THE NUMBER OF M-SEQUENCES AND f-VECTORS

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ABSTRACT. We give a recursive formula for the number of M-sequences (a.k.a. f-vectors for multicomplexes or O-sequences) given the number of variables and a maximum degree. In particular, it is shown that the number of M-sequences for at most 2 variables is a power of two and for at most 3 variables is equal to the Bell numbers. The recursive formula is generalized to the number of f-vectors for general Clements-Lindström complexes and then specialized to the number of f-vectors for simplicial complexes. Keeping the maximum degree fixed we get the number of M-sequences and the number of f-vectors for simplicial complexes as polynomials in the number of variables and it is shown that these numbers are asymptotically equal. A bijection is given from M-sequences to iterated partitions of sets.

1. INTRODUCTION

A *multicomplex* is a collection $\mathcal{M}$ of finite multisets satisfying $B \in \mathcal{M}, A \subseteq B \implies A \in \mathcal{M}$. It is often convenient to think of the underlying ground set as variables and of the sets in $\mathcal{M}$ as monomials. Then a multicomplex is a collection of monomials closed under division. Given a multicomplex $\mathcal{M}$, let $m_i := |\{A \in \mathcal{M} : \deg A = i\}|$. The sequence $m = (m_0, m_1, m_2, \dots)$ is called the M-sequence of $\mathcal{M}$. The purpose of this paper is to study the number of M-sequences given the number of variables and a maximum degree for the monomials and to give a bijection to iterated partitions. The counting technique is generalized to f-vectors of Clements-Lindström complexes in Section 4 and then specialized to f-vectors of simplicial complexes in Section 5. It is also shown in Theorem 5.5 that the number of f-vectors for all these classes have the same asymptotic growth for a fixed maximal degree.

M-sequences play an important role in the theories of polytopes, standard graded algebras and shellable simplicial complexes. The theorem below is a summary of how the cardinality of M-sequences can be interpreted in the different areas. Theorem 1.1 is a consequence of deep theorems (such as Macaulay's theorem, the g-theorem etc.) by Billera, Lee, Macaulay, McMullen and Stanley. We refer to Ziegler [Z, Chapter 8]

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and Stanley [S1] for an account of the underlying definitions and theorems, for the bijection between (v),(vi) and (i) see also Björner [B2]. In this paper we will use (ii) to do the counting.

Theorem 1.1. *Fix $n, p \geq 0$. Then the following are equal.*

(i) *The number of M-sequences with $m_1 \leq p$ and $m_j = 0$ for all $j > n$, not counting $(0, 0, 0, \dots)$.*
(ii) *The number of non-empty compressed multicomplexes on at most p variables and with no monomial of degree higher than n.*
(iii) *The number of f-vectors of $n-1$ dimensional shellable simplicial complexes on at most $n+p$ vertices.*
(iv) *The number of f-vectors of $n-1$ dimensional Cohen-Macaulay simplicial complexes on at most $n+p$ vertices.*
(v) *The number of f-vectors of simplicial $2n$-polytopes with at most $p+2n+1$ vertices.*
(vi) *The number of f-vectors of simplicial $2n+1$-polytopes with at most $p+2n+2$ vertices.*
(vii) *The number of Hilbert functions for standard graded $\boldsymbol{k}$-algebras $R = R_0 + R_1 + \cdots + R_d$, with $d \leq n$ and $\dim R_1 \leq p$.* □

Let $M^p(n) - 1$ denote the common number in Theorem 1.1.

Our basic result on the number of M-sequences, from which the other results will follow, is the recursion in Theorem 2.1. Corollary 2.2 shows that when fixing p and expressing $M^p(n)$ in terms of n, we get the sequence of functions:

constant, linear, powers of 2, Bell numbers, . . . ,

for $p = 0, 1, 2$ and 3 respectively. This sequence is very suggestive, and in Section 3 we give a bijection from M-sequences to what we call iterated partitions of sets.

From Theorem 1.1 we see that every enumerative result about $M^p(n)$ can be interpreted in many ways. Corollary 2.2 implies for example that the number of f-vectors for a simplicial d-polytope with at most $d+3$ vertices is $2^{\lfloor d/2 \rfloor + 1} - 1$ and with at most $d+4$ vertices is $B(\lfloor d/2 \rfloor + 2) - 1$, where $B(n)$ is the Bell-number. It would be interesting if someone could give a direct proof of this, avoiding the g-theorem.

Looking at this sequence of functions, one might suspect combinatorial explosion. However, at the end of Section 2 we give the general upper bound $M^p(n) \leq (n+p-1)^{n(p-1)}$, for $p \geq 3$. It is interesting to compare with known upper and lower bounds for the number of polytopes.

We also prove that for each fixed n we get $M^p(n)$ as a polynomial in p of degree $\binom{n+1}{2}$, Theorem 2.4. This is also true for the number of f-vectors of simplicial complexes, see Theorem 5.3. In Theorem 5.5 we prove the perhaps somewhat surprising result that the number of f-vectors for simplicial complexes and the number of M-sequences for multicomplexes have asymptotically equal growth for each fixed n. From this we can deduce, Theorem 5.6, that for a fixed dimension $n-1$ and a large number of vertices p, almost every f-vector of simplicial complexes is also an f-vector for a shellable simplicial complex.

2. The Number of M-sequences

2.1. Basic recursion. After Theorem 1.1 we defined $M^p(n)$ to be one more than the number of M-sequences for non-empty multicomplexes. We think of this extra one as coming from the sequence $(0,0,0,0,\dots)$ for the empty multi complex. This sequence does not have a proper non-empty counterpart when counting f-vectors of simplicial polytopes, shellable simplicial complexes etc. in Theorem 1.1. We include this extra sequence $(0,0,0,\dots)$ in our count to obtain the nicest looking recursions. Hence, we will have $M^p(0)=2$ for all $p\geq 0$. We also define $M^p(-1):=1$ for all $p\geq 0$. On the other boundary we have $M^0(n)=2$ for all $n\geq 0$.

Given two monomials $x_1^{a_1}\dots x_p^{a_p}$ and $x_1^{b_1}\dots x_p^{b_p}$ we say that $x_1^{a_1}\dots x_p^{a_p}$ comes before $x_1^{b_1}\dots x_p^{b_p}$ in *reverse lexicographic order* if either $a_p<b_p$ or $a_p=b_p, a_{p-1}=b_{p-1},\dots,a_{i+1}=b_{i+1}$, but $a_i<b_i$.

A multicomplex $\mathcal{M}$ is said to be *compressed* if $B\in\mathcal{M}, deg(A)=deg(B)$ and A comes before B in reverse lexicographic order implies that $A\in\mathcal{M}$.

We also need to have a notation for the number of M-sequences corresponding to multicomplexes that for a fixed number of variables have all the monomials up to a fixed degree k. For $n\geq k\geq -1$, $p\geq 1$ define

$L^p(n,k)$:=the number of M-sequences with at most p variables and degree at most n that has maximal value for m_i when $i\leq k$ but not for m_{k+1}, i.e., $m_i=\binom{p+i-1}{i}$ for $i\leq k$ and $m_{k+1}<\binom{k+p}{k+1}$.

The boundary conditions are $L^p(n,n)=L^p(n,-1)=1$ for all $p\geq 1, n\geq -1$. For consistency we define $L^0(n,n):=L^0(n,-1):=1$ and $L^0(n,k):=0$ for $k\neq -1,n$. It follows from these definitions that

$$M^p(n)=\sum_{k=-1}^{n} L^p(n,k), \tag{1}$$

for all $n,p\geq 0$.

