde t_{ij}^{(M)}$ from $\mathrm{U}(\mathfrak{gl}(N)[x])$ onto $\mathrm{gr}_2\mathrm{Y}(N)$. The kernel of this morphism is trivial by Theorem 1.22.

1.27. Remark (cf. Remark 1.25). The algebra $\mathrm{Y}(N)$ may be also regarded as a flat deformation of the algebra $\mathrm{U}(\mathfrak{gl}(N)[x])$. Indeed, let us renormalize the generators of $\mathrm{Y}(N)$ by multiplying $t_{ij}^{(M)}$ by h^{M-1} (instead of h^M as before). This results in the following modification of relations (1.2.1): the numerical factor h will appear in all the terms of the right hand side of (1.2.1) except for the term (1.2.1) corresponding to $r=0$. Let $\mathrm{Y}_h(N)$ denote the algebra defined by these modified relations. Then $\mathrm{Y}_1(N)=\mathrm{Y}(N)$ and $\mathrm{Y}_0(N)=\mathrm{U}(\mathfrak{gl}(N)[x])$. The flatness of the deformation $\{\mathrm{Y}_h(N): h\in\mathbb{C}\}$ is again guaranteed by Theorem 1.22.

1.28. Theorem. *The Yangian $\mathrm{Y}(N)$ is a Hopf algebra with respect to the coproduct $\Delta:\mathrm{Y}(N)\to\mathrm{Y}(N)^{\otimes 2}$ defined by*

$$\Delta(t_{ij}(u)) := \sum_{a=1}^{N} t_{ia}(u)\otimes t_{aj}(u),$$

the antipode S *defined by* (1.13.1) *and the counit* ϵ *defined by* $\epsilon(T(u)) := 1$.

Proof. To work with the coproduct Δ it is more convenient to rewrite its definition in terms of the T-matrix. To do this we will generalize the notation adopted in Subsection 1.6 a little bit.

Suppose that we are dealing with the tensor product of m copies of $\mathrm{Y}(N)[[u^{-1}]]$ and n copies of $\operatorname{End}\mathcal{E}$. Then for any numbers k,l such that $1\le k\le m$ and $1\le l\le n$, set

$$\begin{aligned} T_{[k]l}(u) &:= \sum_{i,j=1}^{N}(1^{\otimes k-1}\otimes t_{ij}(u)\otimes 1^{\otimes m-k})\otimes(1^{\otimes l-1}\otimes E_{ij}\otimes 1^{\otimes n-l}) \\ &\in \mathrm{Y}(N)[[u^{-1}]]^{\otimes m}\otimes(\operatorname{End}\mathcal{E})^{\otimes n}. \end{aligned}$$

Using the informal language of Remark 1.9 one could say that $T_{[k]l}(u)$ is the operator in $W^{\otimes m}\otimes\mathcal{E}^{\otimes n}$ which acts as $T(u)$ in the product of the k-th copy of W and the l-th copy of $\mathcal{E}$ and as 1 in all other copies of these spaces.

When $m=1$, we prefer to abbreviate $T_l(u) := T_{[1]l}(u)$ according to our usual convention, and when $n=1$, we abbreviate $T_{[k]}(u) := T_{[k]1}(u)$. Now we may rewrite the definition of Δ in the form

$$\Delta(T(u)) := T_{[1]}(u)T_{[2]}(u),$$

which is most suitable for our purposes.

Let us verify the 'main' axiom, the compatibility of the product and the coproduct. This means that Δ is an algebra morphism of $\mathrm{Y}(N)$ to $\mathrm{Y}(N)\otimes\mathrm{Y}(N)$ or, by the definition of the Yangian, that $\Delta(T(u))$ satisfies the ternary relation (1.8.1), or else that

(1)
$$R_{12}(u-v)T_{[1]1}(u)T_{[2]1}(u)T_{[1]2}(v)T_{[2]2}(v) = T_{[1]2}(v)T_{[2]2}(v)T_{[1]1}(u)T_{[2]1}(u)R_{12}(u-v).$$

The key observation here is the fact that $T_{[2]1}(u)$ and $T_{[1]2}(v)$, as well as $T_{[1]1}(u)$ and $T_{[2]2}(v)$, commute. Using this, we transform the left hand side of (1) to the right one as follows. We interchange first the commuting T-matrices $T_{[2]1}(u)$ and $T_{[1]2}(v)$, then $T_{[1]1}(u)$ with $T_{[1]2}(v)$ using the ternary relation

$$R_{12}(u-v)T_{[1]1}(u)T_{[1]2}(v) = T_{[1]2}(v)T_{[1]1}(u)R_{12}(u-v),$$

then $T_{[2]1}(u)$ with $T_{[2]2}(v)$ using the ternary relation again, and finally we interchange the commuting T-matrices $T_{[1]1}(u)$ and $T_{[2]2}(v)$. The result of these transformations is the right hand side of (1).

Other axioms follow directly from the definitions of Δ, S and ϵ.

1.29. Remark. It is easily verified that the Yangian $\mathrm{Y}(N)$ is a deformation of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}(N)[x])$ not only as an algebra (Remark 1.27) but as a Hopf algebra too.

1.30. Remark. The coproduct Δ is not cocommutative.

1.31. Comments. The main information about the structure of (general) Yangians is contained in Drinfeld's works [D1, D3, D4].

Concerning the R-matrix formalism see for instance the papers Takhtajan–Faddeev [TF], Kulish–Sklyanin [KS2], Reshetikhin–Takhtajan–Faddeev [RTF].

The fact that the commutation relations of $\mathfrak{gl}(N)$ can be written in an R-matrix form shows that the Yangian $\mathrm{Y}(N)$ is a natural 'superstructure' over $\mathrm{U}(\mathfrak{gl}(N))$. The existence of a projection $\mathrm{Y}(N) \to \mathrm{U}(\mathfrak{gl}(N))$ gives rise to a connection between the Yangians and conventional representation theory (see Cherednik [C3] and Nazarov–Tarasov [NT] for some applications).

Prior to the appearance of the Yangians, the idea of combining generators of a classical Lie algebra into a matrix was used in the work of Perelomov–Popov [PP] and in the works of Bracken–Green [BG] and Green [Gr] on the so-called characteristic identities.

The Poincaré–Birkhoff–Witt theorem (PBW) for general Yangians is due to V.G.Drinfeld. As he has communicated to one of the authors, he derived this theorem from the PBW for quantized loop algebras. Another proof of PBW has recently been given in Levendorskiĭ's note [L1]. Our proof of PBW, presented in Subsection 1.22, follows the approach of Olshanskiĭ's paper [O1]; see especially Lemma 2.1.11 in [O1].

One of Drinfeld's results ([D1, Theorem 2]) shows that the Yangians admit a characterization as the canonical deformations of the current Lie algebras, where 'canonical' means 'satisfying certain natural conditions'.

The coproduct Δ, the antipodal map S and the shift automorphisms of $\mathrm{Y}(N)$ play a key role in constructing the finite-dimensional representations of $\mathrm{Y}(N)$.

2. The quantum determinant qdet $T(u)$ and the center of $\mathrm{Y}(N)$

Here we introduce the quantum determinant of the matrix $T(u)$, which is a formal power series in u^{-1} with coefficients from the Yangian $\mathrm{Y}(N)$. We prove that all the coefficients belong to the center of $\mathrm{Y}(N)$, that they are algebraically independent and generate the whole center. We introduce the Yangian for the Lie algebra $\mathfrak{sl}(N)$ and prove that the algebra $\mathrm{Y}(N)$ is isomorphic to the tensor product of its center and the Yangian for $\mathfrak{sl}(N)$. We will keep to the notation of Section 1.

2.1. Let $u_1, \dots, u_m$ be formal variables. Set

$$R(u_1, \dots, u_m) := (R_{m-1,m})(R_{m-2,m} R_{m-2,m-1}) \cdots (R_{1m} \cdots R_{12}), \tag{1}$$

where we abbreviate $R_{ij} := R_{ij}(u_i - u_j)$.

Proposition. *We have the following fundamental identity:*

$$R(u_1, \dots, u_m)\, T_1(u_1) \cdots T_m(u_m) = T_m(u_m) \cdots T_1(u_1)\, R(u_1, \dots, u_m). \tag{2}$$

Proof. To simplify the notation, set $T_i := T_i(u_i)$. First, let us check the identity

$$(R_{1m} \cdots R_{12}) T_1 (T_2 \cdots T_m) = (T_2 \cdots T_m) T_1 (R_{1m} \cdots R_{12}). \tag{3}$$

Indeed, the left hand side of (3) equals

$$\begin{aligned}(R_{1m} \cdots R_{12}) T_1 (T_2 \cdots T_m) &= R_{1m} \cdots R_{13} (R_{12} T_1 T_2) T_3 \cdots T_m \\ &= R_{1m} \cdots R_{13} (T_2 T_1 R_{12}) T_3 \cdots T_m \\ &= T_2 (R_{1m} \cdots R_{13}) T_1 (T_3 \cdots T_m) R_{12},\end{aligned}$$

where the passage from the first to the second line is based on the ternary relation, and the last transformation is justified by the fact that the matrices R_{ij} and T_k with disjoint indices are pairwise permutable. Repeating the same procedure we can interchange T_1 with $T_3, \dots, T_m$. This proves (3).

Next, observe that

$$R(u_1, \dots, u_m) = R(u_2, \dots, u_m)\, (R_{1m} \dots R_{12}). \tag{4}$$

Using this and (3), we can interchange T_1 with $(T_2 \cdots T_m)$ as follows:

$$\begin{aligned}R(u_1, \dots, u_m) T_1 T_2 \cdots T_m &= R(u_2, \dots, u_m)\, (R_{1m} \cdots R_{12}) T_1 T_2 \cdots T_m \\ &= R(u_2, \dots, u_m)\, (T_2 \cdots T_m) T_1 (R_{1m} \cdots R_{12}).\end{aligned}$$

Similarly we interchange T_2 with $(T_3 \cdots T_m)$ etc. Finally we arrive at the right hand side of (2).

2.2. Remark. Let u and v be formal variables and $c \in \mathbb{C}$ a constant. In contrast to the case of the algebra $\mathrm{Y}(N)[[u, v]]$, in the algebra $\mathrm{Y}(N)[[u^{-1}, v^{-1}]]$ it is possible to perform the specialization $v = u - c$. This means that there exists a natural algebra morphism

$$\mathrm{Y}(N)[[u^{-1}, v^{-1}]] \to \mathrm{Y}(N)[[u^{-1}]] \tag{1}$$

such that

$$\sum_{k,l=0}^{\infty} a_{kl} u^{-k} v^{-l} \mapsto \sum_{k,l=0}^{\infty} a_{kl} u^{-k} (u-c)^{-l} = \sum_{k,l=0}^{\infty} a_{kl} u^{-k-l} (1 + \sum_{r=1}^{\infty} c^r u^{-r}).$$

Also note that this specialization is compatible with the localization relative to $u^{-1} - v^{-1}$ provided $c \neq 0$. This remark will allow us to use the fundamental identity (2.1.2) when $u_1, \ldots, u_m$ are not independent but subject to certain relations with each other.

2.3. Let $\mathfrak{S}_m$ denote the symmetric group realized as the group of permutations of the set $\{1, \ldots, m\}$ and let

$$a_m = \sum_{p \in \mathfrak{S}_m} \operatorname{sgn}(p) \cdot p \in \mathbb{C}[\mathfrak{S}_m] \tag{1}$$

denote the antisymmetrizer in the group ring. Consider the natural action of $\mathfrak{S}_m$ in the tensor space $\mathcal{E}^{\otimes m}$ and denote by A_m the image of the normalized antisymmetrizer $(m!)^{-1} a_m$.

Proposition. *If* $u_i - u_{i+1} = 1$ *for* $i = 1, \ldots, m-1$ *then*

$$R(u_1, \ldots, u_m) = m!\, A_m. \tag{2}$$

Proof. Let $p_{ij} \in \mathfrak{S}_m$ denote the transposition (i, j). Then, in the notation of Proposition 2.1,

$$R_{ij} = \text{ the image of } \quad 1 - \frac{p_{ij}}{u_i - u_j}.$$

Hence (2) is provided by the following 'multiplicative formula' for the antisymmetrizer $a_m \in \mathbb{C}[\mathfrak{S}_m]$:

$$a_m = \prod_{k=1}^{m-1}{}^{\leftarrow} \prod_{l=k+1}^{m}{}^{\leftarrow} (1 - \frac{p_{kl}}{l-k}). \tag{3}$$

Here and below the symbol $\prod^{\leftarrow}$ means that the factors in the product are written from right to left.

Denoting the right hand side of (3) by b_m, let us prove $a_m = b_m$ by induction on m. For $m = 2$ this is obvious:

$$b_2 = 1 - \frac{p_{12}}{2-1} = 1 - p_{12} = a_2.$$

Now, assuming that $m > 2$ and $a_{m-1} = b_{m-1}$, let us check that $a_m = b_m$.

Let $\mathfrak{S}_{m-1}$ be identified with the stabilizer of m in $\mathfrak{S}_m$ so that $\mathbb{C}[\mathfrak{S}_{m-1}]$ is contained in $\mathbb{C}[\mathfrak{S}_m]$. Observe that

$$a_m = (1 - p_{1m} - \cdots - p_{m-1,m}) a_{m-1}. \tag{4}$$

On the other hand, observe that in the double product (3) all the factors with $l = m$ may be moved to the left, so that we obtain

$$b_m = \{ \prod_{k=1}^{m-1} {}^{\leftarrow} (1 - \frac{p_{km}}{m-k}) \} \, b_{m-1} = \{ \prod_{k=1}^{m-1} {}^{\leftarrow} (1 - \frac{p_{km}}{m-k}) \} \, a_{m-1} \, 1; \tag{5}$$

the assumption $a_{m-1} = b_{m-1}$ has been used here.

Now we will consecutively open the brackets in the right hand side of (5). First do this in the factor with $k = 1$ which is the extreme right:

$$b_m = \{ \prod_{k=2}^{m-1} {}^{\leftarrow} (1 - \frac{p_{km}}{m-k}) \} \, a_{m-1} \; - \; \frac{1}{m-1} \{ \prod_{k=2}^{m-1} {}^{\leftarrow} (1 - \frac{p_{km}}{m-k}) \} \, p_{1m} \, a_{m-1}. \tag{6}$$

However, $p_{km} p_{1m} = p_{1m} p_{k1}$ for $k = 2, \ldots, m-1$, so that

$$\frac{1}{m-1} \{ \prod_{k=2}^{m-1} {}^{\leftarrow} (1 - \frac{p_{km}}{m-k}) \} \, p_{1m} = \frac{p_{1m}}{m-1} (1 - \frac{p_{m-1,1}}{1}) \cdots (1 - \frac{p_{21}}{m-2}) \, a_{m-1}. \tag{7}$$

Since $p_{k1} a_{m-1} = -a_{m-1}$ for $k = 2, \ldots, m-1$, the right hand side of (7) equals

$$\frac{p_{1m}}{m-1} (1 + \frac{1}{1}) \cdots (1 + \frac{1}{m-2}) \, a_{m-1} = \frac{p_{1m}}{m-1} \, \frac{2}{1} \, \frac{3}{2} \, \cdots \, \frac{m-1}{m-2} \, a_{m-1} = p_{1m} a_{m-1}.$$

Substituting this into (6), we obtain (compare with (5))

$$b_m = \{ \prod_{k=2}^{m-1} {}^{\leftarrow} (1 - \frac{p_{km}}{m-k}) \} \, a_{m-1} \; - \; p_{1m} a_{m-1}. \tag{8}$$

Next we open the brackets in the factor with $k = 2$, repeat the same transformations etc. Finally we arrive at the last factor $(1 - p_{m-1,m}/1)$ for which no transformations are required, and obtain

$$b_m = (1 - p_{m-1,m} - \cdots - p_{1m}) \, a_{m-1}. \tag{9}$$

Combining (9) with (4) we conclude that $a_m = b_m$.

2.4. Proposition. *The following identity holds:*

$$A_N \, T_1(u) \cdots T_N(u - N + 1) = T_N(u - N + 1) \cdots T_1(u) \, A_N. \tag{1}$$

Moreover, we have

$$(1) = A_N \, T_1(u) \cdots T_N(u - N + 1) \, A_N, \tag{2}$$

$$(1) = A_N \, T_N(u - N + 1) \cdots T_1(u) \, A_N. \tag{3}$$

Proof. Applying Propositions 2.1 and 2.3 to $m = N$, we obtain (1). To prove (2), we have to multiply both sides of (1) by A_N on the right. Then the right hand side will not change since $A_N^2 = A_N$, and the left hand side will turn into the right hand side of (2). Similarly, to prove (2), it suffices to multiply (1) by A_N on the left.

2.5. Proposition. *There exists a formal series*

$$\operatorname{qdet} T(u) := 1 + d_1 u^{-1} + d_2 u^{-2} + \cdots \in \mathrm{Y}(N)[[u^{-1}]] \tag{1}$$

such that (2.4.1) *equals* $\operatorname{qdet} T(u) A_N$.

Proof. Observe that A_N is a one-dimensional projection: it projects $\mathcal{E}^{\otimes N}$ onto $\mathbb{C}\xi$, where

$$\xi := \sum_{p \in \mathfrak{S}_N} \operatorname{sgn}(p) e_{p(1)} \otimes \cdots \otimes e_{p(N)} \tag{2}$$

and $e_1, \dots, e_N$ is the canonical basis of $\mathbb{C}^N$. Hence (2.4.1) equals A_N times a formal series in u^{-1} with coefficients in $\mathrm{Y}(N)$. It remains to check that this series begins with 1; but this follows from the fact that each of the series $T_i(u-i+1)$, $i = 1, \dots, N$, begins with 1.

2.6. Definition. $\operatorname{qdet} T(u)$ is called the *quantum determinant* of the matrix $T(u)$.

2.7. Proposition. *We have*

$$\begin{aligned} \operatorname{qdet} T(u) &= \sum_{p \in \mathfrak{S}_N} \operatorname{sgn}(p)\, t_{p(1),1}(u) \cdots t_{p(N),N}(u-N+1) && (1) \\ &= \sum_{p \in \mathfrak{S}_N} \operatorname{sgn}(p)\, t_{1,p(1)}(u-N+1) \cdots t_{N,p(N)}(u). && (2) \end{aligned}$$

For example, if $N=1$, then $\operatorname{qdet} T(u) = t_{11}(u)$; if $N=2$, then

$$\begin{aligned} \operatorname{qdet} T(u) &= t_{11}(u) t_{22}(u-1) - t_{21}(u) t_{12}(u-1) \\ &= t_{11}(u-1) t_{22}(u) - t_{12}(u-1) t_{21}(u). \end{aligned} \tag{3}$$

Proof. To prove (1), we start with the identity

$$\operatorname{qdet} T(u)\, A_N = A_N\, T_1(u) \cdots T_N(u-N+1). \tag{4}$$

Let us apply both sides of (4) to the vector $e_1 \otimes \cdots \otimes e_N$. Then on the left we obtain

$$\operatorname{qdet} T(u)\, A_N(e_1 \otimes \cdots \otimes e_N) = (N!)^{-1} \operatorname{qdet} T(u) \xi, \tag{5}$$

while on the right we get

$$\begin{aligned} A_N & \sum_{i_1,\dots,i_N,j_1,\dots,j_N=1}^{N} t_{i_1 j_1}(u) \cdots t_{i_N j_N}(u-N+1)\, (E_{i_1 j_1} \otimes \cdots \otimes E_{i_N j_N})(e_1 \otimes \cdots \otimes e_N) \\ &= \sum_{i_1,\dots,i_N=1}^{N} t_{i_1,1}(u) \cdots t_{i_N,N}(u-N+1)\, A_N\, (e_{i_1} \otimes \cdots \otimes e_{i_N}). \end{aligned} \tag{6}$$

If the indices $i_1, \ldots, i_N$ are pairwise distinct, then the vector $A_N(e_{i_1} \otimes \cdots \otimes e_{i_N})$ is equal to $(N!)^{-1} \operatorname{sgn}((i_1, \ldots, i_N))\, \xi$; otherwise it equals 0. Hence (6) equals $(N!)^{-1}\, c\xi$ where c stands for the right hand side of (1). Thus $\operatorname{qdet} T(u)\xi = c\xi$, and (1) is proved.

To prove (2), we start with the identity

$$\operatorname{qdet}\, T(u)\, A_N = T_N(u - N + 1) \cdots T_1(u)\, A_N \tag{7}$$

and apply both of its sides to ξ. Since $A_N \xi = \xi$, we obtain

$$\operatorname{qdet} T(u)\, \xi = T_N(u - N + 1) \cdots T_1(u)\xi. \tag{8}$$

We may decompose the right hand side of (8) relative to the canonical basis of $\mathcal{E}^{\otimes N}$, and a similar calculation shows that the basis vector $e_1 \otimes \cdots \otimes e_N$ enters into this decomposition with coefficient equal to the right hand side of (2). This concludes the proof.

2.8. Remark. Taking the basis vector $e_{q(1)} \otimes \cdots \otimes e_{q(N)}$, $q \in \mathfrak{S}_N$, instead of $e_1 \otimes \cdots \otimes e_N$ in the above proof, one could obtain two other expressions for $\operatorname{qdet} T(u)$, namely

$$\begin{aligned} \operatorname{qdet} T(u) &= \operatorname{sgn}(q) \sum_{p \in \mathfrak{S}_N} \operatorname{sgn}(p)\, t_{p(1),q(1)}(u) \cdots t_{p(N),q(N)}(u - N + 1) && (1)\\ &= \operatorname{sgn}(q) \sum_{p \in \mathfrak{S}_N} \operatorname{sgn}(p)\, t_{q(1),p(1)}(u - N + 1) \cdots t_{q(N),p(N)}(u). && (2) \end{aligned}$$

2.9. Remark. Let $X(u) = (x_{ij}(u))_{i,j=1}^N$ be an arbitrary matrix whose entries are formal power series in u^{-1} with coefficients from $\mathrm{Y}(N)$. Then one can define the *quantum determinant* of the matrix $X(u)$ as follows (cf. (2.7.1)):

$$\operatorname{qdet}\, X(u) = \sum_{p \in \mathfrak{S}_N} \operatorname{sgn}(p)\, x_{p(1),1}(u) \cdots x_{p(N),N}(u - N + 1). \tag{1}$$

In order to avoid an ambiguity we shall sometimes enclose the matrix $X(u)$ in brackets. For example, if $N \geq 2$, then the series

$$\operatorname{qdet} T(-u) = 1 + d_1(-u)^{-1} + d_2(-u)^{-2} + \ldots$$

(see (2.5.1)) does not coincide with the quantum determinant $\operatorname{qdet}\,(T(-u))$ of the matrix $X(u) = T(-u)$.

2.10. Theorem. *$\operatorname{qdet} T(u)$ lies in the center of $\mathrm{Y}(N)$. That is, all of its coefficients are central elements.*

Proof. Consider the auxiliary tensor space $\mathcal{E}^{\otimes(N+1)}$ where the copies of $\mathcal{E}$ are enumerated by the indices $0, \ldots, N$, and consider the $N + 1$ operators

$$T_0 := T_0(v), \quad T_1 := T_1(u), \; \ldots, \; T_N := T_N(u - N + 1). \tag{1}$$

We shall prove the identity

$$T_0(v)\,\mathrm{qdet}\;T(u)\,A_N = \mathrm{qdet}\;T(u)\,T_0(v)\,A_N, \tag{2}$$

where qdet $T(u)$ is built from $T_1,\dots,T_N$ as above and A_N corresponds to the antisymmetrization relative to the indices $1,\dots,N$. It is easy to see that (2) implies the theorem. We shall derive (2) from the fundamental identity (2.1.2) in several steps.

