WEIGHT VECTORS OF THE BASIC $A_1^{(1)}$-MODULE AND THE LITTLEWOOD-RICHARDSON RULE

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Abstract

The basic representation of $A_1^{(1)}$ is studied. The weight vectors are represented in terms of Schur functions. A suitable base of any weight space is given. Littlewood-Richardson rule appears in the linear relations among weight vectors.

§0 INTRODUCTION

The aim of this letter is to give an explicit expression of the weight vectors of the basic $A_1^{(1)}$-module realized on the polynomial ring of infinitely many variables.

In 1978, Lepowsky and Wilson [5] constructed the basic representation of the affine Lie algebra $A_1^{(1)}$ by making use of the vertex operator. This construction was generalized to other types of affine Lie algebras [4] and applied to a study of nonlinear integrable differential equations such as KP, BKP and KdV hierarchies [1]. Roughly speaking, weighted homogeneous polynomial solutions (τ-functions) of these hierarchies are weight vectors whose weights lie on the Weyl group orbit through the highest weight, namely the maximal weights. They are expressed by means of the Schur functions or the Schur Q-functions, reflecting that the formal solutions constitute an infinite dimensional Grassmann manifold or its submanifold.

In this letter we show that the weight space of any weight of the basic $A_1^{(1)}$-module is spanned by 2-reduced Schur functions. We will choose a suitable base of any weight space and discuss the linear relations among the weight vectors. To our surprise, the Littlewood-Richardson rule [2,6] appears in the linear relations. The formula obtained can also be viewed as an identity for 2-modular characters of the symmetric group. We believe that there is a deep connection between affine Lie algebras and modular representations of the symmetric group.

We also believe that the Virasoro algebra is related with this story. Wakimoto [9,10] proved that the sum of weight spaces of weights $\{\Lambda - n\delta; n \in \mathbb{N}\}$ admits a Virasoro action, where Λ is a maximal weight and δ is the fundamental imaginary root. It is irreducible in the case of the basic $A_1^{(1)}$-module. As a corollary, the Schur functions indexed by staircase Young diagrams are characterized as singular vectors of the Virasoro representation. The relation between this Virasoro representation and our weight vectors must be clarified. (*See also* [7,11].)

§1 REVIEW OF THE BASIC $A_1^{(1)}$-MODULE

We first review some ingredients of a realization of the basic $A_1^{(1)}$-module [3,5]. Let $\mathfrak{g} = A_1^{(1)}$ be the affine Lie algebra corresponding to the Cartan matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ with the standard Chevalley generators $\{e_0, e_1, f_0, f_1, \alpha_0^\vee, \alpha_1^\vee\}$. The basic representation of $\mathfrak{g}$ is realized on the space of polynomials of infinitely many variables

$$V = \mathbb{C}[t_1, t_3, t_5, \dots]$$

as follows. For any odd natural number j, let $a_j = \frac{\partial}{\partial t_j}$ and $a_{-j} = jt_j$. Then $\{a_j (j \in \mathbb{Z}, odd), Id\}$ span the infinite dimensional Heisenberg algebra acting on V, which is a subalgebra of $\mathfrak{g}$. The action of $\mathfrak{g}$ is constructed by the so called vertex operator. Let p be an indeterminate and put

$$\xi(t,p) = \sum_{j\geq 1, odd} t_j p^j = \sum_{j\geq 1, odd} \frac{a_{-j}}{j} p^j,$$
$$\xi(\tilde{\partial}, p^{-1}) = \sum_{j\geq 1, odd} \frac{1}{j}\frac{\partial}{\partial t_j} p^{-j} = \sum_{j\geq 1, odd} \frac{a_j}{j} p^{-j}.$$