The numbers $L^p(n,k)$ also have interesting interpretations along the lines of Theorem 1.1. In polytope theory for example we get from the bijection between (i) and (v)-(vi) of Theorem 1.1 that $L^p(n,k)$ is the number of f-vectors for simplicial $2n$-(or $2n+1$)-polytopes with $p+2n+1$ $(p+2n+2)$ vertices that are k-neighborly, i.e., they have all possible r-sets as faces for $r\leq k$, but not $k+1$-neighborly.

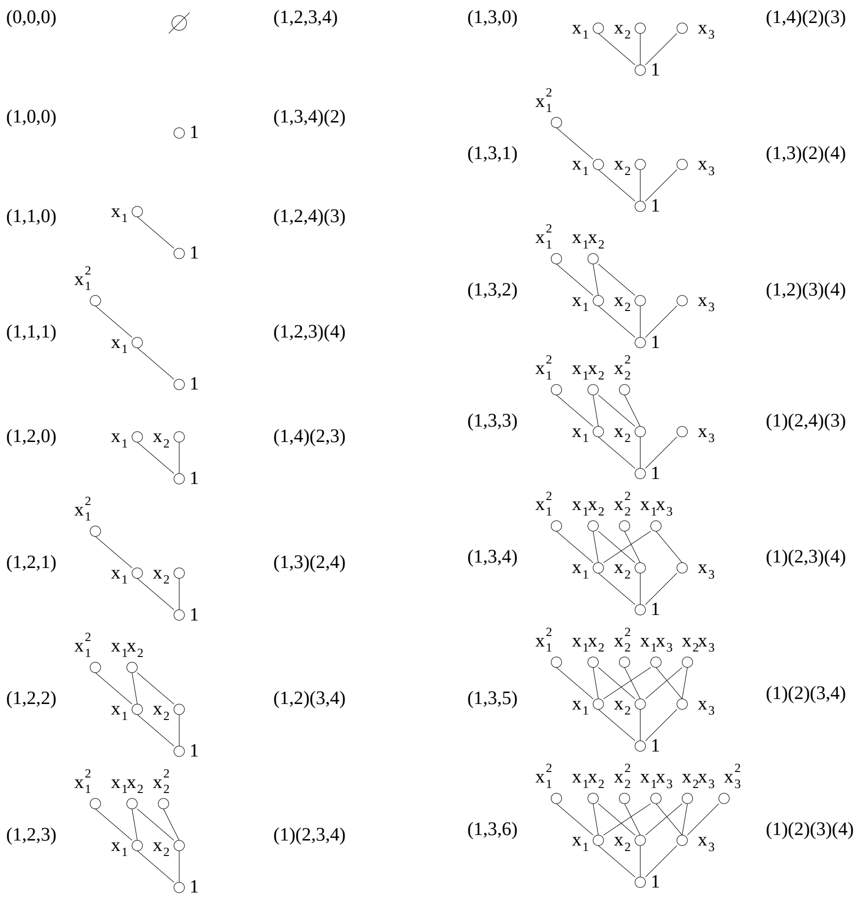

FIGURE 1. All the M-sequences and the corresponding compressed complexes when $p = 3$ and n=2. The third column shows the partition of $\{1,2,3,4\}$ according to the bijection described in Section 3.

The basic theorem from which the other results will follow is the following.

Theorem 2.1. *The number of M-sequences satisfies the following recursions for all $p, n \geq 1, k \geq 0$:*

$$M^p(n) = 1 + \sum_{i=0}^{n} L^{p-1}(n,i) M^p(i-1) \tag{2}$$

and

$$L^p(n,k) = \sum_{i=k}^{n} L^{p-1}(n,i) L^p(i-1,k-1), \tag{3}$$

where the first recursion is implied by the second.

PROOF From Theorem 1.1 we have that when counting M-sequences we can count compressed multicomplexes instead. See Figure 1.

Let $\mathcal{M}$ be a compressed multicomplex on the p variables $x_1, \ldots, x_p$ of degree at most n that is totally filled exactly to level k. Partition the multisets in $\mathcal{M}$ into two disjoint parts depending on whether the multiset contains x_p or not, i.e., $\mathcal{M}_1 := \{A \in \mathcal{M} : x_p \notin A\}$ and $\mathcal{M}_2 := \{A \in \mathcal{M} : x_p \in A\}$. See Figure 2.

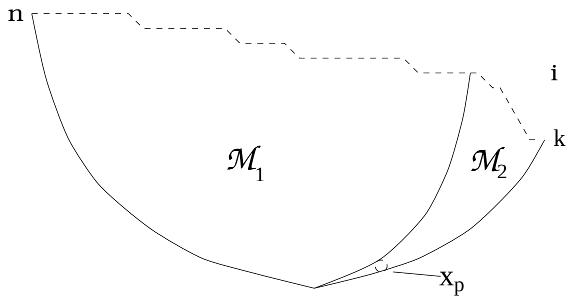

FIGURE 2. The partition of a compressed multicomplex $\mathcal{M}$ as in the proof of Theorem 2.1.

Note that $\mathcal{M}_1$ is a compressed multicomplex on at most $p-1$ variables and that dividing every monomial in $\mathcal{M}_2$ by x_p we get a compressed multicomplex on at most p variables. Let i, $i \geq k$, be the largest level in $\mathcal{M}_1$ that is totally filled. Then there are $L^{p-1}(n,i)$ possibilities for $\mathcal{M}_1$ and $L^p(i-1,k-1)$ possibilities for $\mathcal{M}_2$ and all these possibilities occur for some $\mathcal{M}$. Summing over i we get recursion (3). Recursion (2) follows from (1) and (3). □

Tables of $M^p(n)$ and $L^p(n,k)$ calculated with the recursions in Theorem 2.1 can be found in the Appendix.

2.2. Keeping p fixed. Recall that the Stirling number of the second kind $S(n,k)$ is the number of ways to partition $\{1, 2, \ldots, n\}$ into k blocks, and that the Bell number $B(n) = \sum_{k=1}^{n} S(n,k)$ is the number of all possible partitions. For further details see e.g. [S2].

We get the following result when $p = 1, 2$ and 3.

Corollary 2.2. *For $n \geq k \geq -1$, the number of M-sequences with at most 1,2 and 3 variables are*

$$M^1(n) = n+2, \qquad L^1(n,k) = 1$$

$$M^2(n) = 2^{n+1}, \qquad L^2(n,k) = \binom{n+1}{k+1}$$

$$M^3(n) = B(n+2), \qquad L^3(n,k) = S(n+2,k+2),$$

where $B(n)$ are the Bell numbers and $S(n,k)$ are the Stirling numbers of the second kind.

Proof The values for $p=1$ follow directly from Theorem 2.1 and they are easily seen to be correct.

When $p=2$, the recursion (3) becomes

$$L^2(n,k)=\sum_{i=k}^{n} L^2(i-1,k-1).$$

Since $\sum_{i=k}^{n}\binom{i}{k}=\binom{n+1}{k+1}$, the formulas follow by induction and from formula (1).

When $p=3$, the recursion (3) is equivalent to the known formula

$$S(n+2,k+2)=\sum_{i=k}^{n}\binom{n+1}{i+1}S(i+1,k+1).$$

The result follows again by induction and from formula (1). □

Remark For $p\le 2$ the results in Corollary 2.2 have been previously calculated by Björner [B1]. □

Corollary 2.3. *As special cases we get,*

$$L^p(n,0)=M^{p-1}(n)-1$$

and

$$L^p(n,n-1)=\binom{p+n-1}{n}.$$

Proof Follows directly from Theorem 2.1. Both formulas are also easily understandable directly from the definition of $L^p(n,k)$. □

2.3. Keeping n fixed. Next we calculate formulas for $M^p(n)$ and $L^p(n,k)$ in terms of p while keeping n and k fixed. Surprisingly enough they turn out to be polynomials.

