ON INNER PRODUCT IN MODULAR TENSOR CATEGORIES. II
INNER PRODUCT ON CONFORMAL BLOCKS.

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Introduction.

This is a second part of the paper [K], and we freely use the notations from there. In [K], we defined for every modular tensor category (MTC) an inner product on the space of morphisms and proved that the inner product on the space $\mathrm{Hom}(\bigoplus X_i \otimes X_i^*, U)$ is modular invariant. Also, we have shown that in the case of the MTC arising from representations of quantum group $U_q\mathfrak{sl}_n$ at roots of unity and U being a symmetric power of the fundamental representation, the action of modular group on the space of intertwiners $\mathrm{Hom}(\bigoplus X_i \otimes X_i^*, U)$ can be written explicitly in terms of Macdonald's polynomials at roots of unity.

In this paper, we apply the same construction to the MTC coming from the integrable representations of affine Lie algebras or, equivalently, from Wess-Zumino-Witten model of conformal field theory. We briefly recall construction of this category, first suggested by Moore and Seiberg (see [MS1,2]) and later refined by Kazhdan and Lusztig ([KL1–4]) and Finkelberg ([F]) in Section 9. In particular, spaces of homomorphisms in this category are the spaces of conformal blocks of WZW model. Thus, the general theory developed in Section 2 of [K] gives us an inner product on the space of conformal blocks, and so defined inner product is modular invariant. This definition is constructive: we show how it can be rewritten so that it only involves Drinfeld associator, or, equivalently, asymptotics of solutions of Knizhnik-Zamolodchikov equations. Since there are integral formulas for the solutions of KZ equations, this shows that the inner product on the space of conformal blocks can be written explicitly in terms of certain integrals. In the case $\mathfrak{g} = \mathfrak{sl}_2$ these integrals can be calculated (see [V]), using Selberg integral, and the answer is written in terms of Γ-functions. Thus, in this case we can write explicit formulas for inner product on the space of conformal blocks. These expressions are

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closely related with those suggested by Gawedzki et al. ([G, FGK]) from completely different approach; there is little doubt that the same holds in general case.

It was proved recently in [KL1–4, F] (though this statement was widely believed long before) that as a braided category this category is equivalent to the reduced category of representations of $U_q\mathfrak{g}$ with q being a root of unity, which was discussed in the first part of this paper. Since in both categories the action of modular group is written in terms of braiding, this shows that they are equivalent as modular tensor categories. In particular, this shows that for $\mathfrak{g} = \mathfrak{sl}_n$ the action of the modular group on the space of conformal blocks on the torus with one puncture, to which a symmetric power of fundamental representation is assigned, in some basis has matrix coefficients given by special values of Macdonald's polynomials. Thus, for $\mathfrak{g} = \mathfrak{sl}_2$ the action of modular group on any space of conformal blocks on a torus in some basis can be written in terms of q-ultraspherical polynomials, and this basis is related to the standard one by a matrix of gamma-functions.

8. Integrable representations of affine algebras and conformal blocks

Here we briefly recall the main definitions, referring the reader to [MS1, TUY] for details.

Integrable modules.

As before, let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with the invariant form $(\,,\,)$ normalized so that $(\theta, \theta) = 2$ where θ is the highest root. Let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ be the corresponding affine Lie algebra; as usual, we denote $x[n] = x \otimes t^n$. We also denote by $\hat{\mathfrak{g}}$ its completion: $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}c$. In this whole paper, we fix a non-negative integer k (level). Let $\varkappa = k + h^\vee$, and let $\mathcal{O}_\varkappa$ be the category of integrable finite-length $\hat{\mathfrak{g}}$-modules of level k. It is known that this category is semisimple (which follows from the fact that integrable modules are unitary), and the simple objects in this category are the irreducible modules $L_{\lambda,k}$, $\lambda \in C$, where

$$C = \{\lambda \in P^+ | \langle \lambda, \theta^\vee \rangle \le k\} = \{\lambda \in P^+ | \langle \lambda + \rho, \theta^\vee \rangle < \varkappa\}. \tag{8.1}$$

There exists a natural notion of duality in this category: for $V \in \mathcal{O}_\varkappa$, let DV be the restricted dual space to V (that is, DV is direct sum of spaces dual to homogeneous components of V) and the action of $\tilde{\mathfrak{g}}$ is defined as the usual action in the dual space twisted by the automorphism $\#$ defined by

$$x[n]^\# = (-1)^n x[-n], \quad c^\# = -c.$$

It is easy to check that $DV \in \mathcal{O}_\varkappa$, and $D(DV)$ is canonically isomorphic to V: the usual isomorphism of vector spaces $V^{**} \simeq V$ is $\hat{\mathfrak{g}}$-isomorphism. Also, $DL_{\lambda,k} \simeq L_{\lambda^*,k}$, though the isomorphism is not canonical.

Conformal blocks.

Let $\mathcal{X}$ denote the following collection of data:

(1) X – a non-singular compact complex curve,
(2) $z_1, \dots, z_n$ – distinct point on X divided into two sets In and Out,
(3) w_i – local parameters near the points z_i.

We assume that on each connected component of X there is at least one of the points z_i.

With each point z_i we associate a Lie algebra $\mathfrak{g}\otimes\mathbb{C}((w_i))$. Let

$$
\begin{aligned}
\hat{\mathfrak{g}}_{in} &= \left(\bigoplus_{i\in In}\mathfrak{g}\otimes\mathbb{C}((w_i))\right)\oplus\mathbb{C}c\\
\hat{\mathfrak{g}}_{out} &= \left(\bigoplus_{i\in Out}\mathfrak{g}\otimes\mathbb{C}((w_i))\right)\oplus\mathbb{C}c.
\end{aligned}
\tag{8.2}
$$

and the cocycle defining the central extension in $\hat{\mathfrak{g}}_{in}$ (respectively, $\hat{\mathfrak{g}}_{out}$) is the sum of standard cocycles on each of $\mathfrak{g}\otimes\mathbb{C}((w_i)), i\in In$ (respectively, $i\in Out$).

Also, let us consider the Lie algebra Γ:

(8.3)

$$\Gamma(\mathcal{X})=\{\mathfrak{g}-\text{valued meromorphic functions on } X \text{ regular outside of } z_1,\dots,z_n\}$$

and its central extension $\widehat{\Gamma}=\Gamma\oplus\mathbb{C}c$ with the defining cocycle given by

$$c(f,g)=\sum_{i\in In}\operatorname{Res}_{z_i} f\,dg.$$

Expanding a function $f\in\Gamma$ near a point z_i in a Laurent series in w_i we get an embedding $\pi_i:\Gamma(\mathcal{X})\subset\mathfrak{g}\otimes\mathbb{C}((w_i))$.