Step 1. Applying (2.1.2) to the operators (1), we obtain

$$R(v,u,u-1,\dots,u-N+1)\,T_0T_1\cdots T_N = T_N\cdots T_1T_0\,R(v,u,u-1,\dots,u-N+1). \tag{3}$$

Since

$$R(v,u,u{-}1,\dots,u{-}N{+}1) = R(u,u{-}1,\dots,u{-}N{+}1)\prod_{i=1}^{N}{}^{\leftarrow}R_{0i} = N!\,A_N\prod_{i=1}^{N}{}^{\leftarrow}R_{0i},$$

(3) may be rewritten as follows

$$A_N\,(\prod_{i=1}^{N}{}^{\leftarrow}R_{0i})\,T_0T_1\cdots T_N = T_N\cdots T_1T_0A_N\,(\prod_{i=1}^{N}{}^{\leftarrow}R_{0i}). \tag{4}$$

Step 2. Let us prove that

$$A_N\,R_{0N}\cdots R_{01} = R_{01}\cdots R_{0N}\,A_N. \tag{5}$$

To do this, rewrite (5) as

$$A_N\,R_{01}^{-1}\cdots R_{0N}^{-1} = R_{0N}^{-1}\cdots R_{01}^{-1}\,A_N \tag{6}$$

and observe that the structure of this formula is quite similar to that of (2.4.1).

Now, if we examine the proof of (2.4.1), then we will see that it is based entirely on the identity

$$R_{ij}T_iT_j = T_jT_iR_{ij}.$$

But the same identity holds when T_i, T_j are replaced by R_{0i}^{-1}, R_{0j}^{-1} respectively. Indeed,

$$R_{ij}R_{0i}^{-1}R_{0j}^{-1} = R_{0j}^{-1}R_{0i}^{-1}R_{ij}$$

is simply equivalent to the Yang–Baxter equation (see (1.5.1))

$$R_{0i}R_{0j}R_{ij} = R_{ij}R_{0j}R_{0i}.$$

Hence the proof of (2.4.1) works for (6) as well.

Step 3. Using (5), we transform the left hand side of (4) as follows:

$$\begin{aligned}
&A_N\,(R_{0N}\cdots R_{01})\,T_0T_1\cdots T_N\\
&\quad= A_N^2\,(R_{0N}\cdots R_{01})\,T_0T_1\cdots T_N\\
&\quad= A_N\,(R_{01}\cdots R_{0N})\,A_NT_0T_1\cdots T_N \quad \text{by (5)}\\
&\quad= A_N\,(R_{01}\cdots R_{0N})\,A_N^2T_0T_1\cdots T_N.
\end{aligned}$$

Since T_0 and A_N commute, this equals

$$\begin{aligned} &A_N\,(R_{01}\cdots R_{0N})\,A_N T_0 A_N T_1\cdots T_N \\ &\quad = A_N\,(R_{01}\cdots R_{0N})\,A_N T_0\,\mathrm{qdet}\,T(u)\,A_N. \end{aligned}$$

By applying similar transformations to the right hand side of (4) we arrive at the following identity:

$$\begin{aligned} &A_N\,(R_{01}\cdots R_{0N})\,A_N T_0(v)\,\mathrm{qdet}\,T(u)\,A_N \\ &\quad = \mathrm{qdet}\,T(u)\,A_N T_0(v) A_N\,(R_{01}\cdots R_{0N})\,A_N. \end{aligned} \tag{7}$$

Step 4. Let us prove the identity

$$A_N\,(R_{01}\dots R_{0N})\,A_N = f(u,v)\,A_N, \tag{8}$$

where $f(u,v)$ is a *non zero* element of an appropriate extension of $\mathbb{C}$ containing

$$(v-u_i)^{-1} = (v-u+i-1)^{-1}, \qquad \text{where} \quad i=1,\dots,N.$$

Indeed, write $\mathcal{E}^{\otimes(N+1)}$ as $\mathcal{E}\otimes\mathcal{E}^{\otimes N}$ and recall that A_N is a one–dimensional projection in $\mathcal{E}^{\otimes N}$. It follows that the left hand side of (8) may be written as $X\otimes A_N$, where X is an operator in $\mathcal{E}$ (more correctly, an element of $\mathrm{End}\,\mathcal{E}$ tensored with our extension of $\mathbb{C}$).

On the other hand, the whole picture is clearly equivariant relative to the action of the group $\mathrm{Aut}\,\mathcal{E} = \mathrm{GL}(N,\mathbb{C})$, so that X is a scalar operator, i.e., $X = f(u,v)\cdot 1$.

It remains to check that $f(u,v)\neq 0$. To do this, take as the above-mentioned extension the algebra $\mathbb{C}[u][[v^{-1}]]$ and observe that

$$\begin{aligned} R_{0i} &= 1 - \frac{P_{0i}}{v-u+i-1} \\ &= 1 - P_{0i}v^{-1}(1+(u-i+1)v^{-1}+(u-i+1)^2v^{-2}+\dots). \end{aligned}$$

This implies that $f(u,v)$, as a power series in v^{-1} with coefficients in $\mathbb{C}[u]$, begins with 1. Thus $f(u,v)\neq 0$.

Step 5. Now, by (8) identity (7) reads as follows:

$$T_0(v)\mathrm{qdet}\,T(u)\,A_N = \mathrm{qdet}\,T(u)\,T_0(v)\,A_N.$$

This clearly means that $\mathrm{qdet}\,T(u)$ is central.

2.11. Remark. Theorem 2.10 may be applied to obtain the (well-known) description of the center of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}(N))$. To do this one uses the homomorphism ξ (see (1.16.1)). It is easy to see that

$$u(u-1)\dots(u-N+1)\,\xi(\mathrm{qdet}\,T(u)) =$$

$$(1) \qquad \det \begin{pmatrix} E_{11}+u & E_{12} & \dots & E_{1N} \\ E_{21} & E_{22}+u-1 & \dots & E_{2N} \\ \vdots & \vdots & & \vdots \\ E_{N1} & E_{N2} & \dots & E_{NN}+u-N+1 \end{pmatrix},$$

where the 'determinant' $\det A$ of a noncommutative matrix $A = (a_{ij})_{i,j=1}^N$ is defined as

$$\det A := \sum_{p \in \mathfrak{S}_N} \operatorname{sgn}(p) a_{p(1),1} \dots a_{p(N),N}.$$

Let us denote the right hand side of (1) by $Q(u)$. Then

$$Q(u) = u^N + z_1 u^{N-1} + \dots + z_N, \qquad z_i \in \mathrm{U}(\mathfrak{gl}(N)).$$

Theorem 2.10 implies that all the coefficients z_i belong to the center of $\mathrm{U}(\mathfrak{gl}(N))$. By using the Harish-Chandra homomorphism one can show that the elements $z_1, \dots, z_N$ are algebraically independent and hence generate the whole center of the algebra $\mathrm{U}(\mathfrak{gl}(N))$ (cf. Theorem 2.13). Moreover, the polynomial $\tilde{Q}(u) = Q(-u+N-1)$ may be considered as the 'characteristic polynomial' for the matrix $E = (E_{ij})$ (see Remark 1.18), and the following analogue of the Cayley–Hamilton theorem holds:

$$(2) \qquad \tilde{Q}(E) = 0;$$

see Subsection 2.24 for more references and comments.

2.12. Let x be a formal variable. The following auxiliary assertion will be used in the proof of Theorem 2.13.

Proposition. *Let $\mathfrak{a}$ be a Lie algebra whose center is trivial. Then the center of the universal enveloping algebra $\mathrm{U}(\mathfrak{a}[x])$ is also trivial.*

Proof. Let us use the fact that for any Lie algebra $\mathfrak{g}$ the symmetrization map $\mathrm{S}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})$ yields an isomorphism of $\mathfrak{g}$-modules $\mathrm{U}(\mathfrak{g})$ and $\mathrm{S}(\mathfrak{g})$. Then we have to prove that the symmetric algebra $\mathrm{S}(\mathfrak{a}[x])$ regarded as the adjoint $\mathfrak{a}[x]$-module, has no nontrivial invariant elements. Let $\{e_1, \dots, e_n\}$ be a basis of $\mathfrak{a}$ and

$$[e_i,\ e_j] = \sum_{k=1}^n c_{ij}^k \, e_k,$$

where c_{ij}^k are structure constants. The monomials

$$\prod_{i,r} (e_i \, x^r), \qquad \text{(finite product)}$$

then form a basis of $\mathrm{S}(\mathfrak{a}[x])$. Now let $A \in \mathrm{S}(\mathfrak{a}[x])$ be an $\mathfrak{a}[x]$-invariant element and m be the maximal integer such that the element $e_i\,x^m$ occurs in A for some $i \in \{1, \dots, n\}$. Then A has the form

$$A = \sum_d A_d (e_1\,x^m)^{d_1} \dots (e_n\,x^m)^{d_n},$$

where $d = (d_1, \dots, d_n)$, $d_1 \geq 0, \dots, d_n \geq 0$, and A_d is a polynomial in the variables $e_i\,x^r$ with $r < m$. By the definition of A the following relation holds:

$$\text{(1)} \qquad \mathrm{ad}(e_i\,x)(A) = 0 \quad \text{for} \quad i = 1, \dots, n.$$

The component of the left hand side of (1) that contains the elements of the form $e_k\,x^{m+1}$ must be zero, i.e.,

$$\text{(2)} \qquad \sum_d A_d \sum_{j=1}^n d_j (e_1\,x^m)^{d_1} \dots (e_j\,x^m)^{d_j - 1} \dots (e_n\,x^m)^{d_n} \sum_{k=1}^n c_{ij}^k\, e_k x^{m+1} = 0.$$

Taking the coefficient of $e_k\,x^{m+1}$ in this equality, we obtain:

$$\sum_d A_d \sum_{j=1}^n d_j c_{ij}^k (e_1\,x^m)^{d_1} \dots (e_j\,x^m)^{d_j - 1} \dots (e_n\,x^m)^{d_n} = 0.$$

Thus, for any multi-index $d' = (d'_1, \dots, d'_n)$ with nonnegative components we have

$$\text{(3)} \qquad \sum_{j=1}^n A_{d' + \delta_j} (d'_j + 1) c_{ij}^k = 0, \qquad 1 \leq i, k \leq n,$$

where $d' + \delta_j$ denotes the multi-index $(d'_1, \dots, d'_j + 1, \dots, d'_n)$. Fix d' and observe that the elements

$$e'_j = (d'_j + 1) e_j, \qquad j = 1, \dots, n,$$

also form a basis of $\mathfrak{a}$. Since the center of $\mathfrak{a}$ is trivial, the system of linear equations

$$[e_i, \sum_{j=1}^n x_j e'_j] = 0, \qquad i = 1, \dots, n$$

for the variables x_j, has only trivial solution. This system can be rewritten as the system of n^2 equations

$$\sum_{j=1}^n x_j (d'_j + 1) c_{ij}^k = 0, \qquad i, k = 1, \dots, n.$$

Comparing this with (3) we see that $A_{d' + \delta_j} = 0$. Thus we obtain that $A_d = 0$ for all $d \neq 0$, which proves the proposition.

2.13. Theorem. *The coefficients $d_1, d_2, \dots$ of* $\operatorname{qdet} T(u)$ *are algebraically independent and generate the whole center of* $\mathrm{Y}(N)$.

Proof. The key idea is to reduce this assertion to an analogous one for the algebra $\mathrm{gr}_2\mathrm{Y}(N)$. Recall that $\mathrm{gr}_2\mathrm{Y}(N)$ is isomorphic to $\mathrm{U}(\mathfrak{gl}(N)[x])$; see Theorem 1.26.

Step 1. Set

$$Z := E_{11} + \dots + E_{NN}, \tag{1}$$

so that $\mathfrak{gl}(N) = \mathbb{C}Z \oplus \mathfrak{sl}(N)$. Then for any $M = 1, 2, \dots$, the M-th coefficient d_M of $\operatorname{qdet} T(u)$ has degree $M-1$ relative to $\deg_2(\cdot)$, and its image in the $(M-1)$-th component of $\mathrm{gr}_2\mathrm{Y}(N)$ coincides with Zx^{M-1}.

Indeed, by formula (2.7.1) for $\operatorname{qdet} T(u)$, d_M is a linear combination of monomials of the form

$$t^{(M_1)}_{p(1),1} \cdots t^{(M_N)}_{p(N),N}, \quad \text{where} \quad M_1 + \dots + M_N \le M. \tag{2}$$

By Definition 1.20 of $\deg_2(\cdot)$ it is clear that the degree of (2) is strictly less than $M-1$ with the exception of the case when, for some i, we have $M_j = \delta_{ij}M$, $1 \le j \le N$. Assume this is exactly the case. Then, since $t^{(0)}_{kl} = \delta_{kl}$, the permutation p has to be trivial, otherwise the monomial (2) vanishes. Hence,

$$d_M = t^{(M)}_{11} + \dots + t^{(M)}_{NN} + (\text{terms of degree } < M-1), \tag{3}$$

which proves the assertion. This implies that the elements $d_1, d_2, \dots$ are algebraically independent.

Step 2. It remains to prove the following claim: the center of the algebra $\mathrm{U}(\mathfrak{gl}(N)[x])$ is generated by $Z, Zx, Zx^2, \dots$. Since

$$\mathrm{U}(\mathfrak{gl}(N)[x]) = \mathbb{C}[Z, Zx, Zx^2, \dots] \otimes \mathrm{U}(\mathfrak{sl}(N)[x]),$$

this claim is equivalent to the triviality of the center of $\mathrm{U}(\mathfrak{sl}(N)[x])$. But this follows from Proposition 2.12, because the center of the Lie algebra $\mathfrak{sl}(N)$ is trivial. This concludes the proof of the theorem.

2.14. Definition. Consider the subalgebra in $\mathrm{Y}(N)$

$$\mathrm{SY}(N) := \{y \in \mathrm{Y}(N) \,|\, \mu_f(y) = y \ \text{ for every } \ f\} \tag{1}$$

(see (1.12.2)). This subalgebra is called the *Yangian of the Lie algebra* $\mathfrak{sl}(N)$.

2.15. The following statement will be frequently used later on.

Proposition. *Let $\mathcal{A}$ be an arbitrary commutative associative algebra and u be a formal variable. Then for any series*

$$a(u) = 1 + a_1u^{-1} + a_2u^{-2} + \dots \in \mathcal{A}[[u^{-1}]]$$

and any positive integer N there exists a unique series

$$\tilde{a}(u) = 1 + \tilde{a}_1 u^{-1} + \tilde{a}_2 u^{-2} + \cdots \in \mathcal{A}[[u^{-1}]]$$

such that

$$a(u) = \tilde{a}(u)\tilde{a}(u-1)\dots\tilde{a}(u-N+1). \tag{1}$$

Proof. Write (1) in terms of the coefficients of the series $a(u)$ and $\tilde{a}(u)$:

$$a_k = N\tilde{a}_k + (\dots), \qquad k = 1, 2, \dots,$$

where $(\dots)$ stands for a certain polynomial in the variables $\tilde{a}_1, \dots, \tilde{a}_{k-1}$. Hence, each element $\tilde{a}_k$ may be uniquely expressed as a polynomial in $a_1, \dots, a_k$, which proves the proposition.

2.16. Denote by $\mathrm{Z}(N)$ the center of the algebra $\mathrm{Y}(N)$. We have

Proposition. *The algebra* $\mathrm{Y}(N)$ *is isomorphic to the tensor product of its subalgebras* $\mathrm{Z}(N)$ *and* $\mathrm{SY}(N)$*:*

$$\mathrm{Y}(N) = \mathrm{Z}(N) \otimes \mathrm{SY}(N).$$

In particular, the center of $\mathrm{SY}(N)$ *is trivial.*

Proof. Let us apply Proposition 2.15 to $\mathcal{A} = \mathrm{Z}(N)$ and $a(u) = \operatorname{qdet} T(u)$. The corresponding element $\tilde{a}(u)$ will be denoted by $\tilde{d}(u)$.

Step 1. Consider the automorphism μ_f (see (1.12.2)) and prove that

$$\mu_f(\tilde{d}(u)) = \tilde{d}(u) f(u). \tag{1}$$

It follows from Proposition 2.7 that

$$\mu_f(\operatorname{qdet} T(u)) = \operatorname{qdet} T(u) f(u) f(u-1) \dots f(u-N+1). \tag{2}$$

On the other hand, by the definition of $\tilde{d}(u)$,

$$\mu_f(\operatorname{qdet} T(u)) = \mu_f(\tilde{d}(u))\mu_f(\tilde{d}(u-1))\dots\mu_f(\tilde{d}(u-N+1)).$$

Comparing this with (2) and applying Proposition 2.15 to $a(u) = \mu_f(\operatorname{qdet} T(u))$, we obtain (1).

Step 2. Let us prove that

$$\mathrm{Y}(N) = \mathrm{Z}(N)\,\mathrm{SY}(N). \tag{3}$$

Set

$$\tau_{ij}(u) = \tilde{d}(u)^{-1}\, t_{ij}(u), \qquad 1 \le i, j \le N.$$

Then by (1) we have $\mu_f(\tau_{ij}(u)) = \tau_{ij}(u)$ for any series f. Hence, all the coefficients $\tau_{ij}^{(M)}$ lie in $\mathrm{SY}(N)$, and (3) follows from the decomposition $t_{ij}(u) = \tilde{d}(u)\tau_{ij}(u)$.

Step 3. Let n be the minimum positive integer such that there exists a nonzero polynomial $P \in \mathrm{SY}(N)[x_1, \dots, x_n]$ for which $P(\tilde{d}_1, \dots, \tilde{d}_n) = 0$. Set $f(u) = 1 + au^{-n}$, $a \in \mathbb{C}$. Then

$$\mu_f : P(\tilde{d}_1, \dots, \tilde{d}_n) \mapsto P(\tilde{d}_1, \dots, \tilde{d}_{n-1}, \tilde{d}_n + a).$$

Thus, $P(\tilde{d}_1, \dots, \tilde{d}_{n-1}, \tilde{d}_n + a) = 0$ for any $a \in \mathbb{C}$. This means that the polynomial P does not depend on x_n, which contradicts the choice of n. The proposition is proved.

2.17. Corollary. *The elements* $\tau_{ij}^{(M)}$, $1 \le i, j \le N$, $M = 1, 2, \ldots$, *introduced in the proof of Proposition* 2.16, *are generators of the algebra* $\mathrm{SY}(N)$.

Proof. It follows from the proof of Proposition 2.16 that any element $y \in \mathrm{Y}(N)$ can be uniquely written as

$$y = \sum_a z_a \otimes S_a, \tag{1}$$

where $\{z_a\}$ is the basis of $\mathrm{Z}(N)$, formed by the monomials $\tilde{d}_{i_1} \ldots \tilde{d}_{i_k}$, $1 \le i_1 \le \cdots \le i_k$, and S_a are (non-commutative) polynomials in the variables $\tau_{ij}^{(M)}$. On the other hand, if $y \in \mathrm{SY}(N)$, then (1) obviously must have the form $y = 1 \otimes y$. Hence, y is a polynomial in $\tau_{ij}^{(M)}$.

2.18. Corollary. *The algebra* $\mathrm{SY}(N)$ *is isomorphic to the factor-algebra*

$$\mathrm{Y}(N)/(\mathrm{qdet}\, T(u) = 1) = \mathrm{Y}(N)/(d_1 = d_2 = \cdots = 0).$$

Proof. Let I be the ideal of $\mathrm{Y}(N)$, generated by all the elements $d_1, d_2, \ldots$ (or equivalently, by all the elements $\tilde{d}_1, \tilde{d}_2, \ldots$). Proposition 2.16 immediately proves that

$$\mathrm{Y}(N) = \mathrm{I} \oplus \mathrm{SY}(N).$$

2.19. Proposition. *We have the equality*

$$\Delta(\mathrm{qdet}\, T(u)) = \mathrm{qdet}\, T(u) \otimes \mathrm{qdet}\, T(u). \tag{1}$$

Proof. We will regard Δ as a homomorphism of algebras

$$\Delta : \mathrm{Y}(N) \otimes \mathrm{End}\, \mathcal{E}^{\otimes N} \to \mathrm{Y}(N) \otimes \mathrm{Y}(N) \otimes \mathrm{End}\, \mathcal{E}^{\otimes N},$$

which is identical on $\mathrm{End}\, \mathcal{E}^{\otimes N}$. Using the notation of Subsection 1.28, we obtain

$$\Delta(\mathrm{qdet}\, T(u) A_N) = \Delta(A_N T_1 \ldots T_N) = A_N T_{[1]1} T_{[2]1} \ldots T_{[1]N} T_{[2]N}, \tag{2}$$

where $T_i = T_i(u - i + 1)$. Note that the elements $T_{[i]j}$ and $T_{[k]l}$ commute if $i \ne k$ and $j \ne l$. Hence, (2) may be rewritten as

$$A_N T_{[1]1} \ldots T_{[1]N} T_{[2]1} \ldots T_{[2]N} =$$

$$\mathrm{qdet}\, T_{[1]}(u) A_N T_{[2]1} \ldots T_{[2]N} = \mathrm{qdet}\, T_{[1]}(u) \mathrm{qdet}\, T_{[2]}(u) A_N.$$

This implies (1).

2.20. Corollary. $\Delta(\tilde d(u)) = \tilde d(u)\otimes \tilde d(u)$.

Proof. Recall that $\tilde d(u)$ is defined by the equality

$$\operatorname{qdet} T(u) = \tilde d(u)\tilde d(u-1)\dots \tilde d(u-N+1)$$

(see Subsection 2.16). Hence,

$$\Delta(\operatorname{qdet} T(u)) = \Delta(\tilde d(u))\Delta(\tilde d(u-1))\dots\Delta(\tilde d(u-N+1)).$$

On the other hand, by Proposition 2.19,

$$\Delta(\operatorname{qdet} T(u)) = \tilde d(u)\dots\tilde d(u-N+1)\otimes\tilde d(u)\dots\tilde d(u-N+1) =$$
$$(\tilde d(u)\otimes\tilde d(u))(\tilde d(u-1)\otimes\tilde d(u-1))\dots(\tilde d(u-N+1)\otimes\tilde d(u-N+1)).$$

Applying Proposition 2.15 to the algebra $\mathcal{A} = \mathrm{Z}(N)\otimes\mathrm{Z}(N)$ and the element $a(u) = \Delta(\operatorname{qdet} T(u))\in\mathcal{A}[[u^{-1}]]$, we complete the proof.

2.21. Proposition. *The subalgebra* $\mathrm{SY}(N)\subset\mathrm{Y}(N)$ *is a Hopf algebra, that is, the coproduct, antipode and counit on* $\mathrm{SY}(N)$ *can be obtained by restricting those of the Hopf algebra* $\mathrm{Y}(N)$ *to* $\mathrm{SY}(N)$.

Proof. It follows from Corollary 2.20 that

$$\Delta(\tilde d(u)^{-1}) = \tilde d(u)^{-1}\otimes\tilde d(u)^{-1}.$$

Hence,

$$\begin{aligned}\Delta(\tau_{ij}(u)) =&\Delta(\tilde d(u)^{-1}t_{ij}(u))\\ &= (\tilde d(u)^{-1}\otimes\tilde d(u)^{-1})\sum_{a=1}^{N} t_{ia}(u)\otimes t_{aj}(u) = \sum_{a=1}^{N}\tau_{ia}(u)\otimes\tau_{aj}(u).\end{aligned}$$

By Corollary 2.17 the last equality proves that $\Delta(\mathrm{SY}(N))\subset\mathrm{SY}(N)\otimes\mathrm{SY}(N)$. We omit the verification of the axioms for the antipode and counit, which are obvious.

2.22. Note that $\mathrm{SY}(N)$ inherits both the filtrations of $\mathrm{Y}(N)$ defined in Subsection 1.20. Now we will describe the associated graded algebras $\mathrm{gr}_1\mathrm{SY}(N)$ and $\mathrm{gr}_2\mathrm{SY}(N)$; the result will be analogous to Theorems 1.22 and 1.26.