Theorem 2.4. *$L^p(n,k)$ is a polynomial in p of degree $\binom{n+1}{2}-\binom{k+1}{2}$ and $M^p(n)$ is a polynomial in p of degree $\binom{n+1}{2}$, for each pair $n,k\ge 0$.*

Proof We will prove the theorem by double induction using recursion (3). The statement is trivially true for $n=0$ and for $n=k$ since $L^p(n,n)=1$. Given $n>k\ge 0$, assume that $L^p(s,i)$ is a polynomial of degree $\binom{s+1}{2}-\binom{i+1}{2}$ for all $s<n, 0\le i\le s$. As the second induction hypothesis we assume that $L^p(n,i)$ is a polynomial of degree $\binom{n+1}{2}-\binom{i+1}{2}$ for all $k<i\le n$. Now, write (3) as

$$L^p(n,k)-L^{p-1}(n,k)=\sum_{i=k+1}^{n} L^{p-1}(n,i)L^p(i-1,k-1).$$

By induction we see that if $k > 0$, then $L^p(n,k) - L^{p-1}(n,k)$ is a polynomial of degree

$$\begin{aligned}
\max\left\{\left[\binom{n+1}{2} - \binom{i+1}{2}\right] + \left[\binom{i}{2} - \binom{k}{2}\right]\right\}_{i=k+1}^{n} &= \\
= \max\left\{\binom{n+1}{2} - i - \binom{k}{2}\right\}_{i=k+1}^{n} &= \\
= \binom{n+1}{2} - \binom{k+1}{2} - 1.
\end{aligned}$$

Hence we get that $L^p(n,k)$ is a polynomial of degree $\binom{n+1}{2} - \binom{k+1}{2}$. For $k = 0$, we use that $L^p(i-1,-1) = 1$ for all $p \geq 0$ and similar calculations, to get that $L^p(n,0)$ is a polynomial of degree $\binom{n+1}{2}$. From Corollary 2.3 it follows that $M^p(n)$ is a polynomial of degree $\binom{n+1}{2}$. The explicit polynomials in the theorem are easily calculated using recursion (3) as above. □

Remark A weaker formulation of the polynomial growth of $M^p(n)$ appears without proof in [B3]. □

For small values of n we have the following polynomials for $M^p(n)$.

$$\begin{aligned}
(4)\qquad M^p(0) &= 2, \\
M^p(1) &= p+2, \\
M^p(2) &= \binom{p+2}{3} + \binom{p+2}{1}, \\
M^p(3) &= 4\binom{p+2}{6} + 7\binom{p+2}{5} + 4\binom{p+2}{4} + 2\binom{p+2}{3} + \binom{p+2}{1}, \\
M^p(4) &= 120\binom{p+2}{10} + 483\binom{p+2}{9} + 782\binom{p+2}{8} + 655\binom{p+2}{7} + \\
&\quad + 310\binom{p+2}{6} + 88\binom{p+2}{5} + 16\binom{p+2}{4} + 3\binom{p+2}{3} + \binom{p+2}{1}.
\end{aligned}$$

I have chosen to express the polynomials in a base of binomial coefficients since this makes them easy to evaluate numerically for small values of p.

Proposition 2.5. *For $1 \leq n \geq k \geq 0$, the coefficient for the term of highest degree in $L^p(n,k)$ is*

$$\frac{\prod_{i=k}^{n-2}\binom{\binom{n+1}{2}-\binom{i+1}{2}-1}{i}}{\left(\binom{n+1}{2}-\binom{k+1}{2}\right)!} \tag{5}$$

The coefficient for the term of highest degree in $M^p(n)$ is the same as in $L^p(n,0)$, i.e. take $k=0$ in (5).

Proof Let $c(n,k)$ be the leading coefficient of $L^p(n,k)$. We will again use double induction over n and k. Since $L^p(1,1)=1$, $L^p(1,0)=p$ and $L^p(n,n)=1$, the statement is trivially true for $n=1$ and $n=k$. Assume that the proposition is true for all $c(r,s)$ when $r<n$ or $r=n, s>k$. From the proof of Theorem 2.4 we see that $(\binom{n+1}{2}-\binom{k+1}{2})c(n,k)=c(n,k+1)c(k,k-1)$. From the induction assumptions we get that

$$c(n,k+1)c(k,k-1)=\frac{\prod_{i=k+1}^{n-2}\binom{\binom{n+1}{2}-\binom{i+1}{2}-1}{i}}{\left(\binom{n+1}{2}-\binom{k+2}{2}\right)!}\frac{1}{k!}=\frac{\prod_{i=k}^{n-2}\binom{\binom{n+1}{2}-\binom{i+1}{2}-1}{i}}{\left(\binom{n+1}{2}-\binom{k+1}{2}-1\right)!}.$$

The formula for $c(n,k)$ follows. The result for $M^p(n)$ is easily extracted from (1), (5) and Theorem 2.4. □

2.4. Keeping p and $n-k$ fixed. Now we calculate formulas for $L^p(n,k)$ in terms of n while keeping p and $n-k$ fixed. Once again we get polynomials.

Theorem 2.6. *For fixed $p\geq 2, r\geq 0$, $L^p(n,n-r)$ is a polynomial in n of degree $r(p-1)$, with leading coefficient $1/(p-1)!^r r!$.*

Proof This time we write (3) as

$$L^p(n,n-r)-L^p(n-1,n-r-1)=\sum_{i=n-r}^{n-1}L^{p-1}(n,i)L^p(i-1,n-r-1).$$

Imitating the proof of Theorem 2.4 we get by double induction that $L^p(n,n-r)$ is a polynomial in n of degree $r(p-1)$. The leading coefficient is extracted as in Proposition 2.5. □

2.5. An upper bound. A general upper bound for $M^p(n)$ can be given as follows. For a given M-sequence with $m_1 \le p$, we have that $0 \le m_i \le \binom{i+p-1}{p-1}$. Hence,

$$M^p(n) \le 1 + \prod_{i=1}^{n} \left(\binom{i+p-1}{p-1} + 1 \right) \le (n+p-1)^{n(p-1)}, \tag{6}$$

for $p \ge 3$. For fixed n and large p this bound is larger than the polynomials of degree $\binom{n+1}{2}$ obtained in Theorem 2.4. For fixed p and large n however, the bound is helpful. From the sequence of functions obtained in Corollary 2.2 one might be tempted to suspect combinatorial explosion, but this is not the case. It still remains an interesting problem to give a good lower bound. We have a feeling that (6) is reasonably close to the truth but have not been able to prove so.

To end this section we compare our results with known upper bounds for the number of polytopes. Let $c_s(m,d)$ be the number of different combinatorial types of simplicial d-polytopes on m labelled vertices. Over the years, a lot of attention has been given to the problem of estimating $c_s(m,d)$, see [A][G, pp. 288–290]. Even the asymptotic behavior was a big open question, until Goodman and Pollack [GP] obtained the upper bound $c_s(m,d) \le m^{d(d+1)m}$. The lower bound $\left(\frac{m-d}{d}\right)^{dm/4} \le c_s(m,d)$ is due to Alon [A], who also improved the upper bound and generalized to arbitrary polytopes.