Taking direct sum over all $i\in In$ (respectively, $i\in Out$), we get embeddings

$$
\begin{aligned}
\pi_{in}:\widehat{\Gamma}(\mathcal{X})\subset\hat{\mathfrak{g}}_{in}: &\qquad f\mapsto\oplus_{i\in In}\pi_i(f), \qquad c\mapsto c,\\
\pi_{out}:\widehat{\Gamma}(\mathcal{X})\subset\hat{\mathfrak{g}}_{out}: &\qquad f\mapsto\oplus_{i\in Out}\pi_i(f), \qquad c\mapsto -c.
\end{aligned}
\tag{8.4}
$$

One can easily check that these embeddings are Lie algebra homomorphisms.

Definition 8.1. Let $\mathcal{X}$ be as above, and assume that we are given integrable modules $V_1,\dots,V_n\in\mathcal{O}_\varkappa$ assigned to the points $z_1,\dots,z_n$ respectively. Let us consider V_i as a module over $\mathfrak{g}\otimes\mathbb{C}((w_i))\oplus\mathbb{C}c$. Then the corresponding space of conformal blocks is defined by

$$
\begin{aligned}
W(\mathcal{X};V_1,\dots,V_n)=\{\Phi:\otimes_{i\in In}V_i\to\widehat{\otimes}_{i\in Out}V_i\big|(\pi_{out}(f))^{\#}\Phi=\Phi\pi_{in}(f)\\
\text{for all } f\in\Gamma(\mathcal{X})\}
\end{aligned}
\tag{8.5}
$$

where $\widehat{\otimes}$ is a completion of the tensor product with respect to homogeneous grading.

Remark 8.2. If $i\in Out$ then denote by $\mathcal{X}'$ the same data as $\mathcal{X}$ except that now we consider i as element of In: $In'=In\cup\{i\}, Out'=Out\setminus\{i\}$. Then we have canonical isomorphism:

$$W(\mathcal{X};V_1,\dots,V_n)=W(\mathcal{X}',V_1,\dots,DV_i,\dots V_n).$$

Thus, it suffices to consider conformal blocks when all points are incoming (or all are outgoing), as is done in [TUY]; however, it is more convenient to consider general situation.

In a similar way, it can be shown that if $\mathcal{X}'$ is obtained from $\mathcal{X}$ by marking one more incoming point z_0 and assigning to it the representation $V_0 = L_{0,k}$ then the map

$$\begin{aligned} W(\mathcal{X}'; V_0, V_1, \dots, V_n) &\to W(\mathcal{X}, V_1, \dots, V_n) \\ \Phi &\mapsto \Phi(1, \dots), \end{aligned}$$

where 1 is the highest weight vector in $L_{0,k}$, is an isomorphism. Thus, the condition that there is at least one point on each connected component of X is inessential.

Remark 8.3. In fact, the definition of conformal blocks works in more general situation: we can allow X to be a semi-stable singular curve. This plays a crucial role in proving the gluing axiom ([TUY]) or construction of associativity isomorphism (see below); however, all of the results we need in this paper can be formulated without reference to singular curves.

Example 8.4. Let $n = 3, X = \mathbb{C}P^1$ with global coordinate w, $In = \{\infty\}, Out = \{0, z\}$ with local parameters $1/w, w, w - z$ respectively. Then

$$W(\mathcal{X}; V_0, V_z, V_\infty) = \Big\{ \Phi : V_\infty \to V_0 \hat{\otimes} V_z \Big| \\ \Phi x[n] = \Big(1 \otimes x[n] + \sum_{i \geq 0} \binom{-n}{i} z^{-n-i} x[-i] \otimes 1 \Big) \Phi. \Big\}$$

The following result is well-known; we refer the reader to [TUY] for the proof.

Proposition 8.5. *The spaces of conformal blocks are always finite-dimensional.*

On the sphere the space of conformal blocks can be described more explicitly.

For a simple module $L_{\lambda,k}$ let $L_{\lambda,k}[0]$ be the $\mathfrak{g}$-module generated by the highest weight vector; clearly, $L_{\lambda,k}[0] \simeq L_\lambda$. It is easy to show that this operation can be defined in invariant terms; thus, we have a faithful functor $V \mapsto V[0]$ between $\mathcal{O}_\varkappa$ and the category *Rep* $\mathfrak{g}$ of finite-dimensional representations of $\mathfrak{g}$. Note that we have canonical identifications $(DV)[0] \simeq (V[0])^*, L_{0,k}[0] \simeq \mathbb{C}$. Denote by $i : V[0] \to V$ and $p : V \to V[0]$ the canonical embedding and projection respectively.

For every $\Phi \in W(\mathcal{X}, V_1, \dots, V_n)$ define its correlation function (or, which is the same, its highest term) $\langle \Phi \rangle$ by

$$\langle \Phi \rangle = p \otimes \dots p \Phi(i \otimes \dots \otimes i) \in \mathrm{Hom}_{\mathfrak{g}}(\otimes_{In} V_i[0], \otimes_{Out} V_i[0]). \tag{8.6}$$

Proposition 8.6. *On a sphere, every $\Phi \in W(\mathcal{X}, V_1, \dots, V_n)$ is uniquely defined by its correlation function $\langle\Phi\rangle$. In other words, we have an embedding:*

$$W(\mathcal{X}, V_1, \dots, V_n) \subset \mathrm{Hom}_{\mathfrak{g}}(\bigotimes_{In} V_i[0], \bigotimes_{Out} V_i[0]) \\ \Phi \mapsto \langle\Phi\rangle. \tag{8.7}$$

Remark. This statement is not true on higher genus surfaces.

Note that this embedding depends on the choice of points z_i and local parameters at these points (in fact, it depends only on 1-jet of the local parameter). For irreducible V_i image of this embedding can be explicitly described; we refer the reader to [FSV] (for the case of three-punctured sphere, this description also appears in [TUY]).

Flat connection.

The definition of conformal blocks which we gave depends on $\mathcal{X}$; thus, we can consider space of conformal blocks as finite-dimensional vector bundle over the corresponding moduli space. This vector bundle is called the bundle of conformal blocks.

It turns out that this vector bundle has a natural flat connection, which was first calculated by Knizhnik and Zamolodchikov on the sphere and by Bernard on a torus. This connection can be naturally defined using the Sugawara construction; we refer the reader to [TUY, KL2] for details, giving here only the answer.