Proposition. *The algebra* $\mathrm{gr}_1\mathrm{SY}(N)$ *is commutative and isomorphic to the algebra of polynomials in the generators*

$$(1)\qquad \bar t_{ij}^{(M)}\quad (i\ne j),\quad \bar t_{kk}^{(M)} - \bar t_{k+1,k+1}^{(M)};\qquad k=1,\dots,N-1;\ M=1,2,\dots,$$

where the bar has the same meaning as in Subsection 1.22.

Proof. The commutativity of $\mathrm{gr}_1\mathrm{SY}(N)$ follows from that of $\mathrm{gr}_1\mathrm{Y}(N)$. Furthermore, according to (2.7.1),

$$(2)\qquad \bar d_M = \bar t_{11}^{(M)}+\dots+\bar t_{NN}^{(M)} + (\dots),$$

where $(\dots)$ stands for a sum of (commutative) monomials in letters $\bar t_{ij}^{(L)}$ with degrees $L<M$. Together with the Poincaré–Birkhoff–Witt theorem (1.22), this implies that the elements $\bar d_M$ and (1) form a system of algebraically independent generators of

$\mathrm{gr}_1\mathrm{Y}(N)$. Since the elements d_M are central in $\mathrm{Y}(N)$ it easily follows that the factorization of $\mathrm{Y}(N)$ by them results simply in eliminating the

elements $\bar d_M$ from $\mathrm{gr}_1\mathrm{Y}(N)$.

2.23. Proposition. *The algebra* $\mathrm{gr}_2\mathrm{SY}(N)$ *is isomorphic to* $\mathrm{U}(\mathfrak{sl}(N)[x])$.

Proof. Consider the ideal I of $\mathrm{Y}(N)$, introduced in the proof of Corollary 2.18. By using Step 1 of the proof of Theorem 2.13, we obtain that the image of $\mathrm{gr}_2\mathrm{I}$ under the isomorphism

$$\mathrm{gr}_2\mathrm{Y}(N) \to \mathrm{U}(\mathfrak{gl}(N)[x])$$

(see Theorem 1.26) coincides with the ideal $\tilde{\mathrm{I}}$ of $\mathrm{U}(\mathfrak{gl}(N)[x])$ generated by the elements $Z, Zx, Zx^2, \dots$. It is clear that the factor-algebra $\mathrm{U}(\mathfrak{gl}(N)[x])/\tilde{\mathrm{I}}$ is isomorphic to the algebra $\mathrm{U}(\mathfrak{sl}(N)[x])$. This completes the proof.

2.24. Comments. The definition of the quantum determinant $\mathrm{qdet}\, T(u)$ primarily appeared in Izergin–Korepin [IK] in the case $N = 2$. The basic ideas and formulae associated with the quantum determinant for an arbitrary N are contained in Kulish–Sklyanin's survey paper [KS2]. The quantum determinant has been used in many papers: see Cherednik [C2, C3], Drinfeld [D3], Molev [M2], Nazarov [N1], Nazarov–Tarasov [NT], Tarasov [T1, T2]. However, to our knowledge, no detailed exposition of the properties of the quantum determinant $\mathrm{qdet}\, T(u)$ has been published. In this section we have attempted to fill this gap.

Formula (2.11.1) has a long history. It is closely related to the celebrated Capelli identity [Ca1, Ca2], which is discussed in Weyl's book on classical groups [W, Chapter II, Section 4]. The fact that the determinant (2.11.1) lies in the center of the enveloping algebra $\mathrm{U}(\mathfrak{gl}(N))$ is by no means trivial. A proof of this result proposed in Howe [H] contained an error which was corrected in Howe–Umeda [HU]. The approach to this result based on the Yangian seems to be fruitful, and we hope to return to this subject later. Note that Nazarov [N1] constructed a superanalogue of $\mathrm{qdet}\, T(u)$ (the quantum Berezinian) and applied it to the derivation of a 'super' Capelli identity; in that case the construction is much more complicated because the Berezinian is not a polynomial function of matrix coefficients [B].

Identity (2.11.2) is one of the examples of the polynomial identities satisfied by the generators of a semi-simple Lie algebra. They were investigated in the papers Bracken–Green [BG], Green [Gr], Kostant [K], O'Brien–Cant–Carey [BCC], Gould [G] and others. Nazarov and Tarasov [NT] have proved identity (2.11.2) and its q-analogue by using the properties of the quantum determinant.

The idea of using the reduction to the algebra $\mathrm{gr}_2\mathrm{Y}(N)$ in the proof of Theorem 2.13 was communicated to one of us by V.G.Drinfeld.

Following the general philosophy of Drinfeld [D4], the Yangian of $\mathfrak{sl}(N)$ must be defined as a factor-algebra of $\mathrm{Y}(N)$. The fact that it can also be realized as a subalgebra of $\mathrm{Y}(N)$ was observed by the third author and communicated to V.G.Drinfeld, who proposed in reply an elegant characterization of $\mathrm{Y}(\mathfrak{sl}(N)) \subset \mathrm{Y}(N)$ in terms of the automorphisms μ_f.

3. The twisted Yangian $\mathrm{Y}^{\pm}(N)$

In this section we introduce the twisted Yangians $\mathrm{Y}^{+}(N)$ and $\mathrm{Y}^{-}(N)$. They are defined as certain subalgebras of the Yangian $\mathrm{Y}(N)$ (Definition 3.5). We also find a realization of $\mathrm{Y}^{\pm}(N)$ via generators and defining relations. Finally we prove an analogue of the Poincaré–Birkhoff–Witt theorem for the algebras $\mathrm{Y}^{\pm}(N)$.

3.1. As before, we will denote by $\mathcal{E}$ the vector space $\mathbb{C}^N$ and by $\{e_i\}$ its canonical basis. But from now on it will be convenient to parametrize the basis vectors by the numbers $i = -n, -n+1, \dots, n-1, n$, where $n := [N/2]$ and $i = 0$ is skipped when N is even.

Let us equip $\mathcal{E}$ with a nondegenerate bilinear form $<\cdot,\cdot> = <\cdot,\cdot>_{\pm}$ which may be either symmetric or alternating:

$$< e_i, e_j >_{+} = \delta_{i,-j}, \qquad < e_i, e_j >_{-} = \mathrm{sgn}(i)\delta_{i,-j}.$$

Here the symbol $\mathrm{sgn}(i)$ equals 1 for $i > 0$ and -1 for $i < 0$. Of course, the alternating case may occur only when N is even.

Both of the cases, symmetric and alternating, will be considered simultaneously unless stated otherwise. It will be convenient to use the symbol θ_{ij} which is defined as follows:

$$\theta_{ij} := \begin{cases} 1, & \text{in the symmetric case;} \\ \mathrm{sgn}(i)\mathrm{sgn}(j), & \text{in the alternating case.} \end{cases}$$

Whenever the double sign $\pm$ or $\mp$ occurs, the upper sign corresponds to the symmetric case and the lower sign to the alternating one.

By $A \mapsto A^t$ we will denote the transposition relative to the form $<\cdot,\cdot>$. This is an antihomomorphism of the algebra $\mathrm{End}\,\mathcal{E}$; on the matrix units the transposition acts as follows:

$$(E_{ij})^t = \theta_{ij} E_{-j,-i}. \tag{1}$$

We will often use partial transpositions $t_1, t_2, \dots$ in multiple tensor products of the form $\mathcal{A} \otimes (\mathrm{End}\,\mathcal{E})^{\otimes m}$, where $\mathcal{A}$ is a certain algebra. By definition, t_k denotes the transposition corresponding to the k-th copy of $\mathcal{E}$.

3.2. Let P stand for the permutation in $\mathcal{E}^{\otimes 2}$ as usual.

Proposition. *We have:*

$$Q := P^{t_1} = P^{t_2} = \sum_{i,j} \theta_{ij} E_{-j,-i} \otimes E_{ji}, \tag{1}$$

$$Q^2 = NQ, \tag{2}$$

$$(3)\quad Q\mathcal{E}^{\otimes 2} = \mathbb{C}\xi, \quad \textit{where} \quad \xi := \begin{cases} \sum e_{-j} \otimes e_j, & \textit{in the symmetric case,} \\ \sum \operatorname{sgn}(j) e_{-j} \otimes e_j, & \textit{in the alternating case,} \end{cases}$$

$$(4)\qquad PQ = QP = \pm Q.$$

Proof. We have

$$(5)\qquad Q := P^{t_1} = \sum E^t_{ij} \otimes E_{ji} = \sum \theta_{ij} E_{-j,-i} \otimes E_{ji},$$

$$(6)\qquad P^{t_2} = \sum E_{kl} \otimes E^t_{lk} = \sum \theta_{kl} E_{kl} \otimes E_{-k,-l}.$$

Replacing (k,l) by $(-j,-i)$ in (6) and using the obvious equality $\theta_{ij} = \theta_{-j,-i}$, we see that (5) equals (6). This proves (1).

Further, taking into account (5), we obtain:

$$\begin{aligned} Q^2 &= (\sum_{i,j} \theta_{ij} E_{-j,-i} \otimes E_{ji})(\sum_{k,l} \theta_{kl} E_{-l,-k} \otimes E_{lk}) \\ &= \sum_{i,j,k,l} \theta_{ij}\theta_{kl} (E_{-j,-i} E_{-l,-k} \otimes E_{ji} E_{lk}) \\ &= \sum_{i,j,k,l} \theta_{ij}\theta_{kl}\delta_{il} (E_{-j,-k} \otimes E_{jk}) = N \sum_{k,j} \theta_{kj} (E_{-j,-k} \otimes E_{jk}) = NQ \end{aligned}$$

and

$$\begin{aligned} Qe_k \otimes e_l &= \sum_{i,j} \theta_{ij} (E_{-j,-i} \otimes E_{ji})(e_k \otimes e_l) \\ &= \sum_{i,j} \theta_{ij} \delta_{-i,k}\delta_{il} \cdot e_{-j} \otimes e_j = \delta_{-k,l} \sum_j \theta_{lj} \cdot e_{-j} \otimes e_j = \delta_{-k,l} \operatorname{sgn}(l) \cdot \xi. \end{aligned}$$

This proves (2) and (3). Finally,

$$\begin{aligned} PQ &= (\sum_{i,j} E_{ij} \otimes E_{ji})(\sum_{k,l} \theta_{kl} E_{-l,-k} \otimes E_{lk}) \\ &= \sum_{i,j,k,l} \theta_{kl} (E_{ij} E_{-l,-k} \otimes E_{ji} E_{lk}) = \sum_{i,k} \theta_{ki} E_{i,-k} \otimes E_{-i,k}. \end{aligned}$$

In the symmetric case $\theta_{ki} \equiv 1$, so that $PQ = Q$. In the alternating case $\theta_{ki} = -\theta_{k,-i}$, so that $PQ = -Q$. Thus we have verified (4).

3.3. From (3.2.1) we obtain

$$(1)\qquad R'(u) := R^{t_1}(u) = R^{t_2}(u) = 1 - \frac{Q}{u}.$$

Note that

$$(2)\qquad R(u)R'(v) = R'(v)R(u),$$

since $PQ = QP$.

Proposition. *The T-matrix satisfies the following relations:*

$$T_1^{t_1}(u)R'(u-v)T_2(v) = T_2(v)R'(u-v)T_1^{t_1}(u), \tag{3}$$

$$T_1(u)R'(-u+v)T_2^{t_2}(v) = T_2^{t_2}(v)R'(-u+v)T_1(u), \tag{4}$$

$$R(u-v)T_1^{t_1}(-u)T_2^{t_2}(-v) = T_2^{t_2}(-v)T_1^{t_1}(-u)R(u-v). \tag{5}$$

Proof. Each of these relations is equivalent to the basic ternary relation (1.8.1). Indeed, applying t_1 to both sides of (1.8.1) and repeating the same arguments as in the proof of Proposition 1.12(iv), we obtain (3).

Further, multiply both sides of (3) by P from the left and from the right. Then we obtain

$$T_2^{t_2}(u)R'(u-v)T_1(v) = T_1(v)R'(u-v)T_2^{t_2}(u), \tag{6}$$

or, which is the same,

$$T_1(v)R'(u-v)T_2^{t_2}(u) = T_2^{t_2}(u)R'(u-v)T_1(v).$$

Replacing u, v by v, u we obtain (4).

Finally, applying t_1 to both sides of (4) and replacing u, v by $-u, -v$ we arrive at (5).

3.4. Relation (3.3.5) implies the following analogue of Proposition 1.12(iv) for the transposition (3.1.1).

Corollary. *The mapping $T(u) \mapsto T^t(-u)$ defines an involutive automorphism of the algebra* $\mathrm{Y}(N)$.

3.5. Set

$$S(u) := T(u)T^t(-u). \tag{1}$$

Then

$$S(u) = \sum_{i,j} s_{ij}(u) \otimes E_{ij}, \quad \text{where} \quad s_{ij}(u) := \sum_a \theta_{aj} t_{ia}(u) t_{-j,-a}(-u). \tag{2}$$

Further,

$$s_{ij}(u) = \delta_{ij} + s_{ij}^{(1)} u^{-1} + s_{ij}^{(2)} u^{-2} + \cdots \tag{3}$$

with

$$s_{ij}^{(M)} = \sum_a \sum_{r=0}^{M} \theta_{aj} (-1)^r t_{ia}^{(M-r)} t_{-j,-a}^{(r)}. \tag{4}$$

For example,

$$s_{ij}^{(1)} = t_{ij}^{(1)} - \theta_{ij} t_{-j,-i}^{(1)}, \quad s_{ij}^{(2)} = t_{ij}^{(2)} + \theta_{ij} t_{-j,-i}^{(2)} - \sum_a \theta_{aj} t_{ia}^{(1)} t_{-j,-a}^{(1)}. \tag{5}$$

Definition. The *twisted Yangian* $\mathrm{Y}^{\pm}(N)$ is the subalgebra of the Yangian $\mathrm{Y}(N)$ generated by the entries of the S-matrix, i.e. by the elements $s_{ij}^{(M)}$, where $M = 1, 2, \ldots$ and $-n \le i, j \le n$.

3.6. Set

$$S_1(u) := \sum_{ij} s_{ij}(u) \otimes E_{ij} \otimes 1, \qquad S_2(v) := \sum_{kl} s_{kl}(v) \otimes 1 \otimes E_{kl}. \tag{1}$$

These are elements of the algebra $\mathrm{Y}^{\pm}(N)[[u^{-1}, v^{-1}]] \otimes \operatorname{End} \mathcal{E}^{\otimes 2}$.

Theorem. *The S-matrix satisfies the following relations:*

$$R(u-v)S_1(u)R'(-u-v)S_2(v) = S_2(v)R'(-u-v)S_1(u)R(u-v), \tag{2}$$

$$S^t(-u) = S(u) \pm \frac{S(u) - S(-u)}{2u}. \tag{3}$$

We will refer to (2) and (3) as the *quaternary relation* and the *symmetry relation* respectively.

Proof. The quaternary relation on the S-matrix is derived from the ternary relation on the T-matrix as follows:

$$\begin{aligned}
&R(u-v)S_1(u)R'(-u-v)S_2(v) = \\
&= R(u-v)T_1(u)T_1^{t_1}(-u)R'(-u-v)T_2(v)T_2^{t_2}(-v) \\
&= R(u-v)T_1(u)T_2(v)R'(-u-v)T_1^{t_1}(-u)T_2^{t_2}(-v) \quad \text{by} \quad (3.3.3) \\
&= T_2(v)T_1(u)R(u-v)R'(-u-v)T_1^{t_1}(-u)T_2^{t_2}(-v) \\
&\qquad \text{(by the ternary relation)} \\
&= T_2(v)T_1(u)R'(-u-v)R(u-v)T_1^{t_1}(-u)T_2^{t_2}(-v) \quad \text{by} \quad (3.3.2) \\
&= T_2(v)T_1(u)R'(-u-v)T_2^{t_2}(-v)T_1^{t_1}(-u)R(u-v) \quad \text{by} \quad (3.3.5) \\
&= T_2(v)T_2^{t_2}(-v)R'(-u-v)T_1(u)T_1^{t_1}(-u)R(u-v) \quad \text{by} \quad (3.3.4) \\
&= S_2(v)R'(-u-v)S_1(u)R(u-v).
\end{aligned}$$

To establish the symmetry relation, we will use (3.5.2) and the commutation relations (1.8.2):

$$\begin{aligned}
&(S^t(-u))_{ij} = \theta_{ij} s_{-j,-i}(-u) \\
&= \sum_a \theta_{ij}\theta_{-i,a} t_{-j,a}(-u) t_{i,-a}(u) \quad \text{by} \quad (3.5.2) \\
&= \sum_a \theta_{ja} t_{-j,-a}(-u) t_{i,a}(u), \quad \text{replacing } a \text{ by } -a.
\end{aligned}$$

By (1.8.2), this can be written as

$$\sum_a \theta_{ja} t_{ia}(u) t_{-j,-a}(-u)$$
$$-\frac{1}{2u}\sum_a \theta_{ja} t_{i,-a}(-u) t_{-j,a}(u) + \frac{1}{2u}\sum_a \theta_{ja} t_{i,-a}(u) t_{-j,a}(-u).$$

Observe now that $\theta_{ja} = \pm\theta_{j,-a}$. Substituting this into the second and the third sum, we obtain finally

$$\theta_{ij} s_{-j,-i}(-u) = s_{ij}(u) \pm \frac{s_{ij}(u) - s_{ij}(-u)}{2u} \quad \text{for all} \quad i, j, \tag{4}$$

which is just the symmetry relation.

3.7. Proposition. *The quaternary relation* (3.6.2) *may be written in the form*

$$\begin{aligned}
[S_1(u), S_2(v)] = {} & \frac{1}{u-v}(PS_1(u)S_2(v) - S_2(v)S_1(u)P) \\
& - \frac{1}{u+v}(S_1(u)QS_2(v) - S_2(v)QS_1(u)) \\
& + \frac{1}{u^2-v^2}(PS_1(u)QS_2(v) - S_2(v)QS_1(u)P)
\end{aligned} \tag{1}$$

or else as the following system of relations: for all i, j, k, l

$$\begin{aligned}
[s_{ij}(u), s_{kl}(v)] = {} & \frac{1}{u-v}(s_{kj}(u)s_{il}(v) - s_{kj}(v)s_{il}(u)) \\
& - \frac{1}{u+v}(\theta_{k,-j} s_{i,-k}(u) s_{-j,l}(v) - \theta_{i,-l} s_{k,-i}(v) s_{-l,j}(u)) \\
& + \frac{1}{u^2-v^2}(\theta_{i,-j} s_{k,-i}(u) s_{-j,l}(v) - \theta_{i,-j} s_{k,-i}(v) s_{-j,l}(u)).
\end{aligned} \tag{2}$$

Note that relations (2) are analogous to the relations (1.8.2) for the Yangian $\mathrm{Y}(N)$.

Proof. To derive (1), it suffices to substitute in (3.6.2)

$$R(u-v) = 1 - \frac{P}{u-v}, \qquad R'(-u-v) = 1 + \frac{Q}{u+v}.$$

To derive (2), one simply rewrites (1) in terms of $s_{ij}(u)$ using (3.6.1) and the explicit forms of P and Q:

$$P = \sum E_{ij} \otimes E_{ji}, \quad Q = \sum \theta_{kl} E_{-l,-k} \otimes E_{lk}.$$

3.8. Theorem. *The quaternary relation* (3.6.2) *and the symmetry relation* (3.6.3) *are precisely the defining relations for the twisted Yangian* $\mathrm{Y}^{\pm}(N)$.

Proof. Let us consider two algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ where $\mathcal{A}_1 := \mathrm{Y}^{\pm}(N)$ and $\mathcal{A}_2$ is generated by *arbitrary* generators $s_{ij}^{(M)}$ (where $M = 1, 2, \ldots$ and $-n \le i, j \le n$) subject to relations (3.6.2) and (3.6.3). Then there is an obvious surjective morphism $\mathcal{A}_2 \to \mathcal{A}_1$, and we have to verify that it is injective.

To do this, we endow both of the algebras with filtrations: that of $\mathcal{A}_1 = \mathrm{Y}^{\pm}(N)$ is induced by the *first* filtration of the Yangian (see Definition 1.20) and that of $\mathcal{A}_2$

is defined by setting $\deg(s_{ij}^{(M)}) = M$. The mapping $\mathcal{A}_2 \to \mathcal{A}_1$ preserves filtration, so that it defines a surjective morphism $\mathrm{gr}\mathcal{A}_2 \to \mathrm{gr}\mathcal{A}_1$ of the corresponding graded algebras.

It suffices to show that the latter morphism is injective. To do this we will describe both graded algebras more explicitly.

First consider $\mathrm{gr}\mathcal{A}_1$ which is a subalgebra of $\mathrm{gr}_1 \mathrm{Y}(N)$. Recall that $\mathrm{gr}_1 \mathrm{Y}(N)$ is commutative by Corollary 1.21, hence $\mathrm{gr}\mathcal{A}_1$ is commutative too. Denote by $\bar{t}_{ij}^{(M)}$ and $\bar{s}_{ij}^{(M)}$ the images of $t_{ij}^{(M)}$ and $s_{ij}^{(M)}$ in the M-th homogeneous component of $\mathrm{gr}_1 \mathrm{Y}(N)$ respectively. Then (3.5.4) implies

$$\bar{s}_{ij}^{(M)} = \bar{t}_{ij}^{(M)} + (-1)^M \theta_{ij} \bar{t}_{-j,-i}^{(M)} + \sum_a \sum_{r=1}^{M-1} (-1)^r \theta_{ja} \bar{t}_{ia}^{(M-r)} \bar{t}_{-j,-a}^{(r)}. \tag{1}$$

Note also that

$$\theta_{ij} \bar{s}_{-j,-i}^{(M)} = (-1)^M \bar{s}_{ij}^{(M)}. \tag{2}$$

Indeed, this follows from (1) since we are dealing with a commutative algebra (this also follows from the symmetry relation).

Recall that the generators $\bar{t}_{ij}^{(M)}$ are algebraically independent by Theorem 1.22. Since the sum in (1) involves only generators of degree strictly less than M, we see that the algebra $\mathrm{gr}\mathcal{A}_1$ is isomorphic to the algebra of polynomials in the letters $\bar{s}_{ij}^{(M)}$ subject to the symmetry condition (2).

Now let us turn to the algebra $\mathrm{gr}\mathcal{A}_2$. Here the crucial observation is that this algebra is also commutative. Indeed, to show this it suffices to verify that

$$\deg\,[s_{ij}^{(M)}, s_{kl}^{(L)}] < M + L.$$

Let us examine the commutation relation (3.7.2) and regard its right hand side as an element of the algebra $\mathrm{gr}\mathcal{A}_2 \otimes \mathbb{C}((v^{-1}))[[u^{-1}]]$. In $\mathbb{C}((v^{-1}))[[u^{-1}]]$ we may write

$$\frac{1}{u-v} = u^{-1}(1 - vu^{-1})^{-1} = \sum_{r=0}^{\infty} v^r u^{-r-1},$$

$$\frac{1}{u+v} = u^{-1}(1 + vu^{-1})^{-1} = \sum_{r=0}^{\infty} (-1)^r v^r u^{-r-1}, \tag{3}$$

$$\frac{1}{u^2-v^2} = u^{-2}(1 - v^2 u^{-2})^{-1} = \sum_{r=0}^{\infty} v^{2r} u^{-2r-2}.$$

Substituting these in (3.7.2) and comparing the coefficients of $u^{-M} v^{-L}$ in both the sides, we see that $[s_{ij}^{(M)}, s_{kl}^{(L)}]$ is a finite sum of expressions of degree $M + L - 1$ or $M + L - 2$. (Note that this reasoning is quite similar to that used in the passage from (1.1.1) to (1.2.1).)