Comparing these bounds via Theorem 1.1, we see that for a fixed dimension d, the number of f-vectors for simplicial d-polytopes on m vertices is much smaller than the number of combinatorial types for large m. For fixed $m-d$ and large d however, we have not been able to find any good lower bound for polytopes in the literature to compare with.

3. Bijection to iterated partitions of sets

Inspired by the sequence of functions for $M^p(n)$: constant, linear, powers of 2, Bell numbers,..., for $p = 0, 1, 2$ and 3 respectively, we define in this section a set of objects called *iterated partitions of sets* that will have cardinality $M^p(n)$ for certain parameters n and p. The definition of iterated partitions will be an imitation of a recursive way to get from subsets of a set to partitions of the set as is explained at the end of Section 3.1. We will also for every n and p give a bijection between M-sequences and the iterated partitions of sets in Section 3.2.

3.1. Iterated partitions. We will use the term *first order partitions* for ordinary set partitions of $\{1, \dots, n\}$. Call the blocks of a first order partition *first order blocks.* Order the blocks increasingly by the smallest element in the block.

We will now define the second order partitions of $\{1, \dots, n\}$ by forming first order partitions iteratively, see below for an example.

1. Take a first order partition of $\{1,\ldots,n\}$.
2. Let the first order block containing the smallest element form a second order block.
3. Let k denote the number of first order blocks created in step 1. If $k=1$, then stop. If $k>1$ repeat from step 1 with the set of the $k-1$ first order blocks that do not contain the smallest element as input.

Every possible outcome of steps 1, 2 and 3 above is a second order partition.

Example 1. An example of how a second order partition of $\{1,2,3,4,5,6,7,8,9\}$ is constructed where ( ) denotes a first order block and [ ] denotes a second order block.

$$\begin{array}{ccc}
\{1,2,3,4,5,6,7,8,9\} & & \\
\Big\downarrow{\scriptstyle \text{step 1}} & & \\
(157)(238)(4)(69) & \xrightarrow{\text{step 2}} & [(157)] \\
\Big\downarrow{\scriptstyle \text{step 3 and 1}} & & \\
((238)(69))((4)) & \xrightarrow{\text{step 2}} & [((238)(69))] \\
\Big\downarrow{\scriptstyle \text{step 3 and 1}} & & \\
(((4))) & \xrightarrow{\text{step 2}} & [(((4)))] \\
\Big\downarrow{\scriptstyle \text{step 3}} & & \\
\text{STOP} & &
\end{array}$$

The second order partition formed in this example is $[(157)][((238)(69))][(((4)))]$. □

Properties of second order partitions:

(a) The number of second order blocks is equal to the number of times the steps are iterated.
(b) The order in which the second order blocks are constructed implies a natural ordering of them.
(c) The second order block number i will have i levels of first order partitions inside, where the outmost partition is trivial, i.e. consists of one block.
(d) The number of second order partitions of $\{1,\ldots,n\}$ with k blocks is $L^4(n-2,k-2)$.
(e) The number of second order partitions of $\{1,\ldots,n\}$ is $M^4(n-2)$.

The first three properties are easy consequences of the definition. The last two properties will follow from the bijection established below.

Next we give a tableau for all the six second order partitions of $\{1,2,3\}$. Here we use a shorter notation than in Example 1 above to describe how the second order partitions in the last row are constructed.

Ground set	1 2 3	1 2 3	1 2 3	1 2 3
First Iteration	$[(1,2,3)]$	$[(1,2)](3)$	$[(1,3)](2)$	$[(1)](2,3)$
Second Iteration		$[((3))]$	$[((2))]$	$[((2,3))]$
Third Iteration				
Partition	$[(1,2,3)]$	$[(1,2)][((3))]$	$[(1,3)][((2))]$	$[(1)][((2,3))]$

	1 2 3	1 2 3
	$[(1)](2)(3)$	$[(1)](2)(3)$
	$[((2)(3))]$	$[((2))]((3))$
		$[(((3)))]$
	$[(1)][((2)(3))]$	$[(1)][((2))][(((3)))]$

From properties (a)-(c) stated above we see that all the first order brackets around the numbers 2 and 3 in $[(1)][((2))][(((3)))]$ are not necessary to distinguish between different second order partitions, see Convention below.

There are other ways to define second order partitions, see Remark 2 in Section 6.

Now we extend the definition of second order partitions and second order blocks to general *p:th order partitions* of $\{1,\ldots,n\}$ and *p:th order blocks* with exactly the same recursive steps.

1. Take a $p-1$:th order partition of $\{1,\ldots,n\}$.
2. Let the $p-1$:th order block containing the smallest element form a p:th order block.
3. Let k denote the number of $p-1$:th order blocks created in step 1. If $k=1$, then stop. If $k>1$ repeat from step 1 with the $k-1$ $p-1$:th order blocks that do not contain the smallest element as input.

Every possible outcome of steps $1,2$ and 3 above is a p:th order partition. The five properties (a)-(e) stated above for second order partitions are true also for general p:th order partitions for every $p\geq 2$ (with obvious changes).

CONVENTION Looking at the second order partitions above with all the brackets there, one quickly realizes that higher order partitions will be completely unreadable if we do not do something about our notation. We can avoid this mess of brackets by eliminating *matching* brackets that are directly inside a pair of matching brackets of higher order, for example $[(\ldots)]$ is replaced by $[\ldots]$, if $()$ are matching. By property (c) stated above we know that there are always i levels of $p-1$:th order partitions inside the i:th p:th order block, and which block is the i:th order block can be seen from the integers in the block according to (b). Hence we can reconstruct the original bracketing. $\square$

With this convention we can write the second order partition in Example 1 as $[157][(238)(69)][4]$ and the six second order partitions of $\{1,2,3\}$ in the simpler forms:

$$[1,2,3];\ [1,2][3];\ [1,3][2];\ [1][2,3];\ [1][(2)(3)] \text{ and } [1][2][3]$$

EXAMPLE 2. To give the reader some feeling for the higher order partitions we include also an example of a third order partition with $n = 7$. The third order blocks are embraced by $< >$.

Ground set	1 2 3 4 5 6 7
First iteration	$<[1]>$ $[(2)(6)]$ $[(3,7)(4)]$ $[5]$
Second iteration	$<[(2)(6)]>$ $[([(3,7)(4)])([5])]$
Third iteration	$<[([(3,7)(4)])([5])]>$
Partition	$<1><(2)(6)><([(3,7)(4)])([5])>$

□

In order to describe the bijection below we also need to extend the definition of p:th order partitions to $p = 0$ as follows.

We let the 0:th order partitions of $\{1,\dots,n\}$ be the subsets containing 1. The 0:th order blocks are the integers in the subset. From 0:th order partitions to first order partitions we get by using almost the same recursive steps $1, 2$ and 3, but this time we modify step 2 to

2'. Let the smallest element (a 0:th order block) form a first order block together with the elements not in the 0:th order partition (which is a subset) chosen in step 1.

Note that the change of step 2 does not influence the number of different iterations possible, but it helps to identify the first order partitions with ordinary set partitions which we will use in the definition of the bijection below.

EXAMPLE 3. Using the same notation as above, the recursive steps to create all the five first order partitions from 0:th order partitions are for $n = 3$:

Iteration	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
First	(1)			(1)	2		(1)		3	(1)	2	3	(1)	2	3
Second					(2)				(3)		(2)			(2)	3
Third															(3)
Partition	$(1,2,3)$			$(1,3)(2)$			$(1,2)(3)$			$(1)(2,3)$			$(1)(2)(3)$		

□

3.2. The bijection. We will now recursively define for a given pair $n \geq -1, p \geq 2$ a function $\phi_{n,p}$ from M-sequences with $m_1 \leq p$ and $m_i = 0$ for all $i > n$ (including $(0,0,0,\dots)$), to $p-2$:th order partitions of $\{1,2,\dots,n+2\}$.