From now on we assume the following

$$\begin{gathered} X = \mathbb{C}P^1, w - \text{global coordinate on } X \\ In = \{\infty\}, \quad w_\infty = 1/w, \\ Out = \{z_1, \dots, z_n\}, \quad w_i = w - z_i, \end{gathered} \tag{8.8}$$

As we mentioned before, for each $z_1, \dots, z_n$ the space $W(\mathcal{X}, V_1, \dots, V_n, V_\infty)$ can be identified with a certain subspace in $\mathrm{Hom}_{\mathfrak{g}}(V_\infty[0], V_1[0] \otimes \dots \otimes V_n[0])$. Thus, to define a flat connection in the space of conformal blocks it suffices to define a flat connection in the trivial vector bundle with fiber $\mathrm{Hom}_{\mathfrak{g}}(V_\infty[0], V_1[0] \otimes \dots \otimes V_n[0]) \simeq (V_1[0] \otimes \dots \otimes V_n[0] \otimes V^*_\infty[0])^{\mathfrak{g}}$ which would preserve the subspace of conformal blocks. Such a connection is obtained from the Knizhnik-Zamolodchikov connection on $V_1[0] \otimes \dots \otimes V_n[0] \otimes V^*_\infty[0]$

$$(k + h^\vee)\frac{\partial}{\partial z_i}\psi = \left(\sum_{\substack{j=1\dots n \\ j\neq i}} \frac{\Omega_{ij}}{z_i - z_j}\right)\psi, \tag{8.9}$$

where Ω is the standard $\mathfrak{g}$-invariant element in $\mathfrak{g} \otimes \mathfrak{g}$: if x_i, x^i are dual bases in $\mathfrak{g}$ with respect to the inner product $(\, , \,)$ then $\Omega = \sum x_i \otimes x^i$. As usual, we use the notation $\Omega_{ij} = \pi_i \otimes \pi_j(\Omega), i, j = 1, \dots, n$.

Since this connection commutes with the action of $\mathfrak{g}$, it also defines a connection on the space $(V_1[0] \otimes \dots \otimes V_n[0] \otimes V^*_\infty[0])^{\mathfrak{g}}$, and it can be checked that it preserves the subspace of conformal blocks (see [FSV1,2]).

Example 8.7. Let us consider conformal blocks on a 3-punctured sphere: $in = \{\infty\}, Out = \{z_1, z_2\}$ and assume that the modules V_i are irreducible: $V_\infty = L_{\lambda,k}, V_1 = L_{\mu,k}, V_2 = L_{\nu,k}$. In this case it is easy to check that a section $\Phi(z_1, z_2)$ of the bundle of conformal blocks is flat iff $\langle\Phi(z_1, z_2)\rangle = (z_1 - z_2)^{\Delta_\lambda - \Delta_\mu - \Delta_\nu} g$, where $\Delta_\lambda = \frac{(\lambda,\lambda+2\rho)}{2(k+h^\vee)}$, and $g \in \mathrm{Hom}_{\mathfrak{g}}(L_\lambda, L_\nu \otimes L_\mu)$ does not depend on z_i.

In this case we will use the notation $\Phi^g(z), z = z_1 - z_2$ for such a flat section; the operators $\Phi(z)$ are called (chiral) vertex operators.

9. Category $\mathcal{O}_\varkappa$ as modular category

In this section we recall the construction of tensor (and in fact, modular tensor) structure on the category $\mathcal{O}_\varkappa$. Let us start with defining the space of morphisms.

As before, let us assume that $\mathcal{X}$ is $n+1$-punctured sphere (8.8) and $n \geq 1$. The moduli space of all such punctured spheres is the configuration space

$$X_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n | z_i \neq z_j\}.$$

As discussed above, the space of conformal blocks is a local system over this space with the connection given by KZ equations (8.9). Note also that the KZ equations show that every flat section of this local system is invariant under translations $(z_1, \dots, z_n) \mapsto (z_1 + c, \dots, z_n + c)$ and the local system equivalent to a local system on $X_n^0 = \{\mathbf{z} \in X_n | z_1 = 0\}$. Define

$$\mathcal{D}_n = \{(z_2, \dots, z_n) \in \mathbb{C}^{n-1} | 0 < |z_2| < |z_3| < \dots < |z_n|, z_i \notin \mathbb{R}_{<0}\} \subset X_n, \tag{9.1}$$

Note that $\mathcal{D}$ is contractible.

Now we can formulate the main theorem of this section, which is essentially due to Moore and Seiberg (see also [KL]).

Theorem 9.1. *The category $\mathcal{O}_\varkappa$ can be endowed with a structure of ribbon tensor category such that:*

1. *If we denote the tensor product in this category by $\dot\otimes$ (to avoid confusion with usual tensor product of vector spaces) then for any $n \geq 1$ and for any choice of representations $V_1, \dots, V_n, V_\infty \in \mathcal{O}_\varkappa$*

$$\mathrm{Hom}_{\mathcal{O}_\varkappa}(V_\infty, V_1 \dot\otimes \dots \dot\otimes V_n) = \Gamma(\mathcal{D}_n, W), \tag{9.2}$$

   where $W = W(\mathcal{X}; V_1, \dots, V_\infty)$ is the local system of conformal blocks on $n+1$-punctured sphere (8.8), and Γ stands for the space of global sections of this local system over $\mathcal{D}_n$ defined by (9.1).
2. *The unit object is $\mathbf{1} = L_{0,k}$ and the maps $V \simeq \mathbf{1} \dot\otimes V$ are constructed as in Remark 8.2.*
3. *The dual object is given by $V^* = DV$, and the maps $i_V : \mathbf{1} \to V \dot\otimes V^*$ are defined so that $i_{L_{\lambda,k}} = \Phi^{i_\lambda}$, where Φ^g is defined in Example 8.7, i_λ is the canonical map of $\mathfrak{g}$-modules $\mathbb{C} \to V_\lambda \otimes V_\lambda^*$.*
4. *The isomorphism $\delta_V : V \to V^{**} = D(DV)$ coincides with canonical identification of vector spaces $V \simeq V^{**}$ (recall that as a vector space $DV = V^*$ is*

the restricted dual to V), and the twist θ is given by $\theta = e^{2\pi \mathrm{i} L_0}$, where L_0 is the Sugawara element in the completion of $U\hat{\mathfrak{g}}$; thus,

$$\theta|_{L_{\lambda,k}} = e^{2\pi \mathrm{i}\Delta_\lambda}, \qquad \Delta_\lambda = \frac{(\lambda, \lambda + 2\rho)}{2(k + h^\vee)}. \tag{9.3}$$

Remark. Since $\mathcal{D}_n$ is simply connected, we could just say that we fix some particular choice of points $z_1, \dots, z_n$, say, $\mathbf{z} = (0, 1, \dots, n-1)$ and let $\mathrm{Hom}_{\mathcal{O}_\varkappa}(V_\infty, V_1 \otimes \dots \otimes V_n) = W(\mathcal{X}; V_1, \dots, V_\infty)$.