Thus we have verified the commutativity of $\mathrm{gr}\mathcal{A}_2$. Now let $\bar{s}_{ij}^{(M)}$ have the same meaning as before: the image of the (abstract) generators $s_{ij}^{(M)}$ in the M-th component of $\mathrm{gr}\mathcal{A}_2$. By the symmetry relation (3.6.3), the (abstract) generators $\bar{s}_{ij}^{(M)}$ satisfy the symmetry relation (2). Since the generators of $\mathrm{gr}\mathcal{A}_1$ are algebraically independent, we conclude that the morphism $\mathrm{gr}\mathcal{A}_2 \to \mathrm{gr}\mathcal{A}_1$ is injective.

3.9. Proposition. *The mapping*

$$S(u) \mapsto S^t(u) \tag{1}$$

defines an involutive antiautomorphism of the algebra $\mathrm{Y}^{\pm}(N)$.

Proof. Let us apply the antiautomorphism sign (see (1.11.1)) to $s_{ij}(u)$. Due to (3.5.2) we have

$$\begin{aligned}\mathrm{sign}(s_{ij}(u)) &= \sum_a \theta_{aj} t_{-j,-a}(u) t_{ia}(-u) \\ &= \sum_a \theta_{-a,j} t_{-j,a}(u) t_{i,-a}(-u) \quad \text{(we replaced } a \text{ by } -a) \\ &= \sum_a \theta_{ij}\theta_{-a,i} t_{-j,a}(u) t_{i,-a}(-u) = \theta_{ij} s_{-j,-i}(u).\end{aligned}$$

Thus, the subalgebra $\mathrm{Y}^{\pm}(N)$ is invariant under the antiautomorphism sign, and its restriction to $\mathrm{Y}^{\pm}(N)$ gives the antiautomorphism (1).

3.10. Proposition. *Let* $g(u) = 1 + g_1 u^{-2} + g_2 u^{-4} + \cdots \in \mathbb{C}[[u^{-2}]]$ *be a formal power series. Then the mapping*

$$\nu_g : S(u) \mapsto g(u) S(u) \tag{1}$$

defines an automorphism of the algebra $\mathrm{Y}^{\pm}(N)$.

Proof. The series $g(u)$ may be written in the form $g(u) = f(u)f(-u)$, where $f(u) = 1 + f_1 u^{-1} + f_2 u^{-2} + \cdots \in \mathbb{C}[[u^{-1}]]$. Let us consider the automorphism μ_f of the algebra $\mathrm{Y}(N)$ (see (1.12.2)). By (3.5.1) we have

$$\mu_f(S(u)) = f(u)T(u)f(-u)T^t(-u) = f(u)f(-u)S(u) = g(u)S(u).$$

That is, the subalgebra $\mathrm{Y}^{\pm}(N)$ is invariant under the automorphism μ_f, and its restriction to $\mathrm{Y}^{\pm}(N)$ gives the automorphism ν_g.

3.11. For $-n \le i, j \le n$ set

$$F_{ij} := E_{ij} - \theta_{ij} E_{-j,-i} \tag{1}$$

and denote by $\mathfrak{g}(n)$ the Lie subalgebra of $\mathfrak{gl}(N)$ spanned by the elements (1). Then $\mathfrak{g}(n)$ is isomorphic to $\mathfrak{o}(2n)$ or $\mathfrak{sp}(2n)$ (resp. $\mathfrak{o}(2n+1)$), if $N = 2n$ (resp. $2n+1$). Note that the generators (1) satisfy the following symmetry relations:

$$F_{-j,-i} = -\theta_{ij} F_{ij}. \tag{2}$$

Proposition. *The mapping*

$$\xi : s_{ij}(u) \mapsto \delta_{ij} + (u \pm \frac{1}{2})^{-1} \cdot F_{ij} \tag{3}$$

defines the homomorphism of algebras $\xi : \mathrm{Y}^{\pm}(N) \to \mathrm{U}(\mathfrak{g}(n))$.

Proof. We have to verify that relations (3.7.2) and (3.6.4) hold for

$$s_{ij}(u) = \delta_{ij} + (u \pm \frac{1}{2})^{-1} \cdot F_{ij}. \tag{4}$$

Let us set $u' = u \pm \frac{1}{2}$, $v' = v \pm \frac{1}{2}$ and substitute (4) into (3.7.2). Multiplying both sides by $u'v'$, we obtain:

$$[F_{ij}, F_{kl}] = \frac{1}{u'-v'}((\delta_{kj}u' + F_{kj})(\delta_{il}v' + F_{il}) - (\delta_{kj}v' + F_{kj})(\delta_{il}u' + F_{il}))$$

$$-\frac{1}{u'+v'\mp 1}(\theta_{k,-j}(\delta_{i,-k}u'+F_{i,-k})(\delta_{-j,l}v'+F_{-j,l})-\theta_{i,-l}(\delta_{k,-i}v'+F_{k,-i})(\delta_{-l,j}u'+F_{-l,j}))$$

$$+\frac{1}{(u'-v')(u'+v'\mp 1)}\theta_{i,-j}((\delta_{k,-i}u' + F_{k,-i})(\delta_{-j,l}v' + F_{-j,l}) - (\delta_{k,-i}v' + F_{k,-i})(\delta_{-j,l}u' + F_{-j,l})),$$

which is equal to

$$\delta_{kj}F_{il} - \delta_{il}F_{kj} - \frac{1}{u'+v'\mp 1}((\theta_{k,-j}\delta_{i,-k}F_{-j,l} - \theta_{i,-l}\delta_{-l,j}F_{k,-i})u'$$
$$+(\theta_{k,-j}\delta_{-j,l}F_{i,-k} - \theta_{i,-l}\delta_{k,-i}F_{-l,j})v' - \theta_{i,-j}(\delta_{k,-i}F_{-j,l} - \delta_{-j,l}F_{k,-i})).$$

Here we used the relations

$$\theta_{k,-j}\delta_{i,-k}\delta_{-j,l} - \theta_{i,-l}\delta_{k,-i}\delta_{-l,j} = 0 \quad \text{and} \quad \theta_{k,-j}F_{i,-k}F_{-j,l} - \theta_{i,-l}F_{k,-i}F_{-l,j} = 0.$$

The former is obvious, while the latter follows from (2). Relations (2) also imply that

$$\theta_{k,-j}\delta_{i,-k}F_{-j,l} - \theta_{i,-l}\delta_{-l,j}F_{k,-i} = \theta_{k,-j}\delta_{-j,l}F_{i,-k} - \theta_{i,-l}\delta_{k,-i}F_{-l,j}$$
$$= \pm\theta_{i,-j}(\delta_{k,-i}F_{-j,l} - \delta_{-j,l}F_{k,-i}).$$

Thus, we get the equality

$$[F_{ij}, F_{kl}] = \delta_{kj}F_{il} - \delta_{il}F_{kj} - \theta_{k,-j}\delta_{i,-k}F_{-j,l} + \theta_{i,-l}\delta_{-l,j}F_{k,-i}, \tag{5}$$

which coincides with the commutation relations of the Lie algebra $\mathfrak{g}(n)$.

Now we substitute (4) into (3.6.4). By using (2), we obtain for the left hand side:

$$\theta_{ij}(\delta_{-j,-i} + (-u \pm \frac{1}{2})^{-1} \cdot F_{-j,-i}) = \delta_{ij} + (u \mp \frac{1}{2})^{-1} \cdot F_{ij},$$

and for the right hand side:

$$\delta_{ij} + (u \pm \frac{1}{2})^{-1} \cdot F_{ij} \pm \frac{(u \pm \frac{1}{2})^{-1} - (-u \pm \frac{1}{2})^{-1}}{2u} \cdot F_{ij} = \delta_{ij} + (u \mp \frac{1}{2})^{-1} \cdot F_{ij},$$

and the proof is complete.

3.12. Proposition. *The mapping*

$$\eta : F_{ij} \mapsto s_{ij}^{(1)} \tag{1}$$

defines the inclusion of the algebra $\mathrm{U}(\mathfrak{g}(n))$ *into* $\mathrm{Y}^{\pm}(N)$.

Proof. Using the decompositions (3.8.3), we derive from relations (3.7.2) that

$$[s_{ij}^{(1)}, s_{kl}^{(1)}] = \delta_{kj} s_{il}^{(1)} - \delta_{il} s_{kj}^{(1)} - \theta_{k,-j}\delta_{i,-k} s_{-j,l}^{(1)} + \theta_{i,-l}\delta_{-l,j} s_{k,-i}^{(1)}.$$

Further, (3.6.4) implies the equality

$$-\theta_{ij} s_{-j,-i}^{(1)} = s_{ij}^{(1)}.$$

Comparing this with (3.11.5) and (3.11.2) we conclude that (1) is a homomorphism of algebras. It is clear that $\xi \circ \eta = \mathrm{id}$. Therefore the kernel of η is trivial.

3.13. Remark (cf. Remark 1.18). Denote by F the $N \times N$-matrix formed by the elements F_{ij} and set

$$F(u) := 1 + (u \pm \frac{1}{2})^{-1} \cdot F. \tag{1}$$

Then the previous statements may be formulated as follows: the fact that $\mathcal{S}(u)$ satisfies the quaternary relation (3.6.2) and the symmetry relation (3.6.3) is equivalent to the fact that the elements F_{ij} satisfy relations (3.11.2) and (3.11.6). Note that the summand $\pm\frac{1}{2}$ in (1) is essential. In contrast to the case of the Yangian $\mathrm{Y}(N)$, the mapping $S(u) \mapsto 1 + Fu^{-1}$ does not define a morphism of algebras.

3.14. Remark. There is an analogue of Theorem 1.22 for the twisted Yangian. Namely, it follows from the proof of Theorem 3.8 that the elements

$$\bar{s}_{ij}^{(2k)}, \quad i+j \le 0; \qquad \bar{s}_{ij}^{(2k-1)}, \quad i+j < 0; \qquad k = 1, 2, \ldots,$$

in the case of $\mathrm{Y}^{+}(N)$, and the elements

$$\bar{s}_{ij}^{(2k)}, \quad i+j < 0; \qquad \bar{s}_{ij}^{(2k-1)}, \quad i+j \le 0; \qquad k = 1, 2, \ldots,$$

in the case of $\mathrm{Y}^{-}(N)$, constitute a system of algebraically independent generators of the algebra $\mathrm{gr}_1 \mathrm{Y}^{\pm}(N)$. This fact may be regarded as the Poincaré–Birkhoff–Witt theorem for the algebra $\mathrm{Y}^{\pm}(N)$.

3.15. We will now establish an analogue of Theorem 1.26 for the twisted Yangian. Let us introduce the involutive automorphism σ of the polynomial current Lie algebra $\mathfrak{gl}(N)[x]$:

$$(\sigma(f))(x) = -(f(-x))^{t}, \qquad f \in \mathfrak{gl}(N)[x].$$

This involution determines a Lie subalgebra in $\mathfrak{gl}(N)[x]$; denote it by $\mathfrak{gl}(N)[x]^\sigma$. It will be called the *twisted* polynomial current Lie algebra corresponding to the orthogonal Lie algebra $\mathfrak{o}(N) \subset \mathfrak{gl}(N)$ or to the symplectic Lie algebra $\mathfrak{sp}(N) \subset \mathfrak{gl}(N)$. If

$$f = a_0 + a_1 x + \cdots + a_k x^k$$

is an element of $\mathfrak{gl}(N)[x]^\sigma$, then the coefficients a_{2i} lie in the subalgebra $\mathfrak{o}(N)$ or $\mathfrak{sp}(N)$ of $\mathfrak{gl}(N)$, while the coefficients a_{2i-1} lie in the complement to that subalgebra, defined by the restriction of σ to $\mathfrak{gl}(N)$.

The second filtration of $\mathrm{Y}(N)$ (see Definition 1.20) defines a filtration of the subalgebra $\mathrm{Y}^\pm(N) \subset \mathrm{Y}(N)$. Denote by $\mathrm{gr}_2\mathrm{Y}^\pm(N)$ the corresponding graded algebra.

Theorem. *The graded algebra* $\mathrm{gr}_2\mathrm{Y}^\pm(N)$ *is isomorphic to the universal enveloping algebra* $\mathrm{U}(\mathfrak{gl}(N)[x]^\sigma)$.

Proof. Let us consider the isomorphism $\mathrm{U}(\mathfrak{gl}(N)[x]) \to \mathrm{gr}_2\mathrm{Y}(N)$ constructed in the proof of Theorem 1.26 and find the image of the subalgebra $\mathrm{U}(\mathfrak{gl}(N)[x]^\sigma)$ under this isomorphism. The Lie algebra $\mathfrak{gl}(N)[x]^\sigma$ is the linear span of the elements

$$(E_{ij} + (-1)^M \theta_{ij} E_{-j,-i}) x^{M-1}, \quad -n \le i, j \le n; \ M = 1, 2, \ldots.$$

Their images in $\mathrm{gr}_2\mathrm{Y}(N)$ have the form $\tilde{t}_{ij}^{(M)} + (-1)^M \theta_{ij} \tilde{t}_{-j,-i}^{(M)}$. Formula (3.5.4) implies that they are precisely the images of the generators $s_{ij}^{(M)}$ in the $(M-1)$-th component of $\mathrm{gr}_2\mathrm{Y}(N)$, which proves the theorem.

3.16. Remark (cf. Remark 1.27). The algebra $\mathrm{Y}^\pm(N)$ may be considered as a flat deformation of the algebra $\mathrm{U}(\mathfrak{gl}(N)[x]^\sigma)$. To see this we introduce new generators in $\mathrm{Y}^\pm(N)$:

$$\tilde{s}_{ij}^{(M)} = s_{ij}^{(M)} h^{M-1},$$

where $h \in \mathbb{C} \setminus \{0\}$ is the deformation parameter. Set

$$\tilde{s}_{ij}(u) = \sum_{M=1}^{\infty} \tilde{s}_{ij}^{(M)} u^{-M}.$$

Relations (3.7.2) and (3.6.4) will then take the form

$$\begin{aligned}
&[\tilde{s}_{ij}(u), \tilde{s}_{kl}(v)] = \\
&- \frac{1}{u-v}(\delta_{kj}(\tilde{s}_{il}(u) - \tilde{s}_{il}(v)) - \delta_{il}(\tilde{s}_{kj}(u) - \tilde{s}_{kj}(v))) \\
&+ \frac{1}{u+v}(\delta_{i,-k}(\theta_{i,-l}\tilde{s}_{-l,j}(u) - \theta_{k,-j}\tilde{s}_{-j,l}(v)) - \delta_{-j,l}(\theta_{k,-j}\tilde{s}_{i,-k}(u) - \theta_{i,-l}\tilde{s}_{k,-i}(v))) \\
&+ \frac{h}{u-v}(\tilde{s}_{kj}(u)\tilde{s}_{il}(v) - \tilde{s}_{kj}(v)\tilde{s}_{il}(u)) \\
&- \frac{h}{u+v}(\theta_{k,-j}\tilde{s}_{i,-k}(u)\tilde{s}_{-j,l}(v) - \theta_{i,-l}\tilde{s}_{k,-i}(v)\tilde{s}_{-l,j}(u)) \\
&- \frac{h}{u^2-v^2}\theta_{i,-j}(\delta_{k,-i}(\tilde{s}_{-j,l}(u) - \tilde{s}_{-j,l}(v)) - \delta_{-j,l}(\tilde{s}_{k,-i}(u) - \tilde{s}_{k,-i}(v))) \\
&+ \frac{h^2}{u^2-v^2}\theta_{i,-j}(\tilde{s}_{k,-i}(u)\tilde{s}_{-j,l}(v) - \tilde{s}_{k,-i}(v)\tilde{s}_{-j,l}(u))
\end{aligned}$$

and

$$\theta_{ij}\tilde{s}_{-j,-i}(-u)=\tilde{s}_{ij}(u)\pm h\frac{\tilde{s}_{ij}(u)-\tilde{s}_{ij}(-u)}{2u}.$$

Denote by $\mathrm{Y}_h^{\pm}(N)$ the algebra with abstract generators $\tilde{s}_{ij}^{(M)}$, $-n\le i,j\le n$; $M=1,2,\dots$ and the above relations. Then the algebras $\mathrm{Y}_h^{\pm}(N)$ with $h\ne 0$ are isomorphic to $\mathrm{Y}^{\pm}(N)=\mathrm{Y}_1^{\pm}(N)$, while setting $h=0$ in the above formulae we get the following relations:

$$\begin{aligned}[\tilde{s}_{ij}^{(M)},\tilde{s}_{kl}^{(L)}]&=\delta_{kj}\tilde{s}_{il}^{(M+L-1)}-\delta_{il}\tilde{s}_{kj}^{(M+L-1)}\\&+(-1)^M(\delta_{i,-k}\theta_{k,-j}\tilde{s}_{-j,l}^{(M+L-1)}-\delta_{-l,j}\theta_{i,-l}\tilde{s}_{k,-i}^{(M+L-1)})\end{aligned}$$

and

$$(-1)^M\theta_{ij}\tilde{s}_{-j,-i}^{(M)}=\tilde{s}_{ij}^{(M)}.$$

They coincide with the commutation relations of the Lie algebra $\mathfrak{gl}(N)[x]^{\sigma}$ in the generators $(E_{ij}+(-1)^M\theta_{ij}E_{-j,-i})x^{M-1}$. The flatness of the deformation follows from the Poincaré–Birkhoff–Witt theorem for $\mathrm{Y}^{\pm}(N)$ (see Remark 3.14).

3.17. Comments. In this section we have presented a detailed exposition of some of the results announced in Olshanskiĭ's paper [O2]. The aim of [O2] was to apply the approach of the work [O1] to the orthogonal and symplectic algebras.

4. The Sklyanin determinant $\operatorname{sdet} S(u)$ and the center of $\mathrm{Y}^{\pm}(N)$

In this section we establish several facts about the structure of the algebras $\mathrm{Y}^{\pm}(N)$ introduced in Section 3. We find a system of algebraically independent generators of the center of the algebra $\mathrm{Y}^{\pm}(N)$. We introduce the special twisted Yangian $\mathrm{SY}^{\pm}(N)$ and prove that the algebra $\mathrm{Y}^{\pm}(N)$ is isomorphic to the tensor product of its center and the algebra $\mathrm{SY}^{\pm}(N)$. We will keep to the notation of Section 3 and use the R-matrix formalism of Subsections 1.3 – 1.8 extensively.

4.1. Let $u_1, \dots, u_m$ be formal variables. As in Subsection 2.1 we put

$$R(u_1,\dots,u_m) := (R_{m-1,m})(R_{m-2,m}R_{m-2,m-1})\cdots(R_{1m}\cdots R_{12}),$$

where $R_{ij} := R_{ij}(u_i - u_j)$. We regard R_{ij} as an element of the algebra

$$\mathrm{Y}^{\pm}(N)[[u_1^{-1},\dots,u_m^{-1}]]_{ext} \otimes \operatorname{End}\mathcal{E}^{\otimes m}, \tag{1}$$

where $\mathrm{Y}^{\pm}(N)[[u_1^{-1},\dots,u_m^{-1}]]_{ext}$ is the localization of $\mathrm{Y}^{\pm}(N)[[u_1^{-1},\dots,u_m^{-1}]]$ with respect to the multiplicative family generated by the elements $u_k^{-1} - u_l^{-1}$ and $u_k^{-1} + u_l^{-1}$, $k \neq l$ (cf. (1.7)). We also need the following elements of the algebra (1):

$$S_i := S_i(u_i), \quad 1 \le i \le m \qquad \text{and} \qquad R'_{ij} = R'_{ji} := R'_{ij}(-u_i - u_j), \quad 1 \le i < j \le m$$

(see Subsections 3.3 and 3.5). For an arbitrary permutation $(p_1, \dots, p_m)$ of the numbers $1, \dots, m$, we abbreviate

$$\langle S_{p_1},\dots,S_{p_m}\rangle = S_{p_1}(R'_{p_1p_2}\dots R'_{p_1p_m})S_{p_2}(R'_{p_2p_3}\dots R'_{p_2p_m})\dots S_{p_m}. \tag{2}$$

4.2. Proposition. *We have the fundamental identity* (cf. (2.1.2))

$$R(u_1,\dots,u_m)\langle S_1,\dots,S_m\rangle = \langle S_m,\dots,S_1\rangle R(u_1,\dots,u_m). \tag{1}$$

Proof. We shall prove (1) in several steps.

Step 1. We have the following equalities:

$$R_{ij}S_iR'_{ij}S_j = S_jR'_{ji}S_iR_{ij}, \tag{2}$$

$$R_{ij}R'_{ik}R'_{jk} = R'_{jk}R'_{ik}R_{ij}, \tag{3}$$

where i, j, k are pairwise distinct. Indeed, the equality (2) coincides with the quaternary relation (3.6.2). It follows from the ternary relation that (3) is equivalent to the following fact: the mapping $T(u) \mapsto R'(-u)$, i.e.

$$t_{ij}(u) \mapsto \delta_{ij} + \theta_{ij}E_{-i,-j}u^{-1},$$

defines a homomorphism of algebras $\mathrm{Y}(N) \to \mathrm{U}(\mathfrak{gl}(N))$. However, that has already been proved for the mapping

$$t_{ij}(u) \mapsto \delta_{ij} + E_{ij}u^{-1}$$

(see Proposition 1.16). It remains to note that the mapping $E_{ij} \mapsto \theta_{ij} E_{-i,-j}$ defines an automorphism of the algebra $\mathrm{U}(\mathfrak{gl}(N))$.

Step 2. Observe that (3) can be rewritten as

$$R_{ij} R'_{ki} R'_{kj} = R'_{kj} R'_{ki} R_{ij} \tag{4}$$

since $R'_{ik} = R'_{ki}$ and $R'_{jk} = R'_{kj}$. We shall also need the following generalization of (3):

$$R_{ij}(R'_{ik_1} \dots R'_{ik_r})(R'_{jk_1} \dots R'_{jk_r}) = (R'_{jk_1} \dots R'_{jk_r})(R'_{ik_1} \dots R'_{ik_r}) R_{ij}, \tag{5}$$

provided $i, j, k_1, \dots, k_r$ are pairwise distinct.

To verify (5) we observe that R'_{ik_a} and R'_{jk_b} commute when $a \neq b$, so that (5) can be rewritten as

$$R_{ij} \cdot \prod_{a=1}^{r} (R'_{ik_a} R'_{jk_a}) = \prod_{a=1}^{r} (R'_{jk_a} R'_{ik_a}) \cdot R_{ij},$$

but this equality is an immediate consequence of (3).

Step 3. Let us prove that for any $i = 1, \dots, m-1$ and any permutation $(p_1, \dots, p_m)$ of the numbers $1, \dots, m$

$$R_{p_i p_{i+1}} \langle S_{p_1}, \dots, S_{p_m} \rangle = \langle S_{p_1}, \dots, S_{p_{i-1}}, S_{p_{i+1}}, S_{p_i}, S_{p_{i+2}}, \dots, S_{p_m} \rangle R_{p_i p_{i+1}}. \tag{6}$$

First, we examine the fragment of the product $\langle S_{p_1}, \dots, S_{p_m} \rangle$ (see (4.1.2)) which precedes S_{p_i}. All the factors of this fragment commute with $R_{p_i p_{i+1}}$ except $R'_{p_k p_i}$ and $R'_{p_k p_{i+1}}$, where $k = 1, \dots, i-1$. To permute $R_{p_i p_{i+1}}$ with these factors we use the rule

$$R_{p_i p_{i+1}} R'_{p_k p_i} R'_{p_k p_{i+1}} = R'_{p_k p_{i+1}} R'_{p_k p_i} R_{p_i p_{i+1}}$$

which is a special case of (4). After these transformations the fragment under consideration takes the same form as the corresponding fragment in the right hand side of (6).