We will use the obvious bijection from M-sequences to compressed multicomplexes and thus define $\phi_{n,p}$ on compressed multicomplexes.

DEFINITION For $p = 2$ we define the bijection by drawing a path of length $n+1$ along the contour of the compressed complex $\mathcal{M}$ and marking the segments with $2, 3, \ldots, n+2$ starting from the lower right corner, see Figure 3. The bijection $\phi_{n,2}$ is then defined to take the compressed complex to the subset of $\{1, \ldots, n+2\}$ consisting of 1 plus the numbers on horizontal segments. In particular, let $\phi_{n,2}(\emptyset) := \{1\}$, for all $n \geq 2$.

It is clear from this definition that the number of horizontal segments will be $k+1$ where k is the largest integer such that $x_2^k \in \mathcal{M}$. Hence $\phi_{n,2}(\mathcal{M})$ will consist of $k+2$ integers (0:th order blocks).

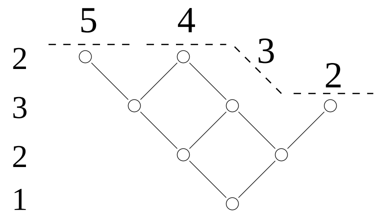

FIGURE 3. $\phi_{3,2}((1, 2, 3, 2)) = \{1, 2, 4, 5\}$.

For every $p \geq 2$, define $\phi_{-1,p}(\emptyset)$ to be the only p:th order partition of $\{1\}$.

Now fix n and p, with $p > 2, n \geq 0$. Assume that $\phi_{r,q}$ has been defined for $q < p, r \leq n$ and for $q = p, n > r$. Assume also that for these cases the number of blocks in the partition $\phi_{r,q}(\mathcal{M})$ is $k+2$, where k is the largest integer such that $x_q^k \in \mathcal{M}$.

Let $\mathcal{M}$ be a compressed multicomplex on at most the p variables $x_1, \ldots, x_p$ of degree at most n. As in the proof of Theorem 2.1, we partition the monomials in $\mathcal{M}$ into two disjoint parts depending on whether the monomial is divisible by x_p or not, i.e., let $\mathcal{M}_1 = \{A \in \mathcal{M} : x_p \notin A\}$ and $\mathcal{M}_2 = \{A \in \mathcal{M} : x_p \in A\}$, see Figure 2 and Figure 4.

We apply $\phi_{n,p-1}$ to $\mathcal{M}_1$, which is a compressed multicomplex on at most $p-1$ variables. This corresponds to step 1 in the definition of iterated partitions and gives a $p-3$:th order partition with $k+2$ blocks, where k is the largest integer such that $x_{p-1}^k \in \mathcal{M}_1$. Order the blocks by the canonical order obtained in the definition, i.e., by the smallest integer in the block.

Dividing every monomial in $\mathcal{M}_2$ with x_p gives a compressed multicomplexes $\mathcal{M}_2'$ on at most p variables of degree at most $k-1$. As in step 2 we now turn the first (first+elements not in the block if $p = 3$) $p-3$:th order block into a $p-2$:th order block. Replace the integers $1, \ldots, k+1$ in the $p-2$:th order partition $\phi_{k-1,p}(\mathcal{M}_2')$

with the last $k+1$ blocks obtained in $\phi_{n,p-1}(\mathcal{M}_1)$. Define $\phi_{n,p}(\mathcal{M})$ to be the $p-2$:th order partition obtained in this way. □

See Figure 1 for an explicit description of $\phi_{2,3}$. To illustrate how $\phi_{n,p}$ is defined we take as an example $n = p = 4$ and the M-sequence $(1,4,9,11,7)$. To construct $\phi_{4,4}((1,4,9,11,7))$, a 2:nd order partition of $\{1,2,3,4,5,6\}$, we first decompose the corresponding compressed complex as shown in Figure 4. We decompose until we reach complexes for which we know the iterated partition. For this example we have that (see Figure 1 and Figure 3)

$$\left.\begin{aligned} \phi_{1,3}((1,3)) =&(1)(2)(3) \\ \phi_{3,2}((1,2,3,2)) =&\{1,2,4,5\} \end{aligned}\right\} \implies \phi_{3,3}((1,3,6,2)) = (13)(2)(4)(5).$$

It is easy to verify that $\phi_{4,2}((1,2,3,4,5)) = \{1,2,3,4,5,6\}$. We can conclude

$$\left.\begin{aligned} \phi_{3,3}((1,3,6,2)) =&(13)(2)(4)(5) \\ \phi_{4,2}((1,2,3,4,5)) =&\{1,2,3,4,5,6\}) \end{aligned}\right\} \implies \phi_{4,3}((1,3,6,10,7)) = (1)(24)(3)(5)(6).$$

Finally we use that (see Figure 1)

$$\phi_{2,3}((1,3,1)) = (13)(2)(4) \implies \phi_{2,4}((1,3,1)) = [13][(2)(4)]$$

to get

$$\phi_{4,4}((1,4,9,11,7)) = [1][(24)(5)][(3)(6)].$$

Theorem 3.1. *For every pair $n \geq -1, p \geq 2$, we have that $\phi_{n,p}$ is a bijection. Hence*

$$M^p(n) = \textit{number of } p-2\textit{:th order partitions of } \{1,2,\dots,n+2\}$$

and

$$\begin{aligned} L^p(n,k) =&\textit{number of } p-2\textit{:th order partitions of } \{1,2,\dots,n+2\} \\ &\textit{with } k+2 \textit{ blocks.} \end{aligned}$$

Proof We easily see that $\phi_{n,2}$ is a bijection. It is also easy to see that each pair of compressed complexes $\mathcal{M}_1, \mathcal{M}'_2$ as defined above corresponds to a unique compressed complex $\mathcal{M}$. Hence $\phi_{n,p}$ is an injection for every pair $n \geq -1, p \geq 2$ by induction. From the definition of iterated partitions, it is clear that step 1 corresponds to choosing $\mathcal{M}_1$. Removing one block in step 2 corresponds to the necessary decrease in degree by one resulting from the division with x_p to create $\mathcal{M}'_2$. An inductive assumption that $\phi_{n,p-1}$ and $\phi_{s,p}$ are surjective for every $s < n$ implies that $\phi_{n,p}$ is also surjective.