We will recall the construction of associativity and commutativity isomorphisms in the next subsection. Note that we have not defined the duality morphisms $V^* \dot{\otimes} V \to \mathbf{1}$: we will not use their explicit form, just their existence (which follows from much more general construction in [KL4]).

Therefore, we can use the language of ribbon graphs for representing morphisms in the category $\mathcal{O}_\varkappa$; unfortunately, associativity morphism (which is highly non-trivial) "does not show" in the pictures, so one must be careful when performing calculations. In particular,we can define the "quantum dimension" of a module V in the same way as we did for arbitrary MTC in Section 1; we will denote it by $\dim_\varkappa V$. Note that it has nothing to do with the usual dimension of V (which is infinite) nor even with the character of V which is a formal power series in q.

Example 9.2. If $n = 1$ then the space of homomorphisms $\mathrm{Hom}_{\mathcal{O}_\varkappa}(V_\infty, V_1 \dot{\otimes} V_2)$ coincides with the space of chiral vertex operators (see Examples 8.4, 8.7) and can be identified with a subspace in the space of $\mathfrak{g}$-homomorphisms: it follows from Proposition 8.6 that we have an embedding

$$\begin{gathered} \mathrm{Hom}_{\mathcal{O}_\varkappa}(V_\infty, V_1 \dot{\otimes} V_2) \subset \mathrm{Hom}_{\mathfrak{g}}(V_\infty[0], V_1[0] \otimes V_2[0]) \\ \Phi^g(z) \mapsto \langle \Phi^g(1) \rangle = g. \end{gathered} \tag{9.4}$$

Now we can describe the action of modular group. Recall the object H defined in Section 1 for any MTC; in our case, it is given by $H = \bigoplus_\lambda L_{\lambda^*,k} \dot{\otimes} L_{\lambda,k}$. The following result immediately follows from gluing axiom for conformal blocks.

Lemma 9.3. *The space*

$$\mathrm{Hom}_{\mathcal{O}_\varkappa}(H, U) = \bigoplus_\lambda \mathrm{Hom}_{\mathcal{O}_\varkappa}(L_{\lambda,k}, L_{\lambda,k} \dot{\otimes} U) \tag{9.5}$$

is isomorphic to the space of conformal blocks on a torus with one puncture to which the representation U is assigned.

The following result clarifies the mysterious action of $SL_2(\mathbb{Z})$ introduced in Section 1.

Theorem 9.4. [MS2] *The category $\mathcal{O}_\varkappa$ is a modular category in the sense of Definition* 1.3*, and the action of $SL_2(\mathbb{Z})$ on $\mathrm{Hom}_{\mathcal{O}_\varkappa}(H, U)$ defined in Theorem* 1.10 *coincides with the natural geometric action of $SL_2(\mathbb{Z})$ on the space of conformal blocks on the torus.*

This theorem can be proved in general context of 2-dimensional modular functor, using Kirby calculus; see [Tu] for this approach. In fact, it is known that in any modular tensor category we can define the action of the mapping class group of any punctured Riemann surface on the appropriate space of homomorphisms: this automatically follows from the possibility to define the action of $SL_2(\mathbb{Z})$.

Associativity and commutativity isomorphisms.

In this section we briefly recall the construction of associativity and commutativity isomorphisms in $\mathcal{O}_\varkappa$.

Let us start with commutativity homomorphism. By definition, this is a functorial isomorphism $V_1 \dot\otimes V_2 \to V_2 \dot\otimes V_1$. To construct such an isomorphism, it suffices to construct isomorphisms

$$\mathrm{Hom}_{\mathcal{O}_\varkappa}(L_{\lambda,k}, V_1 \dot\otimes V_2) \simeq \mathrm{Hom}_{\mathcal{O}_\varkappa}(L_{\lambda,k}, V_2 \dot\otimes V_1).$$

Recalling the definitions (see Theorem 9.1 and following remark) we see that the left-hand side is the space of conformal blocks $W(\mathcal{X}; V_1, V_2, V_\infty)$ where $\mathcal{X}$ is $\mathbb{C}P^1$ with marked points $z_1 = 0, z_2 = 1, \infty$. Similarly, renaming the variables we see that the right-hand side is the space $W(\mathcal{X}; V_1, V_2, V_\infty)$ where now the marked points are $z_1 = 1, z_2 = 0, \infty$. Since W is endowed with a flat connection, we can identify these two spaces if we choose a path in the moduli space connecting $(0, 1)$ with $(1, 0)$. We fix such a path by saying that z_1 must pass z_2 on the left.

If we identify each of these spaces with a subspace in $\mathrm{Hom}_{\mathfrak{g}}(L_\lambda, V_1[0] \otimes V_2[0])$ (see Example 9.2) then commutativity morphism coincides with composition with the map

$$Pe^{\pi \mathrm{i}\Omega/\varkappa} : V_1[0] \otimes V_2[0] \to V_2[0] \otimes V_1[0].$$

Here Ω is as in (8.9), and P is the permutation: $P(v_1 \otimes v_2) = v_2 \otimes v_1$.

In a similar way, constructing the associativity isomorphism $(V_1 \dot\otimes V_2) \dot\otimes V_3 \simeq V_1 \dot\otimes (V_2 \dot\otimes V_3)$ is equivalent to constructing for every λ isomorphisms

$$\mathrm{Hom}(L_{\lambda,k}, (V_1 \dot\otimes V_2) \dot\otimes V_3) \simeq \mathrm{Hom}(L_{\lambda,k}, V_1 \dot\otimes (V_2 \dot\otimes V_3)). \tag{9.6}$$

Using complete reducibility, we can rewrite this as

$$\bigoplus_{\mu \in C} W_\lambda^{\mu, V_3} \otimes W_\mu^{V_1, V_2} \simeq \bigoplus_{\mu' \in C} W_\lambda^{V_1, \mu'} \otimes W_{\mu'}^{V_2, V_3}, \tag{9.7}$$

where we denote

$$W_\mu^{V_1, V_2} = \mathrm{Hom}_{\mathcal{O}_\varkappa}(L_{\mu,k}, V_1 \dot\otimes V_2) \tag{9.8}$$

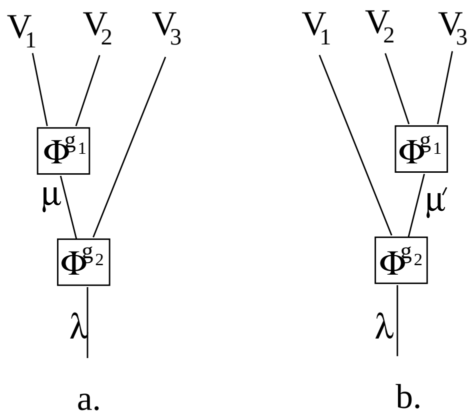


FIGURE 1. Two representations of $\mathrm{Hom}(L_{\lambda,k}, V_1 \dot{\otimes} V_2 \dot{\otimes} V_3)$ as tensor product.

etc. Figure 1 can be useful in visualizing both sides of (9.6).