Second, we examine the fragment

$$S_{p_i} R'_{p_i p_{i+1}} \Big(\prod_{k > i+1} R'_{p_i p_k} \Big) S_{p_{i+1}} \Big(\prod_{k > i+1} R'_{p_{i+1} p_k} \Big). \tag{7}$$

Since $R'_{p_i p_k}$ and $S_{p_{i+1}}$ commute for any $k > i+1$, we may rewrite (7) as

$$S_{p_i} R'_{p_i p_{i+1}} S_{p_{i+1}} \Big(\prod_{k > i+1} R'_{p_i p_k} \Big) \Big(\prod_{k > i+1} R'_{p_{i+1} p_k} \Big). \tag{8}$$

To permute $R_{p_i p_{i+1}}$ with (8), we use the identities

$$R_{p_i p_{i+1}} S_{p_i} R'_{p_i p_{i+1}} S_{p_{i+1}} = S_{p_{i+1}} R'_{p_i p_{i+1}} S_{p_i} R_{p_i p_{i+1}},$$

$$R_{p_i p_{i+1}} \Big(\prod_{k > i+1} R'_{p_i p_k} \Big) \Big(\prod_{k > i+1} R'_{p_{i+1} p_k} \Big) = \Big(\prod_{k > i+1} R'_{p_{i+1} p_k} \Big) \Big(\prod_{k > i+1} R'_{p_i p_k} \Big) R_{p_i p_{i+1}}$$

which are special cases of (2) and (5), respectively. Then we rewrite $R'_{p_ip_{i+1}}$ as $R'_{p_{i+1}p_i}$ and permute S_{p_i} with the product

$$\prod_{k>i+1} R'_{p_{i+1}p_k}.$$

Again after these transformations our fragment takes the same form as the corresponding fragment in the right hand side of (6).

Third, we look at the remaining fragment, which is just $\langle S_{p_{i+2}}, \dots , S_{p_m}\rangle$. All the factors of this fragment commute with $R_{p_ip_{i+1}}$; on the other hand, this fragment appears in the right hand side of (6) in the same form.

Thus the proof of (6) has been completed.

Step 4. Finally, we observe that (1) can be deduced from (6). In fact, using (6) repeatedly, we permute R_{12} with $\langle S_1, \dots , S_m\rangle$, then we permute R_{13} with $\langle S_2, S_1, S_3, \dots , S_m\rangle$ etc. The total effect of the permutation with all the factors R_{ij} occuring in $R(u_1, \dots , u_m)$ clearly amounts to rearranging the factors S_i into reverse order, just as they appear in the right hand side of (1).

4.3. Note that in the case of the twisted Yangian the morphisms like (2.2.1) (see Remark 2.2) may be used as well. So, the fundamental identity (4.2.1) remains true when the variables $u_1, \dots, u_m$ are subjected to certain relations.

Proposition. *The following identity holds in the algebra* (4.1.1) *with* $m{=}N$

(1)

$$A_N S_1 R'_{12} \dots R'_{1N} S_2 \dots S_{N-1} R'_{N-1,N} S_N = S_N R'_{N,N-1} \dots R'_{N1} S_{N-1} \dots S_2 R'_{21} S_1 A_N,$$

where $S_i = S_i(u-i+1)$, $R'_{ij} = R'_{ij}(-2u+i+j-2)$ *and* u *is a formal variable.*

Proof. In the fundamental identity (4.2.1) set $m = N$ and $u_i = u - i + 1$ for $i = 1, \dots, N$. Then using Proposition 2.3, we get the equality (1).

4.4. Proposition. *There exists a formal series*

$$\operatorname{sdet} S(u) := 1 + c_1 u^{-1} + c_2 u^{-2} + \dots \in \mathrm{Y}^{\pm}(N)[[u^{-1}]]$$

such that both sides of (4.3.1) *are equal to* $\operatorname{sdet} S(u) A_N$.

Proof. The proof is similar to that of Proposition 2.5. The required statement follows from the fact that A_N is a one-dimensional projection in $\mathcal{E}^{\otimes N}$ and each of the elements $S_i(u-i+1)$, $1 \le i \le N$ and $R'_{ij}(-2u+i+j-2)$, $1 \le i, j \le N$, $i \ne j$ is a formal series in u^{-1} which begins with 1.

4.5. Definition. The series $\operatorname{sdet} S(u)$ is called the *Sklyanin determinant* of the matrix $S(u)$.

For example, if $N = 2$, then $\operatorname{sdet} S(u)$ is equal to

$$s_{-1,-1}(u)s_{1,1}(u-1) - s_{1,-1}(u)s_{-1,1}(u-1) + \frac{s_{-1,-1}(u) \mp s_{1,1}(u)}{2u-1} s_{1,1}(u-1) =$$

$$s_{-1,-1}(u-1)s_{1,1}(u) - s_{-1,1}(u-1)s_{1,-1}(u) + s_{-1,-1}(u-1)\frac{s_{1,1}(u) \mp s_{-1,-1}(u)}{2u-1}.$$

Using the symmetry relation (3.6.4), we can rewrite this as follows:

$$\begin{aligned}\operatorname{sdet} S(u) &= \frac{2u+1}{2u\pm 1}(s_{-1,-1}(u-1)s_{-1,-1}(-u) \mp s_{-1,1}(u-1)s_{1,-1}(-u))\\ &= \frac{2u+1}{2u\pm 1}(s_{1,1}(-u)s_{1,1}(u-1) \mp s_{1,-1}(-u)s_{-1,1}(u-1)).\end{aligned}$$

An analogue of the latter expression for $\operatorname{sdet} S(u)$ with general N is given in [M3].

4.6. Remark. Let p be an arbitrary element of the symmetric group $\mathfrak{S}_N$. Replace the factors $S_i(u-i+1)$ and $R'_{ij}(-2u+i+j-2)$ in identity (4.3.1) with $S_{p(i)}(u-i+1)$ and $R'_{p(i),p(j)}(-2u+i+j-2)$ respectively. Then Proposition 4.4 holds for the same series $\operatorname{sdet} S(u)$. This follows immediately from the equalities:

$$PA_NP^{-1} = A_N, \quad PS_iP^{-1} = S_{p(i)}, \quad PR'_{ij}P^{-1} = R'_{p(i),p(j)},$$

where P denotes the image of p in $\operatorname{End}\mathcal{E}^{\otimes N}$.

4.7. Theorem. *We have*

$$\operatorname{sdet} S(u) = \gamma_N(u)\operatorname{qdet} T(u)\operatorname{qdet} T(-u+N-1),$$

where $\gamma_N(u) \equiv 1$ *for* $\mathrm{Y}^+(N)$ *and* $\gamma_N(u) = \dfrac{2u+1}{2u-N+1}$ *for* $\mathrm{Y}^-(N)$.

Proof. We use identity (4.3.1).

Step 1. Observe that $S_i = T_iT_i^\sigma$, where $T_i = T_i(u-i+1)$ and σ denotes the involutive automorphism of $\mathrm{Y}(N)$ (see Corollary 3.4): $T^\sigma(u) = T^t(-u)$. Therefore, the left hand side of (4.3.1) takes the form

$$A_NT_1T_1^\sigma R'_{12}\dots R'_{1N}T_2T_2^\sigma R'_{23}\dots R'_{2N}T_3T_3^\sigma\dots T_{N-1}T_{N-1}^\sigma R'_{N-1,N}T_NT_N^\sigma, \tag{1}$$

where $R'_{ij} = R'_{ij}(-2u+i+j-2)$. The equality (3.3.6) implies that for $1\le i,j\le N,\ \ i\ne j$

$$T_i^\sigma R'_{ij}T_j = T_jR'_{ji}T_i^\sigma. \tag{2}$$

Since the elements T_i and T_i^σ commute with R'_{jk} for $i\ne j,k$, we can rewrite (1) in the following way:

$$A_NT_1(T_1^\sigma R'_{12}T_2)R'_{13}\dots R'_{1N}(T_2^\sigma R'_{23}T_3)\dots(T_{N-1}^\sigma R'_{N-1,N}T_N)T_N^\sigma.$$

Applying (2) to the products enclosed in brackets, we obtain the expression

$$A_NT_1T_2R'_{12}(T_1^\sigma R'_{13}T_3)R'_{14}\dots R'_{1N}R'_{23}(T_2^\sigma R'_{24}T_4)\dots(T_{N-2}^\sigma R'_{N-2,N}T_N)R'_{N-1,N}T_{N-1}^\sigma T_N^\sigma.$$

Further applying (2) repeatedly, we bring (1) to the form

$$A_NT_1\dots T_NR'_{12}\dots R'_{1N}R'_{23}\dots R'_{2N}\dots R'_{N-1,N}T_1^\sigma\dots T_N^\sigma.$$

Replacing here A_N by A_N^2 and using Proposition 2.4, we transform this expression into

$$A_N T_N \dots T_1 A_N R'_{12} \dots R'_{1N} R'_{23} \dots R'_{2N} \dots R'_{N-1,N} T_1^\sigma \dots T_N^\sigma. \tag{3}$$

Further we will consider the algebras $\mathrm{Y}^+(N)$ and $\mathrm{Y}^-(N)$ separately.

Step 2. Let us show first that in the case of $\mathrm{Y}^+(N)$

$$A_N R'_{12} \dots R'_{1N} R'_{23} \dots R'_{2N} \dots R'_{N-1,N} = A_N. \tag{4}$$

Indeed, $A_N = \frac{1}{2} A_N (1 - P_{ij})$ for all $i \neq j$. However,

$$(1 - P_{ij}) R'_{ij} = (1 - P_{ij})(1 + \frac{1}{2u - i - j + 2} Q_{ij}) = 1 - P_{ij},$$

since $P_{ij} Q_{ij} = Q_{ij}$ by Proposition 3.2. Therefore, $A_N R'_{ij} = A_N$ and (4) is proved. Hence, (3) takes the form

$$A_N T_N \dots T_1 A_N T_1^\sigma \dots T_N^\sigma.$$

Since σ is an automorphism of $\mathrm{Y}(N)$, this is equal to

$$A_N T_N \dots T_1 (A_N T_1 \dots T_N)^\sigma = A_N \mathrm{qdet}\, T(u)\, \sigma(\mathrm{qdet}\, T(u))$$

by Proposition 2.5. Finally, applying Proposition 4.4, we conclude that

$$\mathrm{sdet}\, S(u) = \mathrm{qdet}\, T(u)\, \sigma(\mathrm{qdet}\, T(u)).$$

Step 3. In the case of $\mathrm{Y}^-(N)$, $\quad N = 2n$, we verify that

$$A_N R'_{12} \dots R'_{1N} R'_{23} \dots R'_{2N} \dots R'_{N-1,N} = \frac{2u+1}{2u-N+1} A_N. \tag{5}$$

First we note that the fundamental identity (4.2.1) implies the relation

$$A_N R'_{12} \dots R'_{1N} \dots R'_{N-1,N} = R'_{N-1,N} \dots R'_{N,1} \dots R'_{21} A_N. \tag{6}$$

To prove this, we consider the trivial homomorphism $\mathrm{Y}^-(N) \to \mathbb{C}$, $\ s_{ij}(u) \mapsto \delta_{ij}$, $-n \leq i,j \leq n$, and put $u_i = u - i + 1$ for $i = 1, \dots, N$. Then identity (4.2.1) becomes (6). Thus,

$$A_N R'_{12} \dots R'_{1N} \dots R'_{N-1,N} = \gamma_N(u) A_N \tag{7}$$

for a certain scalar function $\gamma_N(u)$. To calculate it, we apply the left hand side of (7) to the vector

$$v = e_{-n} \otimes \dots \otimes e_{-1} \otimes e_1 \otimes \dots \otimes e_n \in \mathcal{E}^{\otimes N}.$$

It is clear that $R'_{ij}v=v$ for $n+1\le i<j\le N$. Let $0\le a\le n$. Using induction on a we shall prove that

$$A_N R'_{12}\dots R'_{1N}\dots R'_{a,a+1}\dots R'_{a,N}v=\frac{2u+1}{2u-2a+1}A_N v. \tag{8}$$

Then in the case $a=n$ we shall obtain the required equality $\gamma_N(u)=\dfrac{2u+1}{2u-N+1}$. Let $A^{(i)}_m$ denote the normalized antisymmetrizer over the indices $\{i,i+1,\dots,m\}$ (see Subsection 2.3), so that $A^{(1)}_N$ coincides with A_N. Then $A_N=A_N A^{(2)}_N$ and

$$A^{(2)}_N R'_{12}\dots R'_{1N}=R'_{1N}\dots R'_{12}A^{(2)}_N. \tag{9}$$

Indeed, by Proposition 2.3 we can write

$$A^{(2)}_N=\frac{1}{(N-1)!}R_{N-1,N}R_{N-2,N}R_{N-2,N-1}\dots R_{2N}\dots R_{23}.$$

Using relation (4.2.3), we obtain

$$R_{2N}\dots R_{23}R'_{12}R'_{13}\dots R'_{1N}=R'_{13}\dots R'_{1N}R'_{12}R_{2N}\dots R_{23}.$$

An easy induction argument gives (9). Analogously, $A^{(m+1)}_N=A^{(m+1)}_N A^{(m+2)}_N$ for $m=1,\dots,N-2$ so that the left hand side of (8) can be rewritten as

$$A^{(1)}_N R'_{1N}\dots R'_{12}A^{(2)}_N R'_{2N}\dots R'_{23}A^{(3)}_N\dots A^{(a)}_N R'_{aN}\dots R'_{a,a+1}A^{(a+1)}_N v.$$

Therefore, to verify (8), we may replace v by any vector of the form

$$e_{-n}\otimes\cdots\otimes e_{-(n-a+1)}\otimes e_{p_1}\otimes\cdots\otimes e_{p_{N-a}},$$

where $(p_1,\dots,p_{N-a})$ is a permutation of the indices $(-(n-a),-(n-a)+1,\dots,n)$. Let us fix such a vector v', for which $p_1=n-a+1$. Then $R'_{am}v'=v'$ for $m=a+2,\dots,N$ while $R'_{a,a+1}v'$ equals

$$v'+\frac{1}{2u-2a+1}\sum_k \theta_{k,-(n-a+1)}e_{-n}\otimes\cdots\otimes e_{-(n-a+2)}\otimes e_k\otimes e_{-k}\otimes e_{p_2}\otimes\cdots\otimes e_{p_{N-a}}.$$

Let w denote the right hand side of the latter equality. Note that

$$A^{(a)}_N e_{-n}\otimes\cdots\otimes e_{-(n-a+2)}\otimes e_k\otimes e_{-k}\otimes e_{p_2}\otimes\cdots\otimes e_{p_{N-a}}=0$$

unless $k=\pm(n-a+1)$. Hence, by the induction hypothesis

$$A_N R'_{12}\dots R'_{1N}\dots R'_{a-1,a}\dots R'_{a-1,N}w=\frac{2u+1}{2u-2a+3}A_N w,$$

while $A_N w=\dfrac{2u-2a+3}{2u-2a+1}A_N v'$. Thus, (8) and hence (5) are proved.

Repeating the same arguments as in the case of $\mathrm{Y}^+(N)$, we obtain from (3) and (5) that

$$\operatorname{sdet} S(u) = \frac{2u+1}{2u-N+1} \operatorname{qdet} T(u)\, \sigma(\operatorname{qdet} T(u)).$$

Step 4. It remains to verify that

$$\sigma(\operatorname{qdet} T(u)) = \operatorname{qdet} T(-u+N-1).$$

We shall do this simultaneously for both algebras $\mathrm{Y}^+(N)$ and $\mathrm{Y}^-(N)$. By Proposition 2.7 we have

$$\operatorname{qdet} T(u) = \sum_{p\in\mathfrak{S}_N} \operatorname{sgn}(p) t_{-n,p(-n)}(u-N+1)\dots t_{n,p(n)}(u),$$

hence

$$\sigma(\operatorname{qdet} T(u)) = \sum_{p\in\mathfrak{S}_N} \operatorname{sgn}(p)\theta_{-n,p(-n)}\dots\theta_{n,p(n)} t_{-p(-n),n}(-u+N-1)\dots t_{-p(n),-n}(-u).$$

Let q be the permutation of the indices $(-n,-n+1,\dots,n)$ such that $q(i)=-i$. Then the permutation $p'=(-p(-n),\dots,-p(n))$ equals qp. Since

$$\theta_{-n,p(-n)}\dots\theta_{n,p(n)} = 1,$$

by using Remark 2.8 we obtain

$$\sigma(\operatorname{qdet} T(u)) = \sum_{p'\in\mathfrak{S}_N} \operatorname{sgn}(qp') t_{p'(-n),n}(-u+N-1)\dots t_{p'(n),-n}(-u)$$

$$= \operatorname{sgn}(q) \sum_{p'\in\mathfrak{S}_N} \operatorname{sgn}(p') t_{p'(-n),q(-n)}(-u+N-1)\dots t_{p'(n),q(n)}(-u) = \operatorname{qdet} T(-u+N-1),$$

which proves the theorem.

4.8. Theorem. $\operatorname{sdet} S(u)$ *lies in the center of* $\mathrm{Y}^{\pm}(N)$*, i.e. all its coefficients are central elements.*

Proof. As in the proof of Theorem 2.10 we consider the tensor space $\mathcal{E}^{\otimes(N+1)}$ where the copies of $\mathcal{E}$ are enumerated by the indices $0,1,\dots,N$. Set

$$(1)\qquad S_0 := S_0(v),\quad S_i := S_i(u-i+1),\quad i=1,\dots,N.$$

Then the statement of the theorem follows from the equality

$$(2)\qquad S_0(v)\operatorname{sdet} S(u) A_N = \operatorname{sdet} S(u) S_0(v) A_N.$$

To prove (2), we will use the fundamental identity (4.2.1). We have

$$R(v,u,u-1,\dots,u-N+1) S_0 R'_{01}\dots R'_{0N} S_1 R'_{12}\dots R'_{1N} S_2 \dots S_{N-1} R'_{N-1,N} S_N$$

(3) $$= S_N R'_{N,N-1} \dots R'_{N0} S_{N-1} \dots S_1 R'_{10} S_0 R(v, u, u-1, \dots, u-N+1).$$

It was proved in Subsection 2.10 that

$$R(v, u, u-1, \dots, u-N+1) = N! f(u,v) A_N,$$

where $f(u,v)$ is defined by (2.10.8). Since S_0 and A_N commute, the left hand side of (3) takes the form

(4) $$N! f(u,v) S_0 A_N R'_{01} \dots R'_{0N} S_1 R'_{12} \dots R'_{1N} S_2 \dots S_N.$$

However, (4.7.9) implies that

$$A_N R'_{01} \dots R'_{0N} = A_N^2 R'_{01} \dots R'_{0N} = A_N R'_{0N} \dots R'_{01} A_N.$$

By Remark 1.15,

$$A_N R'_{0N} \dots R'_{01} A_N = A_N (\tilde{R}_{01} \dots \tilde{R}_{0N})^{t_0} A_N = (A_N \tilde{R}_{01} \dots \tilde{R}_{0N} A_N)^{t_0},$$

where $\tilde{R}_{0i} = R_{0i}(-v-u+i-1)$. Repeating the arguments of the proof of the equality (2.10.8), we obtain that

$$A_N \tilde{R}_{01} \dots \tilde{R}_{0N} A_N = g(u,v) A_N,$$

where $g(u,v)$ is a non-zero element of the algebra $\mathbb{C}[u][[v^{-1}]]$. Thus, (4) takes the form

$$N! f(u,v) g(u,v) S_0(v) \operatorname{sdet} S(u) A_N.$$

Similar transformations allow us to rewrite the right hand side of (3) as

$$N! f(u,v) g(u,v) \operatorname{sdet} S(u) S_0(v) A_N,$$

which proves the theorem.

4.9. Remark. One could prove Theorem 4.8 by using the inclusion $\mathrm{Y}^{\pm}(N) \hookrightarrow \mathrm{Y}(N)$. For this, note that Theorem 4.7 implies that all the coefficients of $\operatorname{sdet} S(u)$ belong to the center of the algebra $\mathrm{Y}(N)$ and, therefore, to the center of its subalgebra $\mathrm{Y}^{\pm}(N)$.

4.10. The following generalization of Proposition 2.12 will be used in the proof of Theorem 4.11. Let $\mathfrak{a}$ be a Lie algebra and σ be an involutive automorphism of $\mathfrak{a}$. Denote by $\mathfrak{a}_0$ (resp. $\mathfrak{a}_1$) the set of elements $a \in \mathfrak{a}$ such that $\sigma(a) = a$ (resp. $\sigma(a) = -a$). Let $\mathfrak{a}[x]^\sigma$ denote the corresponding twisted polynomial current Lie algebra:

$$\mathfrak{a}[x]^\sigma = \mathfrak{a}_0 \oplus \mathfrak{a}_1 x \oplus \mathfrak{a}_0 x^2 \oplus \mathfrak{a}_1 x^3 \oplus \cdots.$$

Proposition. *Suppose that the center of the Lie algebra $\mathfrak{a}_0$ is trivial and the $\mathfrak{a}_0$-module $\mathfrak{a}_1$ has no nontrivial invariant elements. Then the center of the universal enveloping algebra $\mathrm{U}(\mathfrak{a}[x]^\sigma)$ is trivial.*

Proof. Let $\{f_1, \dots, f_r\}$ and $\{e_1, \dots, e_n\}$ be bases of $\mathfrak{a}_0$ and $\mathfrak{a}_1$ respectively. Then for $1 \le i \le r$ and $1 \le j \le n$

$$[f_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k,$$

where c_{ij}^k are structure constants. As in the proof of Proposition 2.12 it suffices to verify that if an element $A \in \mathrm{S}(\mathfrak{a}[x]^\sigma)$ is invariant under the adjoint action of $\mathfrak{a}[x]^\sigma$, then $A = 0$. Let m be the maximum integer such that the element $e_i x^m$ occurs in A for some $i \in \{1, \dots, n\}$. Then A may be written in the form

$$A = \sum_d A_d (e_1 \, x^m)^{d_1} \dots (e_n \, x^m)^{d_n},$$

where $d = (d_1, \dots, d_n)$, $d_1 \ge 0, \dots, d_n \ge 0$ and A_d is a polynomial in the variables $e_i x^s$, $s < m$ with coefficients from the subalgebra $\mathrm{S}(\mathfrak{a}_0[x^2])$. Just as in the proof of Proposition 2.12 we deduce the equalities (2.12.2) and (2.12.3) from the relations $\mathrm{ad}(f_i x)(A) = 0$ for $i = 1, \dots, r$. Repeating again the arguments of the proof of Proposition 2.12, we conclude that $A_d = 0$ for all $d \ne 0$, i.e. A belongs to the subalgebra $\mathrm{S}(\mathfrak{a}_0[x^2])$. To complete the proof, it remains to apply Proposition 2.12 to the Lie algebra $\mathfrak{a}_0$.

4.11. It follows from Theorem 4.7 that the Sklyanin determinant of the matrix $S(u)$ satisfies the relation

$$\gamma_N(u) \, \mathrm{sdet}\, S(-u + N - 1) = \gamma_N(-u + N - 1) \, \mathrm{sdet}\, S(u).$$

Therefore, in contrast to the case of $\mathrm{Y}(N)$, the coefficients $c_1, c_2, \dots$ of $\mathrm{sdet}\, S(u)$ are not algebraically independent. The following statement takes the place of Theorem 2.13.