The enumerative results follow from Theorem 2.1. □

The test for the reader is now to find the M-sequence that is mapped by $\phi_{5,5}$ to the third order partition in Example 2 above. (Answer=(1,5,13,14,15,11))

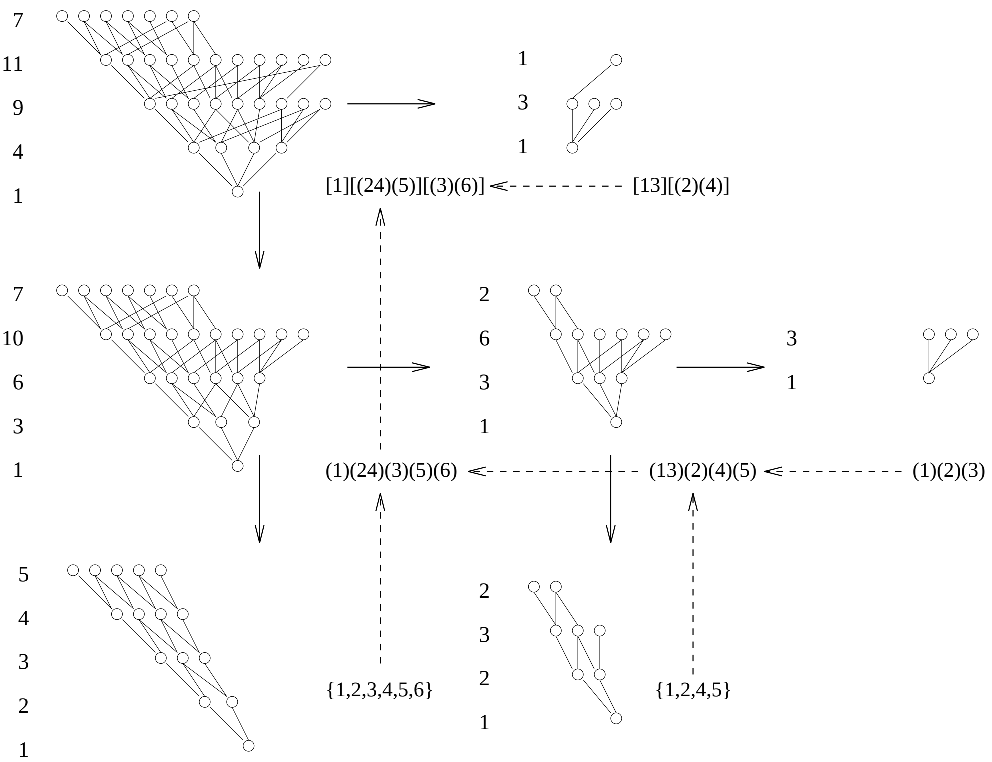

FIGURE 4. Example of a decomposition of a compressed complex as in the definition of the bijection.

4. A generalization of Clements-Lindström type

Assume we are given an integer $p \geq 0$ and $n_1, n_2, \dots, n_p \in \{0, 1, 2, \dots\} \cup \{\infty\}$. Denote the set of all monomials $x_1^{a_1} \dots x_p^{a_p}$ such that $0 \leq a_i \leq n_i$ for all $i = 1, \dots, p$ with $G(n_1, \dots, n_p)$. A set of monomials $\mathcal{CL} \subseteq G(n_1, \dots, n_p)$ closed under division will be called a *Clements-Lindström complex (CL-complex for short) of type* $n_1, \dots, n_p$.

When $n_i = \infty$ for $i = 1, \dots, p$ we have multicomplexes on p variables and when $n_i = 1$ for $i = 1, \dots, p$ we have simplicial complexes on p vertices. Simplicial complexes will be treated in Section 5.

Given a complex $\mathcal{CL}$, let $f_i = |\{A \in \mathcal{CL} : \deg A = i\}|$. The sequence $f = (f_0, f_1, f_2, \dots)$ is called the f-vector of $\mathcal{CL}$. We will count the number of possible f-vectors given $n_1 \geq \dots \geq n_p \geq 0$ and a maximal degree $n \geq -1$. Let

$F(n_1 \geq \dots \geq n_p; n)$:=the number of f-vectors for Clements-Lindström complexes of type $n_1 \geq \dots \geq n_p$ with monomials of degree at most n.

We will include both the f-vector $(0, 0, 0, 0, \dots)$ for the empty complex and the f-vector $(1, 0, 0, 0, \dots)$ for the complex consisting of only a constant. Hence, we will have $F(n_1 \geq \cdots \geq n_p; -1) = 1$ and $F(n_1 \geq \cdots \geq n_p; 0) = 2$ for all $n_1 \geq \cdots \geq n_p$, $p \geq 0$. On the other boundary we have $F(\emptyset; n) = 2$ for all $n \geq 0$.

Remark Note that the number of f-vectors for a CL-complex is the same for any permutation of $n_1, \ldots, n_p$. The condition $n_1 \geq \cdots \geq n_p$ is needed to use the compressing technique (Theorem 4.1). □

The following theorem of Clements and Lindström [CL] is essential to our presentation.

Theorem 4.1 (Clements-Lindström). *Fix p, n and $n_1 \geq \cdots \geq n_p \geq 0$. For every Clements-Lindström complex of type $n_1 \geq \cdots \geq n_p$ there is a compressed complex of the same type having the same f-vector.*

Since the f-vector is different for different compressed complexes, Theorem 4.1 allows us to count the number of compressed complexes instead of the number of f-vectors.

A *facet* in a CL-complex $\mathcal{CL}$ is a maximal monomial in the complex, i.e., it does not divide any other monomial in the complex. Let $v_i := |\{A \in \mathcal{CL} : A \text{ a facet}, \deg A = i\}|$. We call the sequence $v = (v_0, v_1, v_2, \ldots)$ the *facet-vector* of $\mathcal{CL}$. It follows from a theorem by Clements [C] that the number of facet-vectors will be the same as the number of f-vectors. Hence, the number of facet-vectors is also enumerated by $F(n_1 \geq \cdots \geq n_p; n)$. We state a reformulated version of Clements Theorem.

Theorem 4.2 (Clements). *Fix p, n and $n_1 \geq \cdots \geq n_p$. For every Clements-Lindström complex of type $n_1 \geq \cdots \geq n_p$ there is a compressed complex of the same type having the same facet-vector.*

We need the following refinement of $F(n_1 \geq \cdots \geq n_p; n)$. Given $n_1 \geq \cdots \geq n_p \geq 0$ and $n, k \geq -1$, let

$E(n_1 \geq \cdots \geq n_p; n, k)$:=the number of f-vectors for Clements-Lindström complexes of type $n_1 \geq \cdots \geq n_p$ with monomials of degree at most n that contains all the monomials in $G(n_1, \ldots, n_p)$ of degree less than or equal to k, but not all of degree $k+1$.

From the definition we have for every $p \geq 0, n \geq -1$, $n_1 \geq \cdots \geq n_p \geq 0$, that $E(n_1 \geq \cdots \geq n_p; n, -1) = 1$, $E(n_1 \geq \cdots \geq n_p; n, \min\{n, \sum_{i=0}^{p} n_i\}) = 1$ and $E(n_1 \geq \cdots \geq n_p; n, k) = 0$ for $k > \min\{n, \sum_{i=0}^{p} n_i\}$.

With these boundary conditions we get that

$$F(n_1 \geq \cdots \geq n_p; n) = \sum_{k=-1}^{n} E(n_1 \geq \cdots \geq n_p; n, k), \tag{7}$$

for all $n, p \geq 0$, $n_1 \geq \cdots \geq n_p$.

The next theorem is the natural generalization of Theorem 2.1.

Theorem 4.3. *For $p, n \geq 1$, $k \geq 0$ and $n_1 \geq \cdots \geq n_p \geq 1$ we have the recursions*

$$(8) \quad F(n_1 \geq \cdots \geq n_p; n) = \\ 1 + \sum_{i=0}^{n} E(n_1 \geq \cdots \geq n_{p-1}; n, i) F(n_1 \geq \cdots \geq n_{p-1} \geq n_p - 1; i-1)$$

and

$$(9) \quad E(n_1 \geq \cdots \geq n_p; n, k) = \\ \sum_{i=k}^{n} E(n_1 \geq \cdots \geq n_{p-1}; n, i) E(n_1 \geq \cdots \geq n_{p-1} \geq n_p - 1; i-1, k-1),$$

where the first recursion is implied by the second.