We will do it by establishing isomorphism of each of these spaces with the space $\mathrm{Hom}_{\mathcal{O}_\varkappa}(L_{\lambda,k}, V_1 \dot{\otimes} V_2 \dot{\otimes} V_3)$ defined by (9.2).

Proposition 9.5.

(1) *Let $\Phi^1(z) : L_{\mu,k} \to V_1 \dot{\otimes} V_2, \Phi^2(z') : L_{\lambda,k} \to L_{\mu,k} \dot{\otimes} V_3$ be vertex operators* (*see Example* 8.7). *Then for $|z'| > |z|$ the product*

$$\Psi(z,z') = (\Phi^1(z) \otimes 1)\Phi^2(z') : L_{\lambda,k} \to V_1 \widehat{\otimes} V_2 \widehat{\otimes} V_3$$

is well-defined and is a flat section of the bundle $W(\mathcal{X}; V_1, V_2, V_3, L_{\lambda,k})$ where $\mathcal{X}$ is the Riemann sphere with marked points $z_1 = 0, z_2 = z, z_3 = z', \infty$.

(2) *The section Ψ is uniquely defined by the condition that its highest term $\langle \Psi \rangle$ has the following asymptotics as $|z'| \gg |z|$: $\langle \Psi(z,z') \rangle \sim (\langle \Phi^1(z) \rangle \otimes 1)\langle \Phi^2(z') \rangle$. If V_i are irreducible:$V_i = L_{\nu_i,k}, \Phi^1 = \Phi^{g_1}, \Phi^2 = \Phi^{g_2}$* (*see Example* 8.7) *then this can be rewritten as follows:*

$$\langle \Psi(z,z') \rangle \sim z'^{\Delta_\lambda - \Delta_\mu - \Delta_{\nu_3}} z^{\Delta_\mu - \Delta_{\nu_1} - \Delta_{\nu_2}} (g_1 \otimes 1) g_2$$

(3) *The map $\Phi^1 \otimes \Phi^2 \to \Psi$ is an isomorphism:*

$$\bigoplus_{\mu \in C} W_\lambda^{\mu, V_3} \otimes W_\mu^{V_1, V_2} \simeq \mathrm{Hom}_{\mathcal{O}_\varkappa}(L_{\lambda,k}, V_1 \dot{\otimes} V_2 \dot{\otimes} V_3)$$

where the right hand side is defined by (9.2).

In a similar way, we can construct the second isomorphism:

Proposition 9.6.

(1) *Let $\Phi^1(z) : L_{\mu',k} \to V_2 \dot{\otimes} V_3, \Phi^2(z') : L_{\lambda,k} \to V_1 \dot{\otimes} L_{\mu',k}$ be vertex operators. Then for $|z| < |z'|$ the product*

$$\Psi(z,z') = (1 \otimes \Phi^1(z))\Phi^2(z') : L_{\lambda,k} \to V_1 \widehat{\otimes} V_2 \widehat{\otimes} V_3$$

is well-defined and is a flat section of the bundle $W(\mathcal{X}; V_1, V_2, V_3, L_{\lambda,k})$ *where* $\mathcal{X}$ *is the Riemann sphere with marked points* $z_1 = 0, z_2 = z', z_3 = z + z', \infty$.

(2) *The section* Ψ *is uniquely defined by the condition that its highest term* $\langle\Psi\rangle$ *has the following asymptotics as* $|z'| \gg |z|$: $\langle\Psi(z,z')\rangle \sim (1 \otimes \langle\Phi^1(z)\rangle)\langle\Phi^2(z')\rangle$.

Again, if V_i *are irreducible:* $V_i = L_{\nu_i,k}, \Phi^1 = \Phi^{g_1}, \Phi^2 = \Phi^{g_2}$ *(see Example* 8.7*) then this is equivalent to*

$$\langle\Psi(z,z')\rangle \sim z^{\Delta_{\mu'} - \Delta_{\nu_2} - \Delta_{\nu_3}} z'^{\Delta_\lambda - \Delta_{\nu_1} - \Delta_{\mu'}} (1 \otimes g_1) g_2$$

(3) *The map* $\Phi^1 \otimes \Phi^2 \to \Psi$ *is an isomorphism:*

$$\bigoplus_{\mu' \in C} W_\lambda^{V_1, \mu'} \otimes W_{\mu'}^{V_2, V_3} \simeq \mathrm{Hom}_{\mathcal{O}_\varkappa}(L_{\lambda,k}, V_1 \dot{\otimes} V_2 \dot{\otimes} V_3)$$

where the right hand side is defined by (9.2).

Thus, we see that both of the spaces in (9.7) can be identified with the space of flat sections of the bundle $W(\mathcal{X}, V_1, V_2, V_3, L_{\lambda,k})$, where $\mathcal{X}$ is the sphere with marked points $z_1 = 0, z_2, z_3, \infty$. The first space is identified with the flat sections in the region $|z_2| < |z_3|$ and identification is given by considering asymptotics of a section as $|z_2| \ll |z_3|$; the second is identified with flat sections over the region $|z_2 - z_3| < |z_2|$ and identification is given by considering asymptotics of a section as $|z_2 - z_3| \ll |z_2|$. Now we can note that these regions have non-empty intersection: for example, points of the form $(z_1, z_2, z_3) = (0, t, 1), 1/2 < t < 1$ belong to both of them. Thus, we can identify spaces of flat sections over them, which gives isomorphism (9.7).

Using the embedding (8.7) of the space of conformal blocks into the space of $\mathfrak{g}$-intertwiners, we see that associativity isomorphism is obtained by comparing asymptotics of solutions of KZ equations (8.9) in three variables in different asymptotic zones. It can be further simplified: KZ equations in 3 variables is equivalent to just one equation

$$(k + h^\vee)\frac{d}{dt}\psi = \left(\frac{\Omega_{12}}{t} + \frac{\Omega_{23}}{t-1}\right)\psi,$$

where $t = (z_1 - z_2)(z_1 - z_3)$. This gives the standard definition of associativity isomorphism, as in [Dr3, Dr4].