Theorem. *The coefficients $c_2, c_4, c_6, \dots$ of the series $\mathrm{sdet}\, S(u)$ are algebraically independent and generate the center of $\mathrm{Y}^{\pm}(N)$. In particular, $c_1, c_3, \dots$ can be expressed in terms of $c_2, c_4, \dots$.*

Proof. We use the same idea as in the proof of Theorem 2.13. It consists of reducing the assertion to its analogue for the algebra $\mathrm{gr}_2 \mathrm{Y}^{\pm}(N)$, which is isomorphic to $\mathrm{U}(\mathfrak{gl}(N)[x]^\sigma)$ by Theorem 3.15. As in Subsection 2.13 we set

$$Z = E_{-n,-n} + E_{-n+1,-n+1} + \dots + E_{n,n}.$$

Step 1. For any $m = 1, 2, \dots$ the coefficient c_{2m} of $\mathrm{sdet}\, S(u)$ has the degree $2m - 1$ relative to $\deg_2$ and its image in the $(2m-1)$-th component of $\mathrm{gr}_2 \mathrm{Y}^{\pm}(N)$

coincides with Zx^{2m-1}. Indeed, Proposition 4.4 implies that

$$
\begin{aligned}
(1)\quad \operatorname{sdet} S(u)A_N = {} & A_N(S_1(u)\dots S_N(u-N+1)+\\
& \frac{1}{2u-1}S_1(u)Q_{12}S_2(u-1)\dots S_N(u-N+1)+\dots+\\
& \frac{1}{2u-2N+3}S_1(u)\dots S_{N-1}(u-N+2)Q_{N-1,N}S_N(u-N+1)+\dots+\\
& \prod_{1\le i<j\le N}\frac{1}{2u-i-j+1}S_1(u)Q_{12}\dots Q_{1N}\times\\
& S_2(u-1)Q_{23}\dots Q_{2N}S_3(u-2)\dots S_N(u-N+1)).
\end{aligned}
$$

Let us apply both sides of (1) to the vector

$$v = e_{-n}\otimes e_{-n+1}\otimes\dots\otimes e_n \in \mathcal{E}^{\otimes N}.$$

Comparing the coefficients of $u^{-M}A_N v$ in the left and right hand sides, we find that c_M is a linear combination of monomials of the form

$$(2)\qquad s_{i_1j_1}^{(M_1)}\dots s_{i_Nj_N}^{(M_N)},\qquad \text{where}\quad M_1+\dots+M_N\le M.$$

Moreover, the monomials (2) with $M_1+\dots+M_N=M$ arise only from the first summand of the right hand side of (1). It is clear that such monomials have the form

$$s_{p(-n),-n}^{(M_1)}\dots s_{p(n),n}^{(M_N)},\qquad p\in\mathfrak{S}_N.$$

From Definition 1.20 of $\deg_2$ we obtain the following formula:

$$c_M = s_{-n,-n}^{(M)}+\dots+s_{n,n}^{(M)}+(\text{terms of degree } < M-1),$$

which is an analogue of (2.13.3) for the algebra $\mathrm{Y}^{\pm}(N)$. This proves the assertion and the fact that the elements $c_2, c_4,\dots$ are algebraically independent.

Step 2. Now the theorem will follow from the fact that the center of the algebra $\mathrm{U}(\mathfrak{gl}(N)[x]^{\sigma})$ is generated by $Zx, Zx^3, Zx^5,\dots$. To see this, we note that

$$\mathrm{U}(\mathfrak{gl}(N)[x]^{\sigma}) = \mathbb{C}[Zx, Zx^3,\dots]\otimes\mathrm{U}(\mathfrak{sl}(N)[x]^{\sigma}).$$

It remains, therefore, to prove that the center of $\mathrm{U}(\mathfrak{sl}(N)[x]^{\sigma})$ is trivial. But this follows from Proposition 4.10 applied to the Lie algebra $\mathfrak{a}=\mathfrak{sl}(N)$ and the involution σ:

$$\sigma(E_{ij}) = -\theta_{ij}E_{-j,-i}.$$

The theorem is proved.

4.12. Remark. It follows from Theorems 4.7 and 4.11 that the center of the algebra $\mathrm{Y}^{\pm}(N)$ is contained in the center of the algebra $\mathrm{Y}(N)$. Furthermore, if N is even then the centers of $\mathrm{Y}^{+}(N)$ and $\mathrm{Y}^{-}(N)$ coincide with each other as subalgebras in $\mathrm{Y}(N)$.

4.13. Definition. Set

$$\mathrm{SY}^{\pm}(N) := \mathrm{SY}(N)\cap\mathrm{Y}^{\pm}(N).$$

This algebra is called the *special twisted Yangian*. In other words, $\mathrm{SY}^{\pm}(N)$ can be regarded as the subalgebra of $\mathrm{Y}^{\pm}(N)$ consisting of the elements which are stable under all of the automorphisms of the form ν_g (see Subsection 3.10).

4.14. Proposition. *The algebra* $\mathrm{Y}^{\pm}(N)$ *is isomorphic to the tensor product of its center* $\mathrm{Z}^{\pm}(N)$ *and the subalgebra* $\mathrm{SY}^{\pm}(N)$:

$$\mathrm{Y}^{\pm}(N) = \mathrm{Z}^{\pm}(N) \otimes \mathrm{SY}^{\pm}(N). \tag{1}$$

In particular, the center of $\mathrm{SY}^{\pm}(N)$ *is trivial.*

Proof. First we prove that

$$\mathrm{Y}^{\pm}(N) = \mathrm{Z}^{\pm}(N)\,\mathrm{SY}^{\pm}(N). \tag{2}$$

We shall use the notations of Subsection 2.16. Consider the series

$$\sigma_{ij}(u) = (\tilde{d}(u)\tilde{d}(-u))^{-1} s_{ij}(u), \qquad -n \le i, j \le n;$$

by (3.5.2) it coincides with

$$\sum_a \theta_{ja}\tau_{ia}(u)\tau_{-j,-a}(-u).$$

Let us verify that all the coefficients of the series $\tilde{d}(u)\tilde{d}(-u)$ belong to $\mathrm{Z}^{\pm}(N)$. By Theorem 4.7,

$$\gamma_N(u)^{-1}\mathrm{sdet}\, S(u) = \mathrm{qdet}\, T(u)\mathrm{qdet}\, T(-u+N-1)$$
$$= (\tilde{d}(u)\tilde{d}(-u))(\tilde{d}(u-1)\tilde{d}(-u+1))\dots(\tilde{d}(u-N+1)\tilde{d}(-u+N-1)).$$

Proposition 2.15 implies that all the coefficients of the series $\tilde{d}(u)\tilde{d}(-u)$ may be expressed as polynomials in the coefficients of the series $\mathrm{sdet}\, S(u)$. By Theorem 4.8, this proves that $\tilde{d}(u)\tilde{d}(-u) \in \mathrm{Z}^{\pm}(N)[[u^{-1}]]$. Note that $\sigma_{ij}(u) \in \mathrm{SY}^{\pm}(N)[[u^{-1}]]$, since these series are stable under all the automorphisms ν_g (see (2.16.1) and (3.10.1)). Now (2) follows from the decomposition

$$s_{ij}(u) = \tilde{d}(u)\tilde{d}(-u)\sigma_{ij}(u), \qquad -n \le i, j \le n.$$

Finally, the decomposition (1) is a consequence of Proposition 2.16 and the fact that $\mathrm{Z}^{\pm}(N) \subset \mathrm{Z}(N)$ and $\mathrm{SY}^{\pm}(N) \subset \mathrm{SY}(N)$.

4.15. Corollary. *The subalgebra* $\mathrm{SY}^{\pm}(N) \subset \mathrm{SY}(N)$ *is generated by all the coefficients of the series* $\sigma_{ij}(u)$, $\quad -n \le i, j \le n$.

Proof. The proof is the same as that of Corollary 2.17.

4.16. Corollary. $\mathrm{SY}^{\pm}(N)$ *is isomorphic to the factor-algebra*

$$\mathrm{Y}^{\pm}(N)/(\mathrm{sdet}\, S(u) = 1).$$

Proof. Proposition 4.14 implies that $\mathrm{Y}^{\pm}(N) = \mathrm{I}^{\pm} \oplus \mathrm{SY}^{\pm}(N)$, where $\mathrm{I}^{\pm}$ is the ideal of $\mathrm{Y}^{\pm}(N)$ generated by all of the coefficients of the series $\mathrm{sdet}\, S(u)$. This proves the assertion.

4.17. The twisted Yangian $\mathrm{Y}^{\pm}(N)$ seems not to possess any natural Hopf algebra structure. Nevertheless, the following proposition holds.

Proposition. $\mathrm{Y}^{\pm}(N)$ *is a left coideal of the Hopf algebra* $\mathrm{Y}(N)$, *i.e.,*

$$\Delta(\mathrm{Y}^{\pm}(N)) \subset \mathrm{Y}(N) \otimes \mathrm{Y}^{\pm}(N).$$

Moreover,

$$\Delta(s_{ij}(u)) = \sum_{k,l} \theta_{lj} t_{ik}(u) t_{-j,-l}(-u) \otimes s_{kl}(u), \tag{1}$$

where $-n \le i, j \le n$.

Proof. It is enough to prove (1). We use the notation of Subsection 1.28. It is clear that

$$\Delta(T^t(-u)) = T^t_{[2]}(-u) T^t_{[1]}(-u).$$

Therefore,

$$\Delta(S(u)) = \Delta(T(u)T^t(-u))$$

$$= T_{[1]}(u) T_{[2]}(u) T^t_{[2]}(-u) T^t_{[1]}(-u) = T_{[1]}(u) S_{[2]}(u) T^t_{[1]}(-u).$$

Rewriting this in terms of the matrix elements, we obtain (1).

4.18. Corollary. $\mathrm{SY}^{\pm}(N)$ *is a left coideal of the Hopf algebra* $\mathrm{SY}(N)$.

Proof. We have to verify that

$$\Delta(\mathrm{SY}^{\pm}(N)) \subset \mathrm{SY}(N) \otimes \mathrm{SY}^{\pm}(N).$$

Using Corollary 2.20 and Proposition 4.17, we obtain

$$\Delta(\sigma_{ij}(u)) = \Delta((\tilde{d}(u)\tilde{d}(-u))^{-1} s_{ij}(u))$$

$$= ((\tilde{d}(u)\tilde{d}(-u))^{-1} \otimes (\tilde{d}(u)\tilde{d}(-u))^{-1}) \sum_{k,l} \theta_{lj} t_{ik}(u) t_{-j,-l}(-u) \otimes s_{kl}(u)$$

$$= \sum_{k,l} \theta_{lj} \tau_{ik}(u) \tau_{-j,-l}(-u) \otimes \sigma_{kl}(u).$$

The assertion then follows from Corollary 4.15.

4.19. Comments. The results of this section, as those of Section 3, were announced in Olshanskiĭ [O2]. In [O2], the Sklyanin determinant $\operatorname{sdet} S(u)$ was called 'the double quantum determinant' and was denoted by $\operatorname{ddet} S(u)$. We think that the new terminology adopted in the present work is more emphatic. It is motivated by the fact that E.K.Sklyanin was the first to define the new type of determinant involving intermediate factors between matrix coefficients, see [S2]. One of the differences between the algebras studied in [S2] and the twisted Yangians is that here we must use, as the intermediate factors, the R'-matrices instead of Sklyanin's R^{-1}.

5. The quantum contraction and the quantum Liouville formula for the Yangian

Here we develop another approach to the investigation of the structure of the Yangian. This approach is based upon the use of a one-dimensional projection Q different from A_N. We construct a series $z(u)$ (called quantum contraction of the matrix $T(u)$), whose coefficient belong to the center of $\mathrm{Y}(N)$ and generate the center. We establish a link between the quantum contraction and the quantum determinant of the matrix $T(u)$ (the quantum Liouville formula). Then we calculate the square of the antipodal map S.

5.1. We will use here the notation of Sections 1 and 2. Here t will denote the usual transposition, for which $(E_{ij})^t = E_{ji}$. Denote

$$\hat{T}(u) = (T^t(u))^{-1}, \qquad \hat{R}(u) = (R'(u))^{-1} \tag{1}$$

where $R'(u) := R^{t_1}(u) = R^{t_2}(u) = 1 - Qu^{-1}$, and

$$Q := P^{t_1} = P^{t_2} = \sum_{i,j} E_{ij} \otimes E_{ij}.$$

A simple calculation (cf. Proposition 3.2) shows that

$$Q^2 = NQ \qquad \text{and} \qquad Q\mathcal{E}^{\otimes 2} = \mathbb{C}\eta \tag{2}$$

where $\eta = e_1 \otimes e_1 + \cdots + e_N \otimes e_N$, so that $N^{-1}Q$ is a one-dimensional projection in $\mathcal{E}^{\otimes 2}$. It follows from the first equality in (3) that

$$(1 - Qu^{-1})(1 + Q(u-N)^{-1}) = 1.$$

Therefore we have $\hat{R}(u) = 1 + Q(u-N)^{-1}$.

5.2. Proposition. *The following identity holds:*

$$Q\hat{T}_1(u)T_2(u-N) = T_2(u-N)\hat{T}_1(u)Q. \tag{1}$$

Proof. We start with the ternary relation (1.8.1):

$$R_{12}T_1T_2 = T_2T_1R_{12}, \tag{2}$$

where $R_{12} = R_{12}(u_1 - u_2)$, $T_1 = T_1(u_1)$, $T_2 = T_2(u_2)$. Further, we apply the transposition t_1 to both sides of (2). By Remark 1.15, we get

$$T_1^{t_1} R'_{12} T_2 = T_2 R'_{12} T_1^{t_1}.$$

After multiplying both sides of the last equality by $(T_1^{t_1})^{-1}$ and $(R'_{12})^{-1}$, we obtain that

$$T_2\hat{T}_1\hat{R}_{12} = \hat{R}_{12}\hat{T}_1T_2. \tag{3}$$

Now we multiply (3) by $u_1 - u_2 - N$ and put $u_1 = u$, $u_2 = u - N$. Then (3) turns into (1). (In other words, we use the fact that the rational function $\hat{R}(u)$ in the variable u with values in $\operatorname{End}\mathcal{E}^{\otimes 2}$ has a simple pole at the point $u = N$ and $\operatorname{res}_{u=N} \hat{R}(u) = Q$).

5.3. Proposition. *There exists a formal series*

$$z(u) = 1 + z_1 u^{-1} + z_2 u^{-2} + \cdots \in \mathrm{Y}(N)[[u^{-1}]]$$

such that (5.2.1) *equals* $z(u)Q$.

Proof. It follows from the second equality in (5.1.2) that (5.2.1) equals Q times a formal series $z(u)$ in u^{-1} with coefficients in $\mathrm{Y}(N)$. Since the coefficients of u^0 in the series $\hat{T}_1(u)$ and $T_2(u-N)$ are equal to 1, the same is true for $z(u)$.

We will call the series $z(u)$ the *quantum contraction* (of the matrix $T(u)$).

5.4. Let $t'_{ij}(u)$, $1 \le i,j \le N$, denote the image of the series $t_{ij}(u)$ under the antipodal map $\mathrm{S} : T(u) \mapsto T^{-1}(u)$, i.e., $t'_{ij}(u)$ is the matrix element of the matrix $T^{-1}(u)$.

Proposition. *For any* $i = 1, \ldots, N$

$$z^{-1}(u) = \sum_{a=1}^{N} t_{ai}(u) t'_{ia}(u-N) \tag{1}$$

$$= \sum_{a=1}^{N} t'_{ai}(u-N) t_{ia}(u), \tag{2}$$

and hence

$$z^{-1}(u) = \frac{1}{N} \operatorname{tr}\left(T(u)\, T^{-1}(u-N)\right) = \frac{1}{N} \operatorname{tr}\left(T^{-1}(u-N)\, T(u)\right).$$

Proof. Observe that

$$1 = \hat{T}_1(u) T_2(u-N) T_2^{-1}(u-N) T_1^t(u) = T_1^t(u) T_2^{-1}(u-N) T_2(u-N) \hat{T}_1(u).$$

Hence, by Proposition 5.3,

$$Q = Q\hat{T}_1(u) T_2(u-N) T_2^{-1}(u-N) T_1^t(u) = z(u) Q T_2^{-1}(u-N) T_1^t(u).$$

Similarly,

$$T_1^t(u) T_2^{-1}(u-N) Q z(u) = Q.$$

Thus,

$$QT_2^{-1}(u-N)T_1^t(u) = T_1^t(u)T_2^{-1}(u-N)Q = Qz^{-1}(u) = z^{-1}(u)Q. \tag{3}$$

On the other hand, we have

$$T_1^t(u)T_2^{-1}(u-N)Q(e_1 \otimes e_1) = T_1^t(u)T_2^{-1}(u-N)\sum_a e_a \otimes e_a$$

$$= \sum_{a,i,j} t_{ai}(u) t'_{ja}(u-N)(e_i \otimes e_j).$$

Therefore, by (3),

$$\delta_{ij} z^{-1}(u) = \sum_a t_{ai}(u) t'_{ja}(u - N), \tag{4}$$

which is a slight generalization of (1) and will be used later on.

To prove (2), we apply the relation

$$z^{-1}(u) Q = Q T_2^{-1}(u - N) T_1^t(u)$$

to the vector $e_i \otimes e_i$. Then a calculation similar to that performed above shows that the coefficients of the vector η (see (5.1.4)) in the left and right hand sides coincide with those of (2).

5.5. Theorem. *All the coefficients of the series $z(u)$ belong to the center of* $\mathrm{Y}(N)$.

Proof. Consider the tensor space $\mathcal{E}^{\otimes 3}$ where the copies of $\mathcal{E}$ are labelled by the indices $0, 1, 2$ and put $T_i = T_i(u_i)$, $i = 0, 1, 2$ for formal variables u_0, u_1, u_2. The statement of the theorem will follow from the identity

$$T_0 z(u_1) Q_{12} = z(u_1) T_0 Q_{12}. \tag{1}$$

Step 1. We prove the auxiliary identity

$$R_{20} \hat{R}_{10} \hat{R}_{12} = \hat{R}_{12} \hat{R}_{10} R_{20}, \tag{2}$$

where $R_{ij} = R_{ij}(u_i - u_j)$. Applying the transposition t_1 to both sides of the Yang–Baxter identity

$$R_{12} R_{10} R_{20} = R_{20} R_{10} R_{12},$$

we obtain (see Remark 1.15) that

$$R_{10}^{t_1} R_{12}^{t_1} R_{20} = R_{20} R_{12}^{t_1} R_{10}^{t_1}.$$

To get (2), it is sufficient to multiply both sides of this identity by each of $(R_{10}^{t_1})^{-1}$ and $(R_{12}^{t_1})^{-1}$ from the left and from the right.

We will also need another identity

$$R_{20} \hat{R}_{10} Q_{12} = Q_{12} \hat{R}_{10} R_{20} = Q_{12}(1 - (u_0 - u_2)^{-2}), \qquad u_1 - u_2 = N. \tag{3}$$

To prove the first equality in (3), we take the residue of both sides of (2) at $u_1 - u_2 = N$. By (5.1.4), in order to verify the second equality in (3) it suffices to apply $Q_{12} \hat{R}_{10} R_{20}$ to the basis vectors $e_i \otimes e_1 \otimes e_1$, $\quad 1 \le i \le N$. We have

$$Q_{12} \hat{R}_{10} R_{20}(e_i \otimes e_1 \otimes e_1) = Q_{12} \hat{R}_{10}(e_i \otimes e_1 \otimes e_1 - \frac{1}{u_2 - u_0} e_1 \otimes e_1 \otimes e_i)$$

$$= Q_{12}(e_i \otimes e_1 \otimes e_1 - \frac{1}{u_2 - u_0} e_1 \otimes e_1 \otimes e_i + \frac{\delta_{i1}}{u_1 - u_0 - N} \sum_j e_j \otimes e_j \otimes e_1$$

$$-\frac{1}{(u_2-u_0)(u_1-u_0-N)}\sum_j e_j\otimes e_j\otimes e_i)$$

$$= e_i\otimes\eta-\frac{\delta_{i1}}{u_2-u_0}e_1\otimes\eta+\frac{\delta_{i1}}{u_1-u_0-N}e_1\otimes\eta-\frac{1}{(u_2-u_0)(u_1-u_0-N)}e_i\otimes\eta.$$

Since $u_1-N=u_2$, this proves (3).

Step 2. Let us verify that for arbitrary variables u_0,u_1,u_2 the following identity holds:

$$R_{20}\hat R_{10}T_2\hat T_1\hat R_{12}T_0=T_0T_2\hat T_1\hat R_{12}\hat R_{10}R_{20}. \tag{4}$$

Note that if $i\neq j,k$, then T_i and $\hat T_i$ commute with R_{jk} and $\hat R_{jk}$. Therefore, the left hand side of (4) can be transformed in the following way:

$$\begin{aligned}
R_{20}\hat R_{10}T_2\hat T_1\hat R_{12}T_0 &= R_{20}T_2(\hat R_{10}\hat T_1T_0)\hat R_{12}\\
&=(R_{20}T_2T_0)\hat T_1\hat R_{10}\hat R_{12} &&\text{by}\quad(5.2.3)\\
&=T_0T_2R_{20}\hat T_1\hat R_{10}\hat R_{12} &&\text{by}\quad(1.8.1)\\
&=T_0T_2\hat T_1(R_{20}\hat R_{10}\hat R_{12})\\
&=T_0T_2\hat T_1\hat R_{12}\hat R_{10}R_{20} &&\text{by (2)},
\end{aligned}$$

which coincides with the right hand side of (4).

Step 3. Let us take the residue of both sides of (4) at $u_1-u_2=N$. We obtain:

$$R_{20}\hat R_{10}T_2\hat T_1Q_{12}T_0=T_0T_2\hat T_1Q_{12}\hat R_{10}R_{20},\qquad u_1-u_2=N.$$

By Proposition 5.3, we can rewrite this as follows:

$$R_{20}\hat R_{10}z(u_1)Q_{12}T_0=T_0z(u_1)Q_{12}\hat R_{10}R_{20}.$$

By (3), the left hand side is

$$z(u_1)T_0Q_{12}(1-(u_0-u_2)^{-2}),$$

while the right hand side is

$$T_0z(u_1)Q_{12}(1-(u_0-u_2)^{-2}).$$

Thus, the equality (1) is established and Theorem 5.5 is proved.

5.6. Let $\mathcal{A}$ be an arbitrary associative algebra. For any $p=1,2,\dots,m$ we introduce the p-th partial trace as the map

$$\mathrm{tr}_p:\mathcal{A}\otimes\mathrm{End}\,\mathcal{E}^{\otimes m}\to\mathcal{A}\otimes\mathrm{End}\,\mathcal{E}^{\otimes(m-1)}$$

such that

$$\mathrm{tr}_p: E_{i_1j_1}\otimes\cdots\otimes E_{i_pj_p}\otimes\cdots\otimes E_{i_mj_m}\mapsto E_{i_1j_1}\otimes\cdots\otimes\delta_{i_pj_p}\otimes\cdots\otimes E_{i_mj_m}.$$

Furthermore, for any subset $\{p_1, \dots, p_k\} \subset \{1, \dots, m\}$ one defines the map

$$\mathrm{tr}_{p_1,\dots,p_k} : \mathcal{A} \otimes \mathrm{End}\, \mathcal{E}^{\otimes m} \mapsto \mathcal{A} \otimes \mathrm{End}\, \mathcal{E}^{\otimes (m-k)}$$

as the composition of $\mathrm{tr}_{p_1}, \dots, \mathrm{tr}_{p_k}$. We will simply write tr instead of $\mathrm{tr}_{1,\dots,m}$. Let

$$A_r = \sum_{i_1,\dots,i_m,j_1,\dots,j_m} a^{(r)}_{i_1 j_1 \dots i_m j_m} \otimes E_{i_1 j_1} \otimes \dots \otimes E_{i_m j_m}; \qquad r = 1, 2$$

be two arbitrary elements of $\mathcal{A} \otimes \mathrm{End}\, \mathcal{E}^{\otimes m}$. It follows immediately from the definition of the trace, that if the elements $a^{(1)}_{i_1 j_1 \dots i_m j_m}$ and $a^{(2)}_{j_1 i_1 \dots j_m i_m}$ commute for any sets of indices $\{i_1, \dots, i_m\}$ and $\{j_1, \dots, j_m\}$, then

$$\mathrm{tr}(A_1 A_2) = \mathrm{tr}(A_2 A_1). \tag{1}$$

5.7. Theorem. *We have*

$$z(u) = \frac{\mathrm{qdet}\, T(u-1)}{\mathrm{qdet}\, T(u)}. \tag{1}$$

Proof. Consider the auxiliary algebra

$$\mathrm{Y}(N)[[u^{-1}]] \otimes \mathrm{End}\, \mathcal{E}^{\otimes (N+1)}, \tag{2}$$

where the copies of $\mathcal{E}$ are enumerated by the indices $0, 1, \dots, N$. As in Subsection 4.7, $A^{(i)}_m$ denotes the normalized antisymmetrizer over the indices $\{i, i+1, \dots, m\}$.