Proof The theorem is, once discovered, easy to prove. Using the theorem by Clements-Lindström we can compute the number of compressed complexes instead of the number of f-vectors. We can therefore apply exactly the same decomposition technique as in the proof of Theorem 2.1 to prove the theorem. □

If $n_p = 1$, we will obtain types that ends with a zero in (8) and (9). Note that the values for F and E for these types can also be calculated recursively from the theorem, since $F(n_1 \geq \cdots \geq n_{p-1} \geq 0; n) = F(n_1 \geq \cdots \geq n_{p-1}; n)$, and similarly for E.

5. The number of f-vectors for simplicial complexes

A *simplicial complex* is a collection $\mathcal{F}$ of finite sets satisfying $B \in \mathcal{F}, A \subseteq B \implies A \in \mathcal{F}$. A set in $\mathcal{F}$ is called a *simplex* or a *face*. A set in $\mathcal{F}$ of cardinality one is called a *vertex*. A face that is not contained in any other face is called a *facet*. A simplicial complex on p vertices is a CL-complex of type $1 \geq \cdots \geq 1$ (p 1's). Given a simplicial complex $\mathcal{F}$, we have $f_i := |\{A \in \mathcal{F} : |A| = i\}|$. Note that f_i's are indexed by cardinality and not by dimension.

In this section we will study the number of f-vectors given a maximal number of vertices $p \geq 0$ and a maximal cardinality $n \geq 0$ for the simplices. We will also prove that the number of f-vectors for simplicial complexes and the number of M-sequences for multicomplexes have the same asymptotic growth. We may assume that $n \leq p$ without loss of generality.

Let

$F^p(n)$:=the number of f-vectors of simplicial complexes with $f_1 \leq p$ and $f_j = 0$ for all $j > n$, so $F^p(n) = F(1 \geq \cdots \geq 1; n)$. Just as in the definition of $F(n_1 \geq \cdots \geq n_p; n)$, we include both $(0, 0, \dots)$, the f-vector of the empty simplicial complex, and $(1, 0, 0, \dots)$, the f-vector for the simplicial complex containing only the empty set, when computing $F^p(n)$.

Hence we get $F^p(-1) = 1, F^p(0) = 2$ for all $p \geq 0$ and $F^0(n) = 2$ for all $n \geq 0$, see Table 5.

The specialization of the results by Clements and Lindström to simplicial complexes looks as follows.

Theorem 5.1. *Fix $0 \leq n \leq p$. Then the following are equal to $F^p(n)$.*

(i) *The number of f-vectors for simplicial complexes on at most p vertices and with no set of cardinality higher than n.*
(ii) *The number of compressed simplicial complexes on at most p vertices and with no set of cardinality higher than n.*
(iii) *The number of facet-vectors for simplicial complexes on at most p vertices and with no set of cardinality higher than n.* □

For the definition of "compressed" and "facet-vector" see Section 2 and Section 4 respectively.

We also need a notation for $E(1 \geq \cdots \geq 1; n, k)$. Let

$E^p(n, k)$:=the number of f-vectors for simplicial complexes with $f_1 \leq p$ and $f_j = 0$ for all $j > n$ and with maximal value for f_j when $j \leq k$ but not for f_{k+1}, i.e., $f_i = \binom{p}{i}$ for $i \leq k$ but $f_{k+1} < \binom{p}{k+1}$.

From the definition we have for every pair $p \geq 0, n \geq -1$, that $E^p(n, -1) = 1$, $E^p(n, n) = 1$ and $E^p(n, k) = 0$ for $k > n$.

We have the simplicial equivalent of (7).

$$F^p(n) = \sum_{k=-1}^{n} E^p(n, k), \tag{10}$$

for all $p \geq n \geq 0$.

Theorem 5.2. *For $0 \leq k$, $1 \leq n \leq p$ we have the recursions*

$$F^p(n) = \begin{cases} 1 + \sum_{i=0}^{n} E^{p-1}(n, i) F^{p-1}(i-1) & , \text{ if } n < p \\ 1 + F^p(p-1) & , \text{ if } n = p \end{cases} \tag{11}$$

and

$$E^p(n, k) = \begin{cases} \sum_{i=k}^{n} E^{p-1}(n, i) E^{p-1}(i-1, k-1) & , \text{ if } k \leq n < p \\ E^p(p-1, k) & , \text{ if } k < n = p \\ 1 & , \text{ if } k = n = p, \end{cases} \tag{12}$$

where the first recursion is implied by the second.

PROOF Follows from Theorem 4.3. $\square$

As the careful reader has noticed Theorem 2.1 and Theorem 5.2 are remarkably similar. The small difference that exists makes a big difference when keeping p fixed to get expressions in n and k. When keeping n and k fixed the difference between the theorems gives a smaller impact. After a careful study of the proofs of Theorem 2.4 and Proposition 2.5 we see that they are not affected by the slight difference between recursions (3) and (12).

Theorem 5.3. *$E^p(n,k)$ is a polynomial in p of degree $\binom{n+1}{2} - \binom{k+1}{2}$ and $F^p(n)$ is a polynomial in p of degree $\binom{n+1}{2}$, for each fixed $n, k \geq 0$.*

PROOF Identical to the proof of Theorem 2.4. $\square$

Proposition 5.4. *The coefficient for the term of highest degree in $E^p(n,k)$ is*

$$\frac{\prod_{i=k}^{n-2} \binom{\binom{n+1}{2} - \binom{i+1}{2} - 1}{i}}{\left(\binom{n+1}{2} - \binom{k+1}{2}\right)!} \tag{13}$$

If $k \geq n-1$, then the empty product in the numerator should be interpreted as 1. The coefficient for the term of highest degree in $F^p(n)$ is the same as in $E^p(n,0)$, so take $k = 0$ in (13).

PROOF Identical to the proof of Proposition 2.5. $\square$

For small values of n the polynomials are

$$
\begin{aligned}
(14) \qquad & F^p(0) = 2,\\
& F^p(1) = p+2,\\
& F^p(2) = \binom{p+2}{3} - \binom{p+2}{2} + 2\binom{p+2}{1} - 1,\\
& F^p(3) = 4\binom{p+2}{6} + \binom{p+2}{4} + \binom{p+2}{1},\\
& F^p(4) = 120\binom{p+2}{10} + 188\binom{p+2}{9} + 95\binom{p+2}{8} + 18\binom{p+2}{7} +\\
& \qquad + 5\binom{p+2}{6} + \binom{p+2}{4} + \binom{p+2}{1}.
\end{aligned}
$$

The polynomials are expressed in the same base as (4) to make comparisons easier.

Combining the results for f-vectors and M-sequences we get that they have the same asymptotic growth:

Theorem 5.5. *Fix $n \geq 0$. When p is large enough, the number of f-vectors for CL-complexes becomes almost independent of the type $n_1 \geq \cdots \geq n_p$. More precisely $M^p(n) \geq F(n_1 \geq \cdots \geq n_p; n) \geq F^p(n)$ for all $n_1 \geq \cdots \geq n_p$, and*

$$\lim_{p\to\infty} \frac{M^p(n)}{F^p(n)} = 1,$$

for each $n \geq 0$.

Proof Immediate from Theorem 2.4, Proposition 2.5, Theorem 5.3 and Proposition 5.4. □

Finally we will use that by Theorem 1.1 (iii), the number of f-vectors of $n-1$-dimensional shellable simplicial complexes on at most p vertices is equal to $M^{p-n}(n) - 1$.

Theorem 5.6. *Fix $n \geq 0$. When the number of vertices increases, almost every f-vector for an $n-1$-dimensional simplicial complex is also an f-vector for an $n-1$-dimensional shellable simplicial complex.*

The same is true when replacing shellable by Cohen-Macaulay, partitionable, pure or other weaker conditions on simplicial complexes.