The vertex operator is defined by

$$X(p) = -\frac{1}{2} e^{2\xi(t,p)} e^{-2\xi(\tilde{\partial},p^{-1})}.$$

Expanding $X(p)$ as a formal power series of p and p^{-1}:

$$X(p) = \sum_{k\in\mathbb{Z}} X_k p^{-k},$$

we have differential operators X_k $(k \in \mathbb{Z})$ acting on V. It is proved [5] that operators a_j $(j \in \mathbb{Z}$,odd), X_k $(\ k \in \mathbb{Z})$ and identity constitute the affine Lie algebra $\mathfrak{g} = A_1^{(1)}$, namely the basic representation of $\mathfrak{g}$. This is the irreducible highest weight $\mathfrak{g}$-module with highest weight Λ_0, where $\Lambda_0(\alpha_0^\vee) = 1$, $\Lambda_0(\alpha_1^\vee) = 0$. Let α_0 and α_1 be the simple roots of $\mathfrak{g}$, and $\delta = \alpha_0 + \alpha_1$ be the fundamental imaginary root. It is well-known [3] that the set of weights P of the basic $\mathfrak{g}$-module is given by

$$P = \left\{ \Lambda_0 + q\delta + p\alpha_1; p, q \in \mathbb{Z}, q \leq -p^2 \right\}.$$

A weight Λ on the parabola $q = -p^2$ is said to be maximal in the sense that $\Lambda + \delta$ is no longer a weight. Maximal weights consist a single Weyl group orbit. For a maximal weight Λ the weight vector is expressed by the Schur function. For any Young diagram Y of N cells, the Schur function indexed by Y is defined by

$$S_Y(t) = \sum_{\nu_1+2\nu_2+\cdots=N} \chi_Y(\nu) \frac{t_1^{\nu_1} t_2^{\nu_2}\cdots}{\nu_1!\nu_2!\cdots},$$

where $\chi_Y(\nu)$ is the character value of the irreducible representation Y of the symmetric group $\mathfrak{S}_N$, evaluated at the conjugacy class of the cycle type $\nu = (1^{\nu_1}2^{\nu_2}\cdots N^{\nu_N})$ [6]. The Schur function $S_Y(t)$ is obviously a weighted homogeneous ($\deg t_j = j$) polynomial of degree $|Y|$. For a non-negative integer r put $K_r = (r, r-1, r-2, \ldots, 2, 1)$ be the staircase Young diagram of length r. In terminology of modular representations of the symmetric group, K_r are called 2-cores since they do not have 2-hooks [8]. We remark that the Schur functions $S_{K_r}(t)$ $(r = 0, 1, 2, \ldots)$ do not depend on t_{2j} $(j = 1, 2, \ldots)$, namely elements of V, because of the Murnaghan-Nakayama formula [2]. The maximal weight vectors are $S_{K_r}(t)$ for $r = 0, 1, 2, \ldots$ [1]. We denote by Λ_r the maximal weight whose weight vector is $S_{K_r}(t)$. According to the theory of hierarchies of nonlinear integrable systems, these maximal weight vectors exhaust the weighted homogeneous polynomial τ-functions of the KdV hierarchy [1].

The subspace of V consisting of weighted homogeneous polynomials of degree n has dimension $p^{odd}(n)$, the number of partitions of n into odd positive integers. If we put $\phi(q) = \prod_{j\geq 1}(1 - q^j)$, the generating function is

$$\sum_{n=0}^{\infty} p^{odd}(n) q^n = \frac{\phi(q^2)}{\phi(q)}.$$

Let $\mu(n)$ be the weight multiplicity of the weight $\Lambda - n\delta$ for a maximal weight Λ, which is independent of the choice of the maximal weight, since the Weyl group preserves the weight multiplicity. It is obvious that degree of a maximal weight vector is of the form $2m^2 + m$ ($m \in \mathbb{Z}$). Therefore, by using an identity of Gauss [3,p241], we see that

$$\frac{\phi(q^2)}{\phi(q)} = \sum_{m\in\mathbb{Z}} \left(\sum_{n=0}^{\infty} \mu(n) q^{2n} \right) q^{2m^2+m} = \frac{\phi(q^2)^2}{\phi(q)} \sum_{n=0}^{\infty} \mu(n) q^{2n},$$

and hence $\mu(n) = p(n)$, the number of partition of n into positive integers.