Corollary 9.7. *If* V_1, V_2, V_3, λ *are such that on the space of highest weight vectors of weight* λ *in* $V_1[0] \otimes V_2[0] \otimes V_3[0]$ *the operators* Ω_{12}, Ω_{23} *commute then the associativity isomorphism* (9.7) *is trivial, i.e. coincides with the restriction of associativity isomorphism for* $\mathfrak{g}$*-modules* $V_1[0], V_2[0], V_3[0]$.

Equivalence of categories $\mathcal{O}_\varkappa$ and $\mathcal{C}(\mathfrak{g}, \varkappa)$.

The following key theorem, which was widely believed for several years, was proved by M. Finkelberg in his thesis [F].

Theorem 9.8. *There exist numbers* $n(\mathfrak{g})$ *such that for* $k \geq n(\mathfrak{g})$ *the modular tensor category* $\mathcal{O}_\varkappa$ *described above is equivalent as modular tensor category to the reduced* *category $\mathcal{C}(\mathfrak{g}, \varkappa)$ of representations of quantum group in the root of unity, described in Section* 3 *if* $\varkappa = k + h^\vee$.

Finkelberg's proof is based on the series of papers of Kazhdan and Lusztig ([KL1–4]). Since in these papers only the case of simply-laced $\mathfrak{g}$ is considered, his proof formally works only in these cases. However, it turns out that the same arguments work in non simply-laced case as well if we let $q = e^{\pi \mathrm{i}/m\varkappa}$ (note m in denominator!), as we did in Section 3; see [L, Section 8 and Erratum].

The restriction $k \geq n(\mathfrak{g})$ mentioned in the theorem appears only for exceptional root systems (see table in [F]), and for all $\mathfrak{g}$, $n(\mathfrak{g}) \leq 6$; thus, this theorem is automatically satisfied if $k \geq 6$.

In particular, this theorem implies that the quantum dimension of objects in both categories coincide:

$$\dim_\varkappa V = \dim_\varepsilon V[0], \quad \varepsilon = e^{\pi \mathrm{i}/m\varkappa}.$$

Another construction of equivalence of these categories can be obtained from results of Schechtman and Varchenko, who showed that one can identify Verma modules over quantum group with homologies of certain local systems on the configuration space. We will use this approach in the simplest case (for generic $\varkappa$) for $\mathfrak{g} = \mathfrak{sl}_2$ in Section 12. However, doing this for irreducible modules and for integer values of κ requires use of intersection homology (or, equivalently, perverse sheaves), which is done in a recent series of papers of Schechtman and Finkelberg [FS].

10. Hermitian structure on $\mathcal{O}_\varkappa$ and inner product on the space of conformal blocks.

In this section we define a hermitian structure (which is a certain analogue of complex conjugation) in the category $\mathcal{O}_\varkappa$ and later use it to define an inner product on the spaces of morphisms in this category.

Hermitian structure.

Recall (see Section 1) that a hermitian structure on a tensor category is a system of maps $\overline{} : \mathrm{Hom}(V, W) \to \mathrm{Hom}(V^*, W^*)$ which satisfies certain compatibility conditions (1.22). When the category is defined over the field $\mathbb{C}$ we will assume these maps are $\mathbb{C}$-antilinear. Usually, to define such structure we first define a functor $\overline{}$ which is antiequivalence of categories (i.e., we have canonical isomorphisms $\overline{V \otimes W} \simeq \overline{W} \otimes \overline{V}$) and then show that we have isomorphisms $\overline{V} \simeq V^*$. We can formalize this setup in the following Lemma, proof of which is trivial (as any abstract nonsense):

Lemma 10.1. *Let $\mathcal{C}$ be a semisimple ribbon category. Let us assume that we have the following data:*

(1) *A functor $\omega : \mathcal{C} \to \mathcal{C}$ satisfying the following conditions:*
- *It is antilinear: for every $\Phi \in \mathrm{Hom}_\mathcal{C}, \alpha \in \mathbb{C}$, we have $(\alpha\Phi)^\omega = \bar{\alpha}\Phi^\omega$;*
- *We have functorial isomorphisms $V^{\omega\omega} \simeq V^{**}, (V \otimes W)^\omega \simeq W^\omega \otimes V^\omega, (V^\omega)^* \simeq (V^*)^\omega$ and $\mathbf{1}^\omega \simeq \mathbf{1}$ compatible with each other in the natural way.*

- *If α, R are the associativity and commutativity isomorphisms in $\mathcal{C}$ then $\alpha^\omega = \alpha^{-1}, R^\omega = R^{-1}$, i.e. we have*

$$\begin{array}{ccc}
(V\otimes W)^\omega & \xrightarrow{(R_{V,W})^\omega} & (W\otimes V)^\omega \\
\Big\| & & \Big\| \\
W^\omega \otimes V^\omega & \xrightarrow{R^{-1}_{V^\omega,W^\omega}} & V^\omega\otimes W^\omega,
\end{array}$$

and similarly for α.

- *If $\theta : V \to V$ is the universal twist in $\mathcal{C}$ then $\theta^\omega = \theta^{-1}$; similarly, if $i_V : \mathbf{1} \to V \otimes V^*$ and $e_V : V^* \otimes V \to \mathbf{1}$ are the duality maps then $i_V^\omega = (1\otimes\delta^{-1})i_{V^{\omega *}}, e_V^\omega = e_{V^{\omega *}}(\delta^{-1}\otimes 1)$.*

(2) *Isomorphisms $X_i^\omega \simeq X_i^*$ for all simple objects X_i (which, again, are compatible with $V^{\omega\omega} \simeq V^{**}$).*

Then the category $\mathcal{C}$ can be uniquely endowed with a structure of hermitian ribbon category by identifying $V^\omega \simeq V^*$ so that it is compatible with all the structures of ribbon category.

Note that part (2) can be replaced by the following condition:

(2′) For every i, we are given non-zero homomorphisms
$\phi_i : \mathbf{1} \to X_i \otimes X_i^\omega$ such that $\phi^\omega = \phi$ (up to identification $V^{\omega\omega} \simeq V^{**} \simeq V$).