Step 1. We prove the identity

$$P_{0N} A^{(1)}_N \mathrm{qdet}\, T(u-1) T^{-1}_N(u-N) T_0(u)$$

$$= A^{(0)}_{N-1} P_{0N} T_{N-1}(u-N+1) \dots T_0(u) A^{(1)}_{N-1}. \tag{3}$$

By Proposition 2.5,

$$A^{(1)}_N \mathrm{qdet}\, T(u-1) = A^{(1)}_N T_1(u-1) \dots T_N(u-N).$$

Hence, the left hand side of (3) can be rewritten as

$$P_{0N} A^{(1)}_N T_1(u-1) \dots T_{N-1}(u-N+1) T_0(u). \tag{4}$$

Proposition 2.3 and the fundamental identity (2.1.2) imply that

$$A^{(1)}_{N-1} T_1(u-1) \dots T_{N-1}(u-N+1) = T_{N-1}(u-N+1) \dots T_1(u-1) A^{(1)}_{N-1}. \tag{5}$$

It is clear that $A^{(1)}_N = A^{(1)}_N A^{(1)}_{N-1}$, so, making use of (5), we rewrite (4) as follows

$$P_{0N} A^{(1)}_N T_{N-1}(u-N+1) \dots T_1(u-1) A^{(1)}_{N-1} T_0(u).$$

Moving $A^{(1)}_{N-1}$ to the right and using the obvious relation $P_{0N}A^{(1)}_N = A^{(0)}_{N-1}P_{0N}$, we obtain the right hand side of (3).

Step 2. Now we calculate the trace of both sides of (3). To do this, we apply each side of (3) to the vector

$$v = e_{i_0} \otimes e_{i_1} \otimes \cdots \otimes e_{i_N} \in \mathcal{E}^{\otimes(N+1)}$$

and decompose the image with respect to the canonical basis in the space $\mathcal{E}^{\otimes(N+1)}$. We are interested in the coefficient of v in this decomposition. It is clear that the trace is equal to the sum of these coefficients over all the vectors v. For the left hand side we have:

$$\begin{aligned}&\mathrm{tr}\,(P_{0N}A^{(1)}_N \mathrm{qdet}\, T(u-1)T_N^{-1}(u-N)T_0(u))\\ &\qquad\qquad = \mathrm{qdet}\, T(u-1)\mathrm{tr}\,(P_{0N}A^{(1)}_N T_N^{-1}(u-N)T_0(u)),\end{aligned}$$

which by (5.6.1) equals

$$\mathrm{qdet}\, T(u-1)\mathrm{tr}\,(A^{(1)}_N T_N^{-1}(u-N)T_0(u)P_{0N}).$$

Furthermore,

$$\begin{aligned}A^{(1)}_N T_N^{-1}(u-N)T_0(u)P_{0N}v &= A^{(1)}_N T_N^{-1}(u-N)T_0(u)(e_{i_N} \otimes e_{i_1} \otimes \cdots \otimes e_{i_{N-1}} \otimes e_{i_0})\\ &= \sum_{a,b} A^{(1)}_N t'_{bi_0}(u-N)t_{ai_N}(u)(e_a \otimes e_{i_1} \otimes \cdots \otimes e_{i_{N-1}} \otimes e_b).\end{aligned}$$

Hence, the coefficient of v is zero, unless $(i_1, \ldots, i_N)$ is a permutation of the indices $(1, \ldots, N)$. In the latter case, the coefficient is equal to

$$\frac{1}{N!}t'_{i_N i_0}(u-N)t_{i_0 i_N}(u). \tag{6}$$

Using (5.4.2), we obtain that the sum of the elements (6) over all the vectors v equals $z^{-1}(u)$. Thus, the trace of the left hand side of (3) is $\mathrm{qdet}\, T(u-1)z^{-1}(u)$.

Now, again using (5.6.1), for the right hand side of (3) we have:

$$\begin{aligned}&\mathrm{tr}\,(A^{(0)}_{N-1}P_{0N}T_{N-1}(u-N+1)\ldots T_0(u)A^{(1)}_{N-1})\\ &\qquad\qquad = \mathrm{tr}\,(T_{N-1}(u-N+1)\ldots T_0(u)A^{(1)}_{N-1}A^{(0)}_{N-1}P_{0N}).\end{aligned}$$

Since $A^{(1)}_{N-1}A^{(0)}_{N-1} = A^{(0)}_{N-1}$, we transform it as follows:

$$\begin{aligned}\mathrm{tr}\,(T_{N-1}(u-N+1)\ldots T_0(u)A^{(0)}_{N-1}P_{0N}) &= \mathrm{tr}\,(\mathrm{qdet}\, T(u)A^{(0)}_{N-1}P_{0N})\\ &= \mathrm{qdet}\, T(u)\mathrm{tr}\,(A^{(0)}_{N-1}P_{0N}).\end{aligned}$$

Here we used Proposition 2.5. Further,

$$A^{(0)}_{N-1}P_{0N}v = A^{(0)}_{N-1}(e_{i_N} \otimes e_{i_1} \otimes \cdots \otimes e_{i_{N-1}} \otimes e_{i_0}).$$

The coefficient of v in this decomposition is zero, unless $i_0 = i_N$ and $(i_0, \dots, i_{N-1})$ is a permutation of the indices $(1, \dots, N)$. In the latter case it equals $(N!)^{-1}$. Taking the sum over all the vectors v, we find that the trace of the right hand side of (3) is $\operatorname{qdet} T(u)$, which proves the theorem.

5.8. Remark. Relation (5.7.1) may be regarded as a 'quantum analogue' of the classical Liouville formula for the derivative of the determinant of a matrix-valued function. To see this, for each $h \in \mathbb{C} \setminus \{0\}$ consider the algebra $\mathrm{Y}(N, h)$ introduced in Subsection 1.25. Define the generating series $t_{ij}(u)$ for the elements $t_{ij}^{(1)}, t_{ij}^{(2)}, \dots \in \mathrm{Y}(N, h)$ in the same way as it was done in Subsection 1.6 for the algebra $\mathrm{Y}(N)$; then form the matrix $T(u)$. The quantum determinant and the quantum contraction for the new algebra are given by

$$\operatorname{qdet} T(u) = \sum_{p \in \mathfrak{S}_N} \operatorname{sgn}(p)\, t_{p(1),1}(u)\, t_{p(2),2}(u-h) \dots t_{p(N),N}(u - Nh + h), \tag{1}$$

$$z(u) = \left(\frac{1}{N} \operatorname{tr} \left(T(u)\, T^{-1}(u - Nh) \right) \right)^{-1}. \tag{2}$$

Arguments similar to those used in Subsections 2.10 and 5.5 show that $\operatorname{qdet} T(u)$ and $z(u)$ are central in $\mathrm{Y}(N, h)$. The equality (5.7.1) is then generalized to

$$z(u) = \frac{\operatorname{qdet} T(u-h)}{\operatorname{qdet} T(u)}.$$

Due to the definition (2) this equality can be rewritten as

$$\operatorname{tr} \left(T^{-1}(u - Nh) \cdot \frac{T(u) - T(u - Nh)}{Nh} \right)$$

$$= \frac{1}{\operatorname{qdet} T(u-h)} \cdot \frac{\operatorname{qdet} T(u) - \operatorname{qdet} T(u-h)}{h}. \tag{3}$$

In the limit $h \to 0$ the entries of the matrix $T(u)$ become commutative while the quantum determinant (1) tends to the usual $\det T(u)$. In this limit we obtain from (3) the equality

$$\operatorname{tr} \left(T^{-1}(u) \frac{d}{du} T(u) \right) = \frac{1}{\det T(u)} \cdot \frac{d}{du} \det T(u)$$

which is the Liouville formula. For this reason we shall refer to relation (5.7.1) as the *quantum Liouville formula* for the T-matrix.

Note also that the proof of Theorem 5.7 does not use the fact that $z(u)$ is central (Theorem 5.5). Thus, Theorem 5.5 could also be derived from Theorems 2.10 and 5.7.

5.9. Corollary. *The coefficients $z_2, z_3, \dots$ of the series $z(u)$ are algebraically independent and generate the whole center of the algebra* $\mathrm{Y}(N)$.

Proof. By Theorem 5.7

$$(1+z_1u^{-1}+z_2u^{-2}+\dots)(1+d_1u^{-1}+d_2u^{-2}+\dots) = 1+d_1(u-1)^{-1}+d_2(u-1)^{-2}+\dots.$$

As

$$(u-1)^{-k} = \sum_{i=0}^{\infty} \binom{k+i-1}{i} u^{-k-i},$$

we obtain that $z_1 = 0$ and

$$z_k + d_k + \sum_{i=1}^{k-1} z_{k-i} d_i = d_k + \sum_{j=1}^{k-1} \binom{k-1}{j} d_{k-j}$$

for $k \geq 2$. An easy induction shows that for every $n \geq 1$ the coefficient d_n is a polynomial in the variables $z_2, \dots, z_{n+1}$ and the coefficient z_{n+1} may be expressed as $z_{n+1} = nd_n + (\dots)$, where $(\dots)$ stands for a polynomial in the variables $d_1, \dots, d_{n-1}$. By Theorem 2.13 this proves the assertion.

5.10. Remark. All the arguments and results of Subsections 5.1-5.9, in particular the construction of the quantum contraction $z(u)$ and Theorem 5.7, remain valid when the transposition t is changed to the transposition with respect to the forms $< \cdot, \cdot >_+$ and $< \cdot, \cdot >_-$ on the space $\mathcal{E}$ introduced in Subsection 3.1.

5.11. We conclude this section with a theorem which provides a link between the antipode S and the quantum contraction $z(u)$.

Theorem. *We have*

$$\mathrm{S}^2 = \sigma_N \circ \mu_{z(u)},$$

where σ_N and $\mu_{z(u)}$ are the automorphisms of $\mathrm{Y}(N)$ *defined by the formulas* (1.12.1) *and* (1.12.2) *respectively.*

Proof. The equality $T(u)T^{-1}(u) = 1$ implies that

$$\sum_{a=1}^{N} t_{ia}(u) t'_{aj}(u) = \delta_{ij}. \tag{1}$$

By applying the antiautomorphism S to both sides of (1), we get

$$\sum_{a=1}^{N} t''_{aj}(u) t'_{ia}(u) = \delta_{ij}, \tag{2}$$

where $t''_{ij}(u)$ denotes the image of $t_{ij}(u)$ under the automorphism S^2. On the other hand, by relation (5.4.4),

$$\sum_{a=1}^{N} t_{aj}(u+N) t'_{ia}(u) = \delta_{ij} z^{-1}(u+N). \tag{3}$$

Comparing (2) and (3), we find that

$$t''_{ij}(u) = t_{ij}(u+N)z(u+N),$$

which proves the theorem.

5.12. Comments. The approach to the description of the center of the Yangian $\mathrm{Y}(N)$ presented in this chapter was proposed by the second author in [N1]. Theorems 5.5, 5.7, 5.11 and sketches of their proofs are contained in [N1] (in fact they are stated there in a greater generality – for the Yangian of the Lie superalgebra $\mathfrak{gl}(N|M)$).

6. The quantum contraction and the quantum Liouville formula for the twisted Yangian

In this section, we extend the results of Section 5 to the twisted Yangian $\mathrm{Y}^{\pm}(N)$. We start with the introduction of a 'covering' algebra $\tilde{\mathrm{Y}}^{\pm}(N)$ by removing the symmetry relation (3.6.3) from the definition of the twisted Yangian $\mathrm{Y}^{\pm}(N)$. Then we construct certain series $\delta(u)$ whose coefficients are central elements of the 'covering' algebra and prove that the symmetry condition on the S-matrix can be expressed as the equality $\delta(u) = 1$ (Theorem 6.4). By using the series $\delta(u)$ we define an analogue of the quantum contraction $z(u)$ for the twisted Yangian. This is a series $y(u)$ whose coefficients form a new system of generators for the center. We describe the relationship between $y(u)$, $z(u)$ and the Sklyanin determinant $\mathrm{sdet}\,(u)$; see Theorems 6.7, 6.8. The latter theorem is an analogue of the quantum Liouville formula for the twisted Yangian.

6.1. Let us denote by $\tilde{\mathrm{Y}}^{\pm}(N)$ the complex associative algebra with generators $\tilde{s}_{ij}^{(1)}, \tilde{s}_{ij}^{(2)}, \dots$ where $-n \le i, j \le n$; subject to the following relations. Introduce the generating series $\tilde{s}_{ij}(u)$ and form the matrix $\tilde{S}(u)$ in the same fashion as it was done in (3.5.3) and (3.5.2) for the generators $s_{ij}^{(1)}, s_{ij}^{(2)}, \dots$. We impose on the matrix $\tilde{S}(u)$ the quaternary relation but not the symmetry relation (see Subsection 3.6):

$$(1)\qquad R(u-v)\tilde{S}_1(u)R'(-u-v)\tilde{S}_2(v) = \tilde{S}_2(v)R'(-u-v)\tilde{S}_1(u)R(u-v).$$

In the next few subsections we will establish several facts about the structure of the algebra $\tilde{\mathrm{Y}}^{\pm}(N)$.

6.2. Proposition. *There exists a formal series*

$$\delta(u) = 1 + \delta_1 u^{-1} + \delta_2 u^{-2} + \cdots \in \tilde{\mathrm{Y}}^{\pm}(N)[[u^{-1}]]$$

such that

$$(1)\qquad Q\tilde{S}_1(u)R(2u)\tilde{S}_2^{-1}(-u) = \tilde{S}_2^{-1}(-u)R(2u)\tilde{S}_1(u)Q = (1 \mp \frac{1}{2u})\delta(u)Q.$$

Proof. By multiplying both sides of the quaternary relation (6.1.1) by $\tilde{S}_2^{-1}(v)$ we obtain the identity

$$(2)\qquad \tilde{S}_2^{-1}(v)R(u-v)\tilde{S}_1(u)R'(-u-v) = R'(-u-v)\tilde{S}_1(u)R(u-v)\tilde{S}_2^{-1}(v).$$

Note that the rational function $R'(-u) = 1 + Qu^{-1}$ has a simple pole at $u = 0$ and $\operatorname*{res}_{u=0} R'(-u) = Q$. Taking the residue of both sides of (2) at $u + v = 0$, we get the first equality in (1). Now, by Proposition 3.2, the assertion follows from the fact that the coefficients of u^0 in the series $\tilde{S}_1(u)$, $R(2u)$, $\tilde{S}_2^{-1}(-u)$ and $(1 \mp (2u)^{-1})$ are equal to 1.

6.3. Theorem. *All the coefficients of the series $\delta(u)$ belong to the center of the algebra $\tilde{\mathrm{Y}}^{\pm}(N)$.*

Proof. The proof is quite similar to that of Theorem 5.5. Consider the tensor space $\mathcal{E}^{\otimes 3}$, where the copies of $\mathcal{E}$ are enumerated by the indices 0,1,2 and set

$$\tilde{S}_i = \tilde{S}_i(u_i); \qquad i = 0, 1, 2;$$
$$R_{ij} = R_{ij}(u_i - u_j), \quad R'_{ij} = R'_{ij}(-u_i - u_j); \qquad 0 \le i < j \le 2$$

where u_0, u_1, u_2 are formal variables. We shall prove the identity

$$\tilde{S}_0 \delta(u_1) Q_{12} = \delta(u_1) \tilde{S}_0 Q_{12}, \tag{1}$$

which implies the statement of the theorem.

Step 1. We verify that the following auxiliary identities hold provided that $u_1 + u_2 = 0$:

$$Q_{12} R'_{01} R_{02} = R_{02} R'_{01} Q_{12} = Q_{12}(1 - (u_0 + u_1)^{-2}), \tag{2}$$
$$Q_{12} R'_{02} R_{01} = R_{01} R'_{02} Q_{12} = Q_{12}(1 - (u_0 - u_1)^{-2}). \tag{3}$$

Indeed, by (4.2.3)

$$R'_{12} R'_{01} R_{02} = R_{02} R'_{01} R'_{12}.$$

Taking the residue at $u_1 + u_2 = 0$, we obtain the first equality in (2). By Proposition 3.2, to verify the second equality in (2), it suffices to apply $Q_{12} R'_{01} R_{02}$ to the basis vectors $e_i \otimes e_{-1} \otimes e_1, \ i = -n, -n+1, \dots, n$. We have

$$\begin{aligned}
Q_{12} R'_{01} R_{02}(e_i \otimes e_{-1} \otimes e_1) &= Q_{12} R'_{01}(e_i \otimes e_{-1} \otimes e_1 - \frac{1}{u_0 - u_2} e_1 \otimes e_{-1} \otimes e_i) \\
= Q_{12}(e_i \otimes e_{-1} \otimes e_1 &+ \frac{\delta_{i1}}{u_0 + u_1} \sum_j \theta_{ji} e_j \otimes e_{-j} \otimes e_1 - \frac{1}{u_0 - u_2} e_1 \otimes e_{-1} \otimes e_i \\
&- \frac{1}{(u_0 - u_2)(u_0 + u_1)} \sum_j \theta_{j1} e_j \otimes e_{-j} \otimes e_i) \\
= e_i \otimes \xi + \frac{\delta_{i1}\theta_{i1}}{u_0 + u_1} e_1 \otimes \xi &- \frac{\delta_{i1}}{u_0 - u_2} e_1 \otimes \xi - \frac{1}{(u_0 - u_2)(u_0 + u_1)} e_i \otimes \xi.
\end{aligned}$$

Since $u_2 = -u_1$, this implies (2). The proof of (3) is quite similar.

Step 2. We prove that for arbitrary formal variables u_0, u_1, u_2 the following identity holds:

$$R_{01} R'_{02} \tilde{S}_0 R_{02} R'_{01} \tilde{S}_2^{-1} R_{12} \tilde{S}_1 R'_{12} = \tilde{S}_2^{-1} R_{12} \tilde{S}_1 R'_{12} R'_{01} R_{02} \tilde{S}_0 R'_{02} R_{01}. \tag{4}$$

We use the fact that $\tilde S_i$ and $\tilde S_i^{-1}$ commute with R_{jk} and R'_{jk}, if $i \neq j,k$. Let us transform the left hand side of (4) in the following way:

$$\begin{aligned} R_{01}(R'_{02}\tilde S_0 R_{02}\tilde S_2^{-1})R'_{01}R_{12}\tilde S_1 R'_{12} &= R_{01}\tilde S_2^{-1}R_{02}\tilde S_0(R'_{02}R'_{01}R_{12})\tilde S_1 R'_{12} && \text{by (6.2.2)}\\ &= R_{01}\tilde S_2^{-1}R_{02}\tilde S_0 R_{12}R'_{01}R'_{02}\tilde S_1 R'_{12} && \text{by (4.2.3)}\\ &= \tilde S_2^{-1}(R_{01}R_{02}R_{12})\tilde S_0 R'_{01}\tilde S_1 R'_{02}R'_{12} && \\ &= \tilde S_2^{-1}R_{12}R_{02}(R_{01}\tilde S_0 R'_{01}\tilde S_1)R'_{02}R'_{12} && \text{by (1.5.2)}\\ &= \tilde S_2^{-1}R_{12}R_{02}\tilde S_1 R'_{01}\tilde S_0(R_{01}R'_{02}R'_{12}) && \text{by (6.1.1)}\\ &= \tilde S_2^{-1}R_{12}R_{02}\tilde S_1 R'_{01}\tilde S_0 R'_{12}R'_{02}R_{01} && \text{by (4.2.3)}\\ &= \tilde S_2^{-1}R_{12}\tilde S_1(R_{02}R'_{01}R'_{12})\tilde S_0 R'_{02}R_{01}, && \end{aligned}$$

which coincides with the right hand side of (4) by (4.2.3).

Step 3. Let us take the residues of both sides of (4) at $u_1+u_2=0$. We obtain the equality

$$R_{01}R'_{02}\tilde S_0 R_{02}R'_{01}(\tilde S_2^{-1}R_{12}\tilde S_1 Q_{12}) = (\tilde S_2^{-1}R_{12}\tilde S_1 Q_{12})R'_{01}R_{02}\tilde S_0 R'_{02}R_{01},$$

provided $u_1+u_2=0$. By Proposition 6.2, this may be rewritten as follows:

$$R_{01}R'_{02}\tilde S_0 R_{02}R'_{01}Q_{12}\delta(u_1) = \delta(u_1)Q_{12}R'_{01}R_{02}\tilde S_0 R'_{02}R_{01}. \tag{5}$$

By (2) and (3), the left hand side equals

$$(1-(u_0+u_1)^{-2})R_{01}R'_{02}Q_{12}\tilde S_0\delta(u_1) = (1-(u_0+u_1)^{-2})(1-(u_0-u_1)^{-2})\tilde S_0\delta(u_1)Q_{12}.$$

Similarly, the right hand side of (5) is

$$(1-(u_0+u_1)^{-2})(1-(u_0-u_1)^{-2})\delta(u_1)\tilde S_0 Q_{12},$$

which proves (1) and the theorem.

6.4. Theorem. *The relation $\delta(u)=1$ is equivalent to the symmetry relation*

$$\tilde S(-u) = \tilde S(u) \pm \frac{\tilde S(u)-\tilde S(-u)}{2u}. \tag{1}$$

Proof. We will make use of Proposition 6.2. Let us apply both sides of the equality

$$(1 \mp \frac{1}{2u})\delta(u)Q = Q\tilde S_1(u)R(2u)\tilde S_2^{-1}(-u)$$

to the vector $e_{-i}\otimes e_j \in \mathcal{E}^{\otimes 2}$. By Proposition 3.2, we have

$$(1 \mp \frac{1}{2u})\delta(u)Q(e_{-i}\otimes e_j) = \delta_{ij}(1 \mp \frac{1}{2u})\theta_{i1}\delta(u)\xi.$$

Denote by $s'_{ij}(u),\ -n \le i,j \le n$ the matrix elements of the matrix $\tilde S^{-1}(u)$. Then

$$Q\tilde S_1(u)R(2u)\tilde S_2^{-1}(-u)(e_{-i}\otimes e_j) = Q\tilde S_1(u)R(2u)\sum_k s'_{kj}(-u)(e_{-i}\otimes e_k)$$

$$= Q\tilde S_1(u)\sum_k s'_{kj}(-u)(e_{-i}\otimes e_k - \frac{1}{2u}e_k\otimes e_{-i})$$

$$= Q(\sum_{k,l}\tilde s_{l,-i}(u)s'_{kj}(-u)(e_l\otimes e_k) - \frac{1}{2u}\sum_{k,l}\tilde s_{kl}(u)s'_{kj}(-u)(e_l\otimes e_{-i}))$$

$$= (\sum_k \theta_{k1}\tilde s_{-k,-i}(u)s'_{kj}(-u) - \frac{1}{2u}\theta_{-i,1}\sum_k \tilde s_{ik}(u)s'_{kj}(-u))\xi.$$

Thus,

$$\delta_{ij}(1\mp\frac{1}{2u})\delta(u) = \sum_k(\theta_{ik}\tilde s_{-k,-i}(u)\mp\frac{1}{2u}\tilde s_{ik}(u))s'_{kj}(-u).$$

On the other hand,

$$\delta_{ij} = \sum_k \tilde s_{ik}(-u)s'_{kj}(-u).$$

By comparing the last two equalities, we obtain that

$$(1\mp\frac{1}{2u})\tilde s_{ij}(-u)\delta(u) = \theta_{ij}\tilde s_{-j,-i}(u)\mp\frac{1}{2u}\tilde s_{ij}(u). \tag{2}$$

Hence, the relation $\delta(u)=1$ implies that

$$\theta_{ij}\tilde s_{-j,-i}(u) = \tilde s_{ij}(-u)\pm\frac{\tilde s_{ij}(u)-\tilde s_{ij}(-u)}{2u},$$

which essentially coincides with (6.4.1); see the relation (3.6.4).