§2 BASES FOR WEIGHT SPACES

We define the 2-quotient for a given Young diagram [2,8]. Let $Y = (y_1, \ldots, y_n)$ $(y_1 \geq \cdots \geq y_n \geq 0)$ be a Young diagram. We always assume n to be even. Consider the " Maya diagram" or the "β-set" $X = (x_1, \ldots, x_n)$ where $x_j = y_j + (n-j)$ for $1 \leq j \leq n$. For $i = 0, 1$ let

$$X^{(i)} = \left\{ \xi^{(i)} \in \mathbb{N}; 2\xi^{(i)} + i = x_j \text{ for some } j \right\}.$$

If we have $X^{(i)} = \left\{ \xi_1^{(i)}, \ldots, \xi_{m^{(i)}}^{(i)} \right\}$ $(\xi_1^{(i)} > \cdots > \xi_{m^{(i)}}^{(i)} \geq 0)$, then we define the Young diagram $Y^{(i)}$ by

$$Y^{(i)} = \left(\xi_1^{(i)} - (m^{(i)} - 1), \xi_2^{(i)} - (m^{(i)} - 2), \ldots \xi_{m^{(i)}}^{(i)} \right).$$

The pair $(Y^{(0)}, Y^{(1)})$ of Young diagrams is called the 2-quotient of Y. The 2-core of Y is described as follows. If $|X^{(0)}| - |X^{(1)}| = r \geq 1$, then the 2-core of Y is K_{r-1} and if $|X^{(1)}| - |X^{(0)}| = r \geq 0$, then it is K_r. In this fashion we can attach a triplet $(K; Y^{(0)}, Y^{(1)})$ of Young diagrams for any Young diagram Y, where K is the 2-core of Y and $(Y^{(0)}, Y^{(1)})$ is the 2-quotient of Y. It is easily shown that this correspondence is one-to-one and $|Y| = 2(|Y^{(0)}| + |Y^{(1)}|) + |K|$. Denote by $\tau(Y)$ the triplet $(K; Y^{(0)}, Y^{(1)})$ corresponding to Y.

Example.

$$\begin{aligned}
&Y = (4,3,1^2), && X = (7,5,2,1),\\
&X^{(0)} = (1), && X^{(1)} = (3,2,0),\\
&Y^{(0)} = (1), && Y^{(1)} = (1^2,0),\\
&K = K_2 = (2,1).
\end{aligned}$$

We now describe the weight vectors of the basic $A_1^{(1)}$-module by means of the Schur functions. To this end we denote by $S_Y^{red}(t) \in V$ the 2-reduced Schur function indexed by Y, which is by definition,

$$S_Y^{red}(t) = S_Y(t)|_{t_2=t_4=\cdots=0}.$$

By using the Boson-Fermion correspondence established by Date et al.[1], we can see the following.

Proposition 1. *The 2-reduced Schur function $S_Y^{red}(t)$ is a weight vector of weight $\Lambda_r - n\delta$ if $\tau(Y) = (K_r; Y^{(0)}, Y^{(1)})$ with $|Y^{(0)}| + |Y^{(1)}| = n$.*