We have implicitly used this construction when defining the hermitian structure on representations of quantum group in Section 4. Now we do the same for the category $\mathcal{O}_\varkappa$. Let $\omega : \mathfrak{g} \to \mathfrak{g}$ be the $q = 1$ limit of the antilinear involution defined in Section 4; on the generators it is given by

$$\begin{aligned}
\omega : & e_i \mapsto e_{i^\vee}, \quad i = 1, \dots, r \\
& f_i \mapsto f_{i^\vee}, \quad i = 1, \dots, r \\
& h \mapsto -w_0(h), \quad h \in \mathfrak{h}.
\end{aligned} \tag{10.1}$$

We extend it to an involution on $\hat{\mathfrak{g}}$ by the rule

$$\omega(x[n]) = (-1)^n(\omega(x))[n], \quad \omega(c) = c. \tag{10.2}$$

Obviously, ω is an automorphism of Lie algebras. Now, for every $V \in \mathcal{O}_\varkappa$ define V^ω to be the same set as V but with the action of $\hat{\mathfrak{g}}$ twisted by ω. Similarly, for every $\hat{\mathfrak{g}}$-homomorphism $\Phi : V \to W$ between modules in $\mathcal{O}_\varkappa$ let Φ^ω be the same homomorphism but considered as the map $V^\omega \to W^\omega$. It is easy to see that this operation preserves composition of morphisms. One easily checks that we have canonical identification $L_{0,k}^\omega \simeq L_{0,k}, (DV)^\omega \simeq D(V^\omega)$ (since ω commutes with the automorphism # used in the definition of the dual – see beginning of Section 8). Also, let us define isomorphism $V^{\omega\omega} \simeq V$ to be given by $V \mapsto Zv$, where Z is the central element in (completion of) $U\mathfrak{g}$ constructed in Theorem 7.3.

The most difficult part is to prove that ω is tensor functor, i.e. to construct isomorphisms $(V \dot\otimes W)^\omega = W^\omega \dot\otimes V^\omega$. To do it, let us return to definition of conformal blocks on Riemann surfaces.

Let $\mathcal{X}$ be a Riemann sphere with n marked points $z_1, \dots, z_n$ (see (8.8)) and representations $V_1, \dots, V_n, V_\infty \in \mathcal{O}_\varkappa$ assigned to these points, and let $\Phi : V_\infty \to V_1 \hat\otimes \dots \hat\otimes V_n$ be a conformal block, i.e. a linear operator satisfying commutation relations (8.5).

Theorem 10.2. *Let $\mathcal{X}, \Phi$ be as above and denote by Φ^ω the same Φ considered as a map $V_\infty^\omega \to V_1^\omega \hat\otimes \dots \hat\otimes V_n^\omega$, where V^ω is defined above. Then Φ^ω is a conformal block on the Riemann surface $\mathcal{X}^\omega = \mathbb{C}P^1$ with marked points $z_1' = -\bar z_1, \dots, z_n' = -\bar z_n, \infty$, local parameters $w_i = w - z_i', w_\infty = 1/w$ at $z_1', \dots, z_n', \infty$ and representations $V_1^\omega, \dots, V_n^\omega, V_\infty^\omega$ assigned to these points.*

Proof. Follows from the fact that the map $f(w) \mapsto \omega(f(-\overline{w}))$ is isomorphism of the algebras of the rational functions $\Gamma(\mathcal{X}) \simeq \Gamma(\mathcal{X}^\omega)$ which were defined by (8.3). $\square$

Recalling that we have identified the space of morphisms $V_\infty \to V_1 \dot\otimes V_2$ in $\mathcal{O}_\varkappa$ with the space of conformal blocks on the sphere with marked points $0, 1, \infty$ (see Theorem 9.1 and remark after it), we immediately deduce from Theorem 10.2 the following result:

Theorem 10.3. *The map $\Phi \mapsto \Phi^\omega$ defined in Theorem 10.2 gives rise to canonical antilinear isomorphism*

$$\mathrm{Hom}_{\mathcal{O}_\varkappa}(V_\infty, V_1 \dot\otimes V_2) \simeq \mathrm{Hom}_{\mathcal{O}_\varkappa}(V_\infty^\omega, V_2^\omega \dot\otimes V_1^\omega)$$

and thus gives rise to an isomorphism $(V_1 \dot\otimes V_2)^\omega \simeq V_2^\omega \otimes V_1^\omega$.

Example 10.4. Let $\Phi = \Phi^g \in \mathrm{Hom}_{\mathcal{O}_\varkappa}(V_\infty, V_1 \dot\otimes V_2)$ (see Example 8.7). Then $(\Phi^g)^\omega = \Phi^{(g^\omega)}$, where

$$g^\omega : (V_\infty[0])^\omega \to V_2[0]^\omega \otimes V_1[0]^\omega$$

is defined using the involution ω on $\mathfrak{g}$.

Finally, let us fix for every λ homomorphisms of $\mathfrak{g}$-modules $i_\lambda : \mathbb{C} \to L_\lambda \otimes L_\lambda^\omega$ such that they are symmetric: $i_\lambda(1)^\omega = i_\lambda(1)$. This defines i_λ uniqueley up to a real constant (in fact, it follows from arguments of Section 7 that there is a way to define them uniquely up to a positive constant). Let $\mathcal{O}_\varkappa$-morphisms $\phi_\lambda : \mathbf{1} \to L_{\lambda,k} \dot\otimes L_{\lambda,k}^\omega$ by $\phi_\lambda = \Phi^{i_\lambda}$.

Proposition 10.5. *The functor ω and the system of maps $\phi_\lambda : \mathbf{1} \to L_{\lambda,k} \dot\otimes L_{\lambda,k}^\omega$ defined above satisfy the axioms of Lemma 10.1 and thus endow $\mathcal{O}_\varkappa$ with a structure of hermitian category.*

Lemma 10.6. *Equivalence of categories $\mathcal{C}(\mathfrak{g}, \kappa) \simeq \mathcal{O}_\varkappa$ constructed in [F] (see Theorem 9.8) preserves the hermitian structure.*

Proof. Proof is based on the analysis of formulas defining the equivalence of categories in [KL3].

Remark. As in quantum group case, the involution defined above can be related with the usual compact involution on $\hat{\mathfrak{g}}$, which is much more usual in physical literature. We discuss this in Appendix.

Existence theorem.

Now that we have defined the structure of hermitian modular tensor category on $\mathcal{O}_\varkappa$, the general theory developed in Section 2 immediately gives the existence of modular invariant inner product on the spaces of intertwining operators. We formulate this result as a theorem. As before, we denote

$$H = \bigoplus_{\lambda \in C} DL_{\lambda,k} \dot{\otimes} L_{\lambda,k}. \tag{10.2}$$

Recall the notation (9.8):

$$W_\lambda^{\mu\nu} = \mathrm{Hom}_{\mathcal{O}_\varkappa}(L_{\lambda,k}, L_{\mu,k} \dot{\otimes} L_{\nu,k}).$$

Theorem 10.7. *There exists a hermitian inner product on each of the spaces $W_\lambda^{\mu\nu}$ such that the resulting inner product on the space*

$$\mathrm{Hom}_{\mathcal{O}_\varkappa}(H, L_{\mu,k}) = \bigoplus_{\lambda \in C} W^{\mu}_{\lambda^*\lambda} = \bigoplus_{\lambda \in C} W_\lambda^{\lambda\mu} \tag{10.3}$$

is modular invariant.