Conversely, if the symmetry relation (6.4.1) is valid, then (2) for $i=j=-1$ becomes

$$(1\mp\frac{1}{2u})\tilde s_{-1,-1}(-u)\delta(u) = \tilde s_{1,1}(u)\mp\frac{1}{2u}\tilde s_{-1,-1}(u) = (1\mp\frac{1}{2u})\tilde s_{-1,-1}(-u).$$

As $\tilde s_{-1,-1}(-u)$ is invertible, this implies that $\delta(u)=1$. The theorem is proved.

6.5. Propoition. *The mapping*

$$\mathrm{inv}:\tilde S(u)\mapsto \tilde S^{-1}(-u-\frac{N}{2})$$

defines an involutive automorphism of the algebra $\tilde{\mathrm Y}^{\pm}(N)$.

Proof. By inverting the left and right hand sides of the quaternary relation (6.1.1), we obtain

$$\tilde S_2^{-1}(v)(R'(-u-v))^{-1}\tilde S_1^{-1}(u)R^{-1}(u-v) = R^{-1}(u-v)\tilde S_1^{-1}(u)(R'(-u-v))^{-1}\tilde S_2^{-1}(v). \tag{1}$$

Furthermore, we observe that

$$R^{-1}(u-v)=\frac{(u-v)^2}{(u-v)^2-1}R(v-u),$$

$$(R'(-u-v))^{-1}=R'(u+v-N).$$

Therefore, replacing (u,v) by $(-u-N/2,-v-N/2)$, we transform (1) into the quaternary relation for the matrix $\tilde S^{-1}(-u-N/2)$.

It remains to verify that $\operatorname{inv}\circ\operatorname{inv}=1$. Let us apply inv to both sides of the relation

$$(\operatorname{inv}\tilde S(u))\tilde S(-u-\frac{N}{2})=1.$$

We obtain that

$$(\operatorname{inv}\circ\operatorname{inv})(\tilde S(u))\tilde S^{-1}(u)=1,\quad \text{i.e.,}\quad (\operatorname{inv}\circ\operatorname{inv})(\tilde S(u))=\tilde S(u).$$

6.6. Let us consider the image $\operatorname{inv}(\delta(u))$ of the element $\delta(u)$ under the automorphism inv of the algebra $\tilde{\mathrm{Y}}^{\pm}(N)$. Denote by $y(u)$ the image of $\operatorname{inv}(\delta(u-N/2))$ under the factorization map $\tilde{\mathrm{Y}}^{\pm}(N)\to\mathrm{Y}^{\pm}(N)$. By Theorem 6.3, all the coefficients of the series $\operatorname{inv}(\delta(u))$ belong to the center of $\tilde{\mathrm{Y}}^{\pm}(N)$, hence, all the coefficients of $y(u)$ are central elements in the algebra $\mathrm{Y}^{\pm}(N)$. We shall call the series $y(u)$ the *quantum contraction* (of the matrix $S(u)$).

Proposition. *The following identities hold in the algebra* $\mathrm{Y}^{\pm}(N)$*:*

$$(1)\quad QS_1^{-1}(-u)R(2u-N)S_2(u-N)$$
$$=S_2(u-N)R(2u-N)S_1^{-1}(-u)Q=(1\mp\frac{1}{2u-N})y(u)Q.$$

Proof. It suffices to apply the automorphism inv to each of the parts of identity (6.2.1), replace u by $u-N/2$, and take their images in the algebra $\mathrm{Y}^{\pm}(N)$.

6.7. Let us define the quantum contraction $z(u)$ corresponding to the transposition t that was used in Section 3: $(E_{ij})^t=\theta_{ij}E_{-j,-i}$; see Proposition 5.3 and Remark 5.10. Then we obtain the following

Theorem. *We have the equality* $y(u)=z(u)z^{-1}(-u+N)$.

Proof. Recall that $S(u)=T(u)T^t(-u)$. It follows from Proposition 6.6 that

$$(1\mp\frac{1}{2u-N})y(u)Q=Q(T_1(-u)T_1^t(u))^{-1}R(2u-N)T_2(u-N)T_2^t(-u+N)$$

$$(1)\qquad =Q\hat T_1(u)T_1^{-1}(-u)R(2u-N)T_2(u-N)T_2^t(-u+N),$$

where $\hat T(u)$ denotes the matrix $(T^t(u))^{-1}$. However,

$$T_1^{-1}(-u)R(2u-N)T_2(u-N)=T_2(u-N)R(2u-N)T_1^{-1}(-u),$$

which is an immediate consequence of the ternary relation written in the form

$$R(u-v)T_2(u)T_1(v) = T_1(v)T_2(u)R(u-v).$$

Indeed, it suffices to multiply both sides of the latter relation by $T_1^{-1}(v)$ from the left and from the right and to replace (u,v) by $(u-N,-u)$. Therefore, the right hand side of (1) takes the form:

$$Q\hat{T}_1(u)T_2(u-N)R(2u-N)T_1^{-1}(-u)T_2^t(-u+N).$$

By Proposition 5.3 (see Remark 5.10) the last expression equals

$$Qz(u)R(2u-N)T_1^{-1}(-u)T_2^t(-u+N) = z(u)QR(2u-N)T_1^{-1}(-u)T_2^t(-u+N).$$

Now, using (3.2.4), we obtain

$$QR(2u-N) = Q(1-\frac{P}{2u-N}) = (1\mp\frac{1}{2u-N})Q.$$

Hence, by (5.4.3), we have

$$\begin{aligned}(1\mp\frac{1}{2u-N})y(u)Q &= (1\mp\frac{1}{2u-N})z(u)QT_1^{-1}(-u)T_2^t(-u+N)\\ &= (1\mp\frac{1}{2u-N})z(u)z^{-1}(-u+N)Q,\end{aligned}$$

which proves the assertion.

6.8. Now we prove an analogue of the quantum Liouville formula for the twisted Yangian.

Theorem. *In the algebra* $\mathrm{Y}^{\pm}(N)$ *we have*

$$y(u) = \varepsilon_N(u)\frac{\operatorname{sdet} S(u-1)}{\operatorname{sdet} S(u)}, \tag{1}$$

where $\varepsilon_N(u)\equiv 1$ *for* $\mathrm{Y}^+(N)$ *and* $\varepsilon_N(u) = \dfrac{(2u+1)(2u-N-1)}{(2u-1)(2u-N+1)}$ *for* $\mathrm{Y}^-(N)$.

Proof. It follows from Theorems 6.7 and 5.7 that

$$y(u) = \frac{\operatorname{qdet} T(u-1)}{\operatorname{qdet} T(u)}\,\frac{\operatorname{qdet} T(-u+N)}{\operatorname{qdet} T(-u+N-1)}.$$

Furthermore, by Theorem 4.7 we get

$$y(u) = \frac{\gamma_N(u)}{\gamma_N(u-1)}\,\frac{\operatorname{sdet} S(u-1)}{\operatorname{sdet} S(u)} = \varepsilon_N(u)\frac{\operatorname{sdet} S(u-1)}{\operatorname{sdet} S(u)},$$

and the theorem is proved.

6.9. Corollary. *The coefficients* $y_1, y_2, \dots$ *of the quantum contraction* $y(u)$ *generate the center of the algebra* $\mathrm{Y}^{\pm}(N)$.

Proof. By using Theorem 4.11 and repeating the arguments of the proof of Corollary 5.9, we obtain that the coefficients of the series $\operatorname{sdet} S(u-1)(\operatorname{sdet} S(u))^{-1}$ generate the center of $\mathrm{Y}^{\pm}(N)$. Since $\varepsilon_N(u)$ has the form $1+a_1u^{-1}+a_2u^{-2}+\dots,\ \ a_i\in\mathbb{C}$, the same is true for the series $y(u)$.

6.10. Comments. Theorem 6.4 was announced in Olshanskiĭ [O2]; it has some similarity to Theorem 6 in Drinfeld [D1].

7. The quantum determinant and the Sklyanin determinant of block matrices

Here we will gather several results related to dividing the T- and S-matrices into rectangular blocks.

7.1. Let us fix a partition of the number N into a sum of two nonnegative integers, $N = r + s$. For any $N \times N$-matrix A we will denote by ${}^{11}A$, ${}^{12}A$, ${}^{21}A$, ${}^{22}A$ the blocks of the matrix A with respect to this partition so that

$$A = \begin{pmatrix} {}^{11}A & {}^{12}A \\ {}^{21}A & {}^{22}A \end{pmatrix}.$$

7.2. Proposition. *The matrix elements of the matrices* ${}^{11}T(u)$ *and* ${}^{22}(T^{-1}(v))$ *commute with each other.*

Proof. Multiplying both sides of the ternary relation (1.8.1) by $T_2^{-1}(v)$ from the left and from the right, we obtain

$$T_2^{-1}(v)R(u-v)T_1(u) = T_1(u)R(u-v)T_2^{-1}(v).$$

Since $R(u-v) = 1 - P(u-v)^{-1}$, this may be expressed as

$$[T_1(u), T_2^{-1}(v)] = \frac{1}{u-v}(T_1(u)PT_2^{-1}(v) - T_2^{-1}(v)PT_1(u)).$$

Rewriting this in terms of matrix elements (see the proof of Proposition 1.8), we obtain that

$$[t_{ij}(u), t'_{kl}(v)] = \frac{1}{u-v}(\delta_{kj}\sum_a t_{ia}(u)t'_{al}(v) - \delta_{il}\sum_a t'_{ka}(v)t_{aj}(u)),$$

where, as before, $t'_{ij}(u)$ denotes the matrix element of the matrix $T^{-1}(u)$. Thus, if $1 \le i,j \le r$ and $r < k,l \le N$, then $[t_{ij}(u), t'_{kl}(v)] = 0$, which proves the proposition.

7.3. For an invertible $N \times N$-matrix A over $\mathbb{C}$ one has the following formula for the determinant of A:

$$\det\, A \,\det\, {}^{11}(A^{-1}) = \det\, {}^{22}A.$$

Here we prove an analogue of this formula for the quantum determinant of the T-matrix. Denote by $t^*_{ij}(u)$, $1 \le i,j \le N$, the image of the series $t_{ij}(u)$ under the automorphism inv of the algebra $\mathrm{Y}(N)$. That is, $t^*_{ij}(u)$ is the matrix element of the matrix $T^*(u) := T^{-1}(-u)$. Since $T^*(u)$ satisfies the ternary relation (see Subsection 1.12), we can repeat the construction of the quantum determinant from Section 2 for the matrix $T^*(u)$. In particular, analogues of formulae (2.8.1) and (2.8.2) hold for $\operatorname{qdet} T^*(u)$. We shall use the following one below: for $q \in \mathfrak{S}_N$

$$\text{(1)} \qquad \operatorname{qdet} T^*(u) = \operatorname{sgn}(q) \sum_{p \in \mathfrak{S}_N} \operatorname{sgn}(p) t^*_{p(1),q(1)}(u) \ldots t^*_{p(N),q(N)}(u-N+1).$$

Theorem. *We have*

$$\operatorname{qdet} T(u)\ \operatorname{qdet}{}^{11}(T^*)(-u+N-1) = \operatorname{qdet}{}^{22}T(u).$$

Proof. By Proposition 2.5,

$$\operatorname{qdet} T(u) A_N = A_N T_1 \dots T_N, \tag{2}$$

where $T_i = T_i(u-i+1)$ for $i = 1, \dots, N$. Let us multiply both sides of (2) by $T_N^{-1} \dots T_{s+1}^{-1}$ from the right. Then (2) takes the form

$$\operatorname{qdet} T(u) A_N T_N^{-1} \dots T_{s+1}^{-1} = A_N T_1 \dots T_s. \tag{3}$$

Now we apply both the sides of (3) to the basis vector

$$v_0 = e_{r+1} \otimes \dots \otimes e_N \otimes e_1 \otimes \dots \otimes e_r \in \mathcal{E}^{\otimes N}.$$

For the right hand side of (3) we have

$$A_N T_1 \dots T_s v_0 = A_N \sum_{i_1, \dots, i_s} t_{i_1, r+1}(u) \dots t_{i_s, N}(u-s+1) e_{i_1} \otimes \dots \otimes e_{i_s} \otimes e_1 \otimes \dots \otimes e_r.$$

The coefficient of v_0 in this decomposition equals

$$\frac{1}{N!} \sum_p \operatorname{sgn}(p) t_{p(r+1), r+1}(u) \dots t_{p(N), N}(u-s+1),$$

where p runs over all the permutations of the set of indices $\{r+1, \dots, N\}$. By (2.7.1), this is nothing else but $(N!)^{-1} \operatorname{qdet}{}^{22}T(u)$.

Similarly, for the left hand side of (3) we obtain

$$\begin{aligned} &A_N T_N^{-1} \dots T_{s+1}^{-1} v_0 \\ &\quad = A_N \sum_{i_{s+1}, \dots, i_N} t'_{i_N, r}(u-N+1) \dots t'_{i_{s+1}, 1}(u-s) e_{r+1} \otimes \dots \otimes e_N \otimes e_{i_{s+1}} \otimes \dots \otimes e_{i_N}. \end{aligned}$$

It is clear that the coefficient of v_0 in this expression equals

$$\begin{aligned} &\frac{1}{N!} \sum_{p \in \mathfrak{S}_r} \operatorname{sgn}(p) t'_{p(r), r}(u-N+1) \dots t'_{p(1), 1}(u-s) \\ &\qquad = \frac{1}{N!} \sum_{p \in \mathfrak{S}_r} \operatorname{sgn}(p) t^*_{p(r), r}(-u+N-1) \dots t^*_{p(1), 1}(-u+s). \end{aligned} \tag{4}$$

It follows immediately from (1) that (4) coincides with $(N!)^{-1} \operatorname{qdet}{}^{11}(T^*)(-u+N-1)$ and the theorem is proved.

7.4. Now we shall prove analogues of Proposition 7.2 and Theorem 7.3 for the twisted Yangian $\mathrm{Y}^{\pm}(N)$. We will keep using the notation of Sections 3, 4 and 6.

Let us fix a nonnegative integer $M \le N$ such that $N - M$ is even. Put $m = [M/2]$. For a $N \times N$-matrix A denote by ${}^{11}A$ and ${}^{22}A$ the submatrices of A whose rows and columns are enumerated by the indices $\{-m, -m+1, \dots, m\}$ and $\{-n, -n+1, \dots, -m-1, m+1, \dots, n\}$ respectively.

7.5. Proposition. *The matrix elements of the matrices* ${}^{11}S(u)$ *and* ${}^{22}(S^{-1}(v))$ *commute with each other.*

Proof. Multiplying both sides of the quaternary relation (3.6.2) by $S_2^{-1}(v)$ from the left and from the right, we obtain the relation

$$S_2^{-1}(v)R(u-v)S_1(u)R'(-u-v) = R'(-u-v)S_1(u)R(u-v)S_2^{-1}(v).$$

Using the equalities

$$R(u-v) = 1 - \frac{P}{u-v} \qquad \text{and} \qquad R'(-u-v) = 1 + \frac{Q}{u+v}$$

we rewrite the last relation as follows:

$$\begin{aligned}
[S_1(u), S_2^{-1}(v)] =& \frac{1}{u-v}(S_1(u)PS_2^{-1}(v) - S_2^{-1}(v)PS_1(u)) \\
&- \frac{1}{u+v}(QS_1(u)S_2^{-1}(v) - S_2^{-1}(v)S_1(u)Q) \\
&+ \frac{1}{u^2-v^2}(QS_1(u)PS_2^{-1}(v) - S_2^{-1}(v)PS_1(u)Q),
\end{aligned}$$

or in terms of the matrix elements:

$$\begin{aligned}
[s_{ij}(u), s'_{kl}(v)] =& \frac{1}{u-v}(\delta_{kj}\sum_a s_{ia}(u)s'_{al}(v) - \delta_{il}\sum_a s'_{ka}(v)s_{aj}(u)) \\
&- \frac{1}{u+v}(\delta_{i,-k}\sum_a \theta_{ak}s_{-a,j}(u)s'_{al}(v) - \delta_{j,-l}\sum_a \theta_{al}s'_{ka}(v)s_{i,-a}(u)) \\
&+ \frac{1}{u^2-v^2}(\delta_{i,-k}\theta_{kj}\sum_a s_{-j,a}(u)s'_{al}(v) - \delta_{j,-l}\theta_{il}\sum_a s'_{ka}(v)s_{a,-i}(u)).
\end{aligned}$$

Thus, if $-m \le i,j \le m$ and $m < |k|, |l| \le n$, then $[s_{ij}(u), s'_{kl}(v)] = 0$, and the proposition is proved.

7.6. Set $S^*(u) := \mathrm{inv}(\tilde{S}(u)) = \tilde{S}^{-1}(-u - N/2)$ and denote by $s^*_{ij}(u)$ the matrix elements of the matrix $S^*(u)$. Note that in the proof of the fundamental identity (4.2.1) and hence in the construction of the Sklyanin determinant of the S-matrix (Propositions 4.3 and 4.4) we have only used the quaternary relation (3.6.2) and have never used the symmetry relation (3.6.3). Therefore, these results remain valid for the algebra $\tilde{\mathrm{Y}}^{\pm}(N)$; see Subsection 6.1. In particular, all of the coefficients of $\mathrm{sdet}\, \tilde{S}(u)$ belong to the center of $\tilde{\mathrm{Y}}^{\pm}(N)$. Since inv is an automorphism of the algebra $\tilde{\mathrm{Y}}^{\pm}(N)$, we can repeat the construction of the Sklyanin determinant for the matrix $S^*(u)$ and obtain the analogues of Propositions 4.3 and 4.4 for this matrix. By applying the factorization map $\tilde{\mathrm{Y}}^{\pm}(N) \to \mathrm{Y}^{\pm}(N)$ to the equality (1) below, we will obtain an analogue of Theorem 7.3 for the twisted Yangian.

Theorem. *In the algebra* $\tilde{\mathrm{Y}}^{\pm}(N)$ *we have*

$$\operatorname{sdet} \tilde{S}(u) \operatorname{sdet} {}^{22}(S^*)(-u + \frac{N}{2} - 1) = \operatorname{sdet} {}^{11}\tilde{S}(u). \tag{1}$$

Proof. By the analogue of Proposition 4.3 for the matris $\tilde{S}(u)$ we have

$$\operatorname{sdet} \tilde{S}(u) A_N = A_N \tilde{S}_1 R'_{12} \dots R'_{1N} \tilde{S}_2 \dots \tilde{S}_{N-1} R'_{N-1,N} \tilde{S}_N, \tag{2}$$

where $\tilde{S}_i = \tilde{S}_i(u - i + 1)$ for $1 \le i \le N$ and $R'_{ij} = R'_{ij}(-2u + i + j - 2)$ for $1 \le i, j \le N,\ i \ne j$. Using the fact that $\tilde{S}_i$ and R'_{jk} commute provided $i \ne j, k$, we rewrite the right hand side of (2) in the form

$$A_N \tilde{S}_1 R'_{12} \dots R'_{1M} \tilde{S}_2 \dots \tilde{S}_{M-1} R'_{M-1,M} \tilde{S}_M R'_{1,M+1} \cdots R'_{1N} \dots R'_{M,M+1} \cdots R'_{MN}$$

$$\tilde{S}_{M+1} R'_{M+1,M+2} \cdots R'_{M+1,N} \tilde{S}_{M+2} \dots \tilde{S}_{N-1} R'_{N-1,N} \tilde{S}_N.$$

Since all of the matrices $\tilde{S}_i$ and R'_{ij} are invertible, relation (2) is equivalent to the following one:

(3)

$$\operatorname{sdet} \tilde{S}(u) A_N \tilde{S}_N^{-1} (R'_{N-1,N})^{-1} \tilde{S}_{N-1}^{-1} \dots \tilde{S}_{M+2}^{-1} (R'_{M+1,N})^{-1} \dots (R'_{M+1,M+2})^{-1} \tilde{S}_{M+1}^{-1}$$

$$= A_N \tilde{S}_1 R'_{12} \dots R'_{1M} \tilde{S}_2 \dots \tilde{S}_{M-1} R'_{M-1,M} \tilde{S}_M R'_{1,M+1} \cdots R'_{1N} \dots R'_{M,M+1} \cdots R'_{MN}.$$

Now we compare the diagonal elements of the matrices in the left and right hand sides of (3) corresponding to the vector

$$v_0 = e_{-m} \otimes e_{-m+1} \otimes \cdots \otimes e_m \otimes e_{-n} \otimes \cdots \otimes e_{-m-1} \otimes e_{m+1} \otimes \cdots \otimes e_n \in \mathcal{E}^{\otimes N}.$$

It is clear that $R'_{ij} v_0 = v_0$, if $i \le M$ and $j > M$. Thus, for the right hand side of (3) we have:

(4)

$$A_N \tilde{S}_1 R'_{12} \dots R'_{1M} \tilde{S}_2 \dots \tilde{S}_{M-1} R'_{M-1,M} \tilde{S}_M R'_{1,M+1} \cdots R'_{1N} \dots R'_{M,M+1} \cdots R'_{MN} v_0$$

$$= A_N \tilde{S}_1 R'_{12} \dots R'_{1M} \tilde{S}_2 \dots \tilde{S}_{M-1} R'_{M-1,M} \tilde{S}_M v_0$$

$$= A_N \sum_{i_1,\dots,i_M} a_{i_1,\dots,i_M}(u) e_{i_1} \otimes \cdots \otimes e_{i_M} \otimes e_{-n} \otimes \cdots \otimes e_{-m-1} \otimes e_{m+1} \otimes \cdots \otimes e_n,$$

where $a_{i_1,\dots,i_M}(u)$ are certain elements of $\tilde{\mathrm{Y}}^{\pm}(N)[[u^{-1}]]$. To calculate the coefficient of v_0 in this decomposition we may take into account only those summands for which the sequence $(i_1, \dots, i_M)$ is obtained by a permutation of the indices $(-m, -m + 1, \dots, m)$. It is not difficult to see that this allows us to replace in (4) the matrix $\tilde{S}$ by its submatrix ${}^{11}\tilde{S}$; A_N by the antisymmetrizer $(N!)^{-1} M! A_M$ in the space $(\mathbb{C}^M)^{\otimes M}$ where $\mathbb{C}^M$ is spanned by the basis vectors $e_{-m}, e_{-m+1}, \dots, e_m$; and R' by its restriction to the space $\mathbb{C}^M \otimes \mathbb{C}^M$. Therefore, by Proposition 4.3, the coefficient of v_0 in (4) equals $(N!)^{-1} \operatorname{sdet} {}^{11}\tilde{S}(u)$.

Furthermore, set $\tilde{u}_i = -u + i - \dfrac{N}{2} - 1$ for $i = M+1, \dots, N$. Since

$$(R'_{ij})^{-1} = R'_{ij}(2u - i - j + N + 2),$$

we can express the left hand side of (3) as follows:

$$\text{(5)} \qquad \operatorname{sdet} \tilde{S}(u) A_N S^*_N R^*_{N,N-1} \dots R^*_{N,M+1} S^*_{N-1} \dots S^*_{M+2} R^*_{M+2,M+1} S^*_{M+1},$$

where $S^*_i = S^*_i(\tilde{u}_i)$ and $R^*_{ij} = R'_{ij}(-\tilde{u}_i - \tilde{u}_j)$. Repeating the same arguments as for the right hand side of (3) and using Remark 4.6, we obtain that the matrix element of the operator (5) on the basis vector v_0 equals $(N!)^{-1} \operatorname{sdet} \tilde{S}(u) \operatorname{sdet}^{22}(S^*)(\tilde{u}_N)$, which proves the theorem.

7.7. Comments. Proposition 7.2 (which in fact holds for even more general R-matrices) is due to Cherednik [C1, Theorem 2.4]. The use of quantum determinants of certain submatrices of $T(u)$ was significant in Drinfeld [D3], Nazarov–Tarasov [NT], Molev [M2].

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