As we have seen in §1, the multiplicity of each weight is expressed by the number of partitions for any maximal weight Λ_r, *i.e.*, $mult(\Lambda_r - n\delta) = p(n)$. Hence the weight vectors found above satisfy linear relations in general. For example, the Young diagrams $(4), (3,1), (2^2), (2,1^2), (1^4)$ determine the 2-reduced Schur functions of the same weight $\Lambda_0 - 2\delta$ whose multiplicity equals $p(2) = 2$. It is easily checked that

$$\begin{cases}
S_{(4)}^{red}(t) = S_{(1^4)}^{red}(t) = \dfrac{1}{24}t_1^4 + t_1t_3,\\
S_{(3,1)}^{red}(t) = S_{(2,1^2)}^{red}(t) = \dfrac{1}{8}t_1^4,\\
S_{(2^2)}^{red}(t) = S_{(2,1^2)}^{red}(t) - S_{(4)}^{red}(t) = \dfrac{1}{12}t_1^4 - t_1t_3.
\end{cases}$$

Therefore the next problem is to find a suitable base for each weight space. The following theorem gives an answer.

Theorem 2. *The 2-reduced Schur functions*

$$\left\{S_Y^{red}(t); \tau(Y) = (K_r; \phi, Y^{(1)}) \text{ with } |Y^{(1)}| = n\right\}$$

are linearly independent and hence constitute a base for the weight space of weight $\Lambda_r - n\delta$.

Any weight vector $S_Y^{red}(t)$ can be expressed uniquely as a linear combination of the base vectors obtained above. We now focus on the coefficients of these expressions. Suppose that the Young diagram Y corresponds to the triplet $\tau(Y) = (K; Y^{(0)}, Y^{(1)})$. The 2-sign $\delta_2(K; Y^{(0)}, Y^{(1)})$ is defined as follows. If the 2-core K is obtained from Y by removing a sequence of 2-hooks, where q of them are column 2-hooks and the others are row 2-hooks, then

$$\delta_2(K; Y^{(0)}, Y^{(1)}) = (-1)^q.$$

It can be proved that $\delta_2(K; Y^{(0)}, Y^{(1)})$ does not depend on the choice of 2-hooks being removed. The following is our main result.

Theorem 3. *For such a Young diagram Y that $\tau(Y) = (K; Y^{(0)}, Y^{(1)})$ with $n = |Y^{(0)}| + |Y^{(1)}|$, we have*

$$S_Y^{red}(t) = (-1)^{|Y^{(0)}|}\delta_2(K; Y^{(0)}, Y^{(1)}) \sum_{Z^{(1)}} LR^{Z^{(1)}}_{Y^{(0)\prime}Y^{(1)}} \delta_2(K; \phi, Z^{(1)}) S_Z^{red}(t),$$

where the summation runs over all Young diagrams $Z^{(1)}$ of size n, the Young diagram Z corresponds to $(K; \phi, Z^{(1)})$ and $Y^{(0)\prime}$ denotes the transpose of $Y^{(0)}$. We also denote by LR the Littlewood-Richardson coefficient.

The proof of this theorem is performed by analysing the Schur functions, apart from the affine Lie algebra $A_1^{(1)}$. Details will be published elsewhere.

References

1. E. Date, M. Jimbo, M.Kashiwara and T. Miwa, Publ. RIMS. **18** (1982), 1077–1110.
2. G. James and A. Kerber, *The Representation Theory of the Symmetric Groups*, Addison-Wesley, 1981.
3. V. Kac, *Infinite Dimensional Lie Algebras, 3rd ed.*, Cambridge, 1990.
4. V. Kac, D. Kazhdan, J. Lepowsky and R. L. Wilson, Adv. Math. **42** (1981), 83–112.
5. J. Lepowsky and R. L. Wilson, Comm. Math. Phys. **62** (1978), 43–53.
6. I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford, 1979.
7. T. Nakajima and H. Yamada, J. Phys. **A27** (1994), L171–L176.
8. J. B. Olsson, *Combinatorics and Representations of Finite Groups*, Lecture Notes, University of Essen **20** (1993).

9. M. Wakimoto, RIMS Kokyuroku **503** (1983), 36–46.

10. M. Wakimoto, Jap. J. Math. **12** (1986), 379–400.

11. H. Yamada, Lett. Math. Phys. **9** (1985), 133–137.