Proof We have the following chain of inequalities:

$$M^{p-n}(n) \leq F^p(n) - F^p(n-1) \leq F^p(n) \leq M^p(n).$$

The result follows, since for a fixed n, all four are polynomials in p of degree $\binom{n+1}{2}$ by Theorem 2.4 and Theorem 5.3. $\square$

Note that in this paper "shellable" complexes are always pure. If we use "shellable" in the generalized nonpure sense, then something much stronger is true. It follows from Theorem 3.1 in [B2] that every f-vector of a simplicial complex is the f-vector of a nonpure shellable complex.

6. Remarks and open Problems

Remark 1: The total number of possible f-vectors of a simplicial complex with at most p vertices, i.e. $F^p(p)$, gives rise to an interesting sequence. This sequence starts $2, 3, 5, 10, 26, 96, 553, 5461, 100709, \ldots$, see Table 5 in the Appendix. We found the reference to [C] in [SP], where this sequence appear as the number of facet-vectors for simplicial complexes on p vertices. However, neither the sequence nor the recurrence can be found in [C]. It seems as if the sequence has been calculated by Knuth and it is said to appear in [K]. We are waiting impatiently.

Remark 2: There are at least two more ways to describe the second order partitions of $\{1, \ldots, n\}$ defined in Section 3. We state them here without proof of the equivalence with the original definition.

The first of the alternative definitions is more explicit and is in terms of multichains in the partition lattice. Let Π_n be the partition lattice of all ordinary partitions of $\{1, \ldots, n\}$ ordered by refinement. Given a partition σ, order the blocks increasingly by smallest element and let B_j^σ denote block number j in σ.

- Let $\mathcal{C}(n, k)$ be the set of all multichains $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_k$, $\sigma_i \in \Pi_n$ for $i = 1, \ldots, k$ such that $B_j^{\sigma_i} = B_j^{\sigma_{i+1}}$, for $j = 1, \ldots, i$ and such that σ_k has k blocks. The choice of σ_i is equivalent to the i:th iteration in the original definition of second order partitions and hence there is an easy bijection from the multichains in $\mathcal{C}(n, k)$ to the second order partitions of $\{1, \ldots, n\}$ with k blocks.

The second alternative definition is an algorithmic definition.

- Start by forming an ordinary partition and enclose the blocks with second order brackets [ ]. Order the blocks increasingly by smallest element. Leave the first block as it is. Partition the elements of the second block with an ordinary partition, this time using (). Partition the elements of the third block and then for each block created, partition its elements with an ordinary partition. Continue in the same manner and form $i-1$ consecutive partitions of the elements in block i. Now place an extra pair of brackets () in every block enclosing all the elements.
  This algorithm will construct precisely all second order partitions (with the first notation used in Section 3).

Problem 1: It is unclear whether any of the two definitions of second order partitions given in Remark 2 above can be generalized to p:th order partitions.

Problem 2: Does there exist some poset structure for p:th order partitions that naturally generalizes Π_n and that makes it possible to generalize the description in Remark 2 of second order partitions as multichains in Π_n to p:th order partitions?

Problem 3: More generally: Is there a way to describe p:th order partitions that is not of a recursive nature?

Problem 4: We have obtained a good knowledge of the asymptotic behavior of $M^p(n)$ when n is fixed. It still remains an interesting problem to give such results when keeping p fixed, $p \geq 4$. The best upper bound obtained here is (6), see the end of Section 2. Is this close to the truth?

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Appendix: Tables

$p\backslash n$	-1	0	1	2	3	4	5	6	7
0	1	2	2	2	2	2	2	2	2
1	1	2	3	4	5	6	7	8	9
2	1	2	4	8	16	32	64	128	256
3	1	2	5	15	52	203	877	4140	21147
4	1	2	6	26	152	1144	10742	122772	1673856
5	1	2	7	42	392	5345	102050	2632429	89026966
6	1	2	8	64	904	20926	753994	40412530	3099627142
7	1	2	9	93	1899	70506	4486435	463830877	74407044672

Table 1. Table of $M^p(n)$, the number of M-sequences with $m_1 \leq p$ and $m_i = 0$ for $i > n$.

$n\backslash k$	-1	0	1	2	3	4	5	6
-1	1	0	0	0	0	0	0	0
0	1	1	0	0	0	0	0	0
1	1	3	1	0	0	0	0	0
2	1	7	6	1	0	0	0	0
3	1	15	25	10	1	0	0	0
4	1	31	90	65	15	1	0	0
5	1	63	301	350	140	21	1	0
6	1	127	966	1701	1050	266	28	1
7	1	255	3025	7770	6951	2646	462	36

Table 2. Table of $L^3(n,k) = S(n+2,k+2)$, the refinement of $M^3(n)$.

$n\backslash k$	-1	0	1	2	3	4	5	6
-1	1	0	0	0	0	0	0	0
0	1	1	0	0	0	0	0	0
1	1	4	1	0	0	0	0	0
2	1	14	10	1	0	0	0	0
3	1	51	79	20	1	0	0	0
4	1	202	611	294	35	1	0	0
5	1	876	4934	4020	854	56	1	0
6	1	4139	42568	55257	18622	2100	84	1
7	1	21146	395364	788788	395676	68182	4578	120

Table 3. Table of $L^4(n,k)$, the refinement of $M^4(n)$.

$n\backslash k$	-1	0	1	2	3	4	5	6
-1	1	0	0	0	0	0	0	0
0	1	1	0	0	0	0	0	0
1	1	5	1	0	0	0	0	0
2	1	25	15	1	0	0	0	0
3	1	151	204	35	1	0	0	0
4	1	1143	3107	1023	70	1	0	0
5	1	10741	55983	31361	3837	126	1	0
6	1	122771	1208256	1079958	209415	11817	210	1
7	1	1673855	31171031	42783118	12308618	1058497	31515	330

TABLE 4. Table of $L^5(n,k)$, the refinement of $M^5(n)$.

$p\backslash n$	-1	0	1	2	3	4	5	6	7	8
0	1	2	2	2	2	2	2	2	2	2
1	1	2	3	3	3	3	3	3	3	3
2	1	2	4	5	5	5	5	5	5	5
3	1	2	5	9	10	10	10	10	10	10
4	1	2	6	16	25	26	26	26	26	26
5	1	2	7	27	70	95	96	96	96	96
6	1	2	8	43	190	457	552	553	553	553
7	1	2	9	65	471	2246	4908	5460	5461	5461
8	1	2	10	94	1060	9705	48230	95248	100708	100709
9	1	2	11	131	2189	35926	398663	2016372	3617645	3718353

TABLE 5. Table of $F^p(n)$, the number of f-vectors for simplicial complexes of dimension at most $n-1$ and with at most p vertices.

$p\backslash k$	-1	0	1	2	3	4	5	6	7	8	$F^p(p)$
0	1	1	0	0	0	0	0	0	0	0	2
1	1	1	1	0	0	0	0	0	0	0	3
2	1	2	1	1	0	0	0	0	0	0	5
3	1	4	3	1	1	0	0	0	0	0	10
4	1	9	10	4	1	1	0	0	0	0	26
5	1	25	43	20	5	1	1	0	0	0	96
6	1	95	267	147	35	6	1	1	0	0	553
7	1	552	2662	1775	406	56	7	1	1	0	5461
8	1	5460	47018	38525	8645	966	84	8	1	1	100709

TABLE 6. Table of $E^p(p,k)$, the refinement of $F^p(p)$.