Proof. Follows from the fact that $\mathcal{O}_\varkappa$ has a structure of modular tensor category and constructions of Section 2. □

Note that definition of this inner product defines on the choice of the maps $\phi_\lambda : \mathbf{1} \to L_{\lambda,k} \dot{\otimes} L^\omega_{\lambda,k}$ whihc were used in the definition of hermitian structure. However, the inner product on the space $\mathrm{Hom}_{\mathcal{O}_\varkappa}(H, L_{\mu,k})$ in fact depends only on the choice of identification ϕ_μ and does not depend on the choice of ϕ_λ for each λ. Thus, the inner product on this space is defined uniquely up to a constant factor.

Recall (see Theorem 9.1) that the spaces $W_\lambda^{\mu\nu} = \mathrm{Hom}_{\mathcal{O}_\varkappa}(L_{\lambda,k}, L_{\mu,k} \dot{\otimes} L_{\nu,k})$ are identified with spaces of conformal blocks for 3-punctured sphere; thus, the construction above defines an inner product on the spaces of conformal blocks on 3-punctured sphere.

To define inner product on other Riemann surfaces with marked points, we use the gluing, which allows us to represent the space of conformal blocks assigned to a Riemann surface as a sum of tensor products of the 3-point conformal blocks. Once we choose such a representation, we define the norm by the rule $\|\Phi_1 \otimes \Phi_2\| = \|\Phi_1\| \cdot \|\Phi_2\|$.

However, it is not clear why this is well-defined. Even for a 3-point conformal blocks, we so far have defined the inner product not for all 3-punctured spheres but only for some contractible subspace $\mathcal{D}$ in the moduli space. For other Riemann surfaces, it is even more complicated, since there are different ways to cut a surface in trinions. For example, there are different ways to represent a torus with one puncture as a result of gluing together two holes of a three-punctured sphere. Each of these ways gives an identification of the space of conformal blocks with the space $\bigoplus_{\lambda \in C} W_\lambda^{\lambda\mu}$ (see Lemma 9.3), and different identifications give rise to an action of the modular group on this space – cf. Theorem 9.4.

Now we come to the main result of our paper:

Theorem 10.8. *There exists a positive definite inner product on the spaces of conformal blocks on arbitrary Riemann surface such that with respect to this inner product, the natural flat connection on the space of conformal blocks is unitary.*

Proof. It follows from general result of Moore and Seiberg (see [MS1, Section 5]) that it suffices to define such an inner product on the sphere with n punctures and on the torus with one puncture. Moreover, to define the latter inner products it suffices to check that the commutativity and associativity morphisms, as well as the action of the modular group are unitary with respect to the inner product defined on $W_\lambda^{\mu\nu}$. But we have proved that in any modular tensor category, the inner product defined in Section 2 satisfies these conditions (see Theorems 2.4, 2.5). $\square$

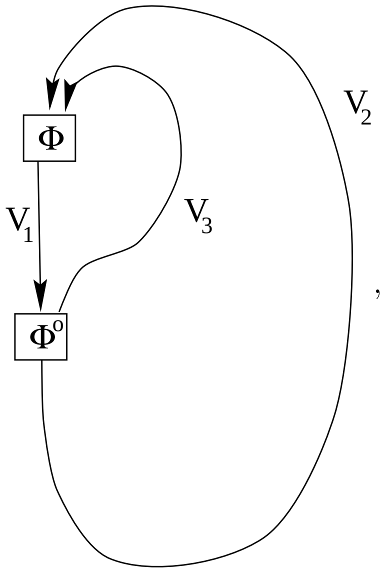
This theorem is very important for Conformal Field Theory, since existence of such an inner product is one of the axioms of CFT; thus, Theorem 10.8 claims that this axiom is satisfied in the WZW model. To the best of my knowledge, this has not been proved in general case so far. We refer the reader to the paper [FG] for review of known results and explicit constructions related to the inner products in higher genera.

11. Explicit formulas

Our next goal is to give as explicit formulas as possible for this inner product. Let us consider a for simplicity $\mathcal{O}_\varkappa$-homomorphisms of the form $V_1 \to V_2 \dot{\otimes} V_3$; let $\Phi(z)$ be such a morphism in $\mathcal{O}_\varkappa$. First note that we can rewrite the definition of $\|\Phi\|^2$ given in Section 2 in the following form:

Lemma 11.1. *Let* $\Phi \in \mathrm{Hom}_{\mathcal{O}_\varkappa}(V_1, V_2 \dot{\otimes} V_3)$. *Then*

(11.1) $$\|\Phi\|^2 = \frac{1}{(\dim_\varkappa V_1 \dim_\varkappa V_2 \dim_\varkappa V_3)^{1/2}}$$



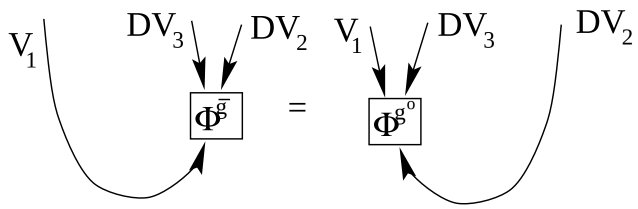
,

where $\Phi^o(z) \in W_{V_2}^{V_1, DV_3}$ *is defined as follows: if* $\Phi = \Phi^g$ *for some* $g \in \mathrm{Hom}_{\mathfrak{g}}(V_1[0], V_2[0] \otimes V_3[0])$ *then* $\Phi^o = \Phi^{g^o}$, *where* $g^o : V_2[0] \to V_1[0] \otimes V_3[0]^*$ *is the image of* $\bar{g}$ *defined in Example* 10.4 *under the isomorphism*

$$\mathrm{Hom}_{\mathfrak{g}}(V_1[0]^*, V_3[0]^* \otimes V_2[0]^*) \simeq \mathrm{Hom}_{\mathfrak{g}}(V_2[0], V_1[0] \otimes V_3[0]^*)$$

Proof. Recalling the definition of inner product (2.2) and Example 10.4, we see that it suffices to prove the following lemma:

Lemma 11.2.



This Lemma follows from the fact that on the space $(V_1[0] \otimes V_2[0] \otimes V_3[0])^{\mathfrak{g}}$ the operators Ω_{12} and Ω_{23} commute and thus, according to Corollary 9.7, the associativity isomorphism is trivial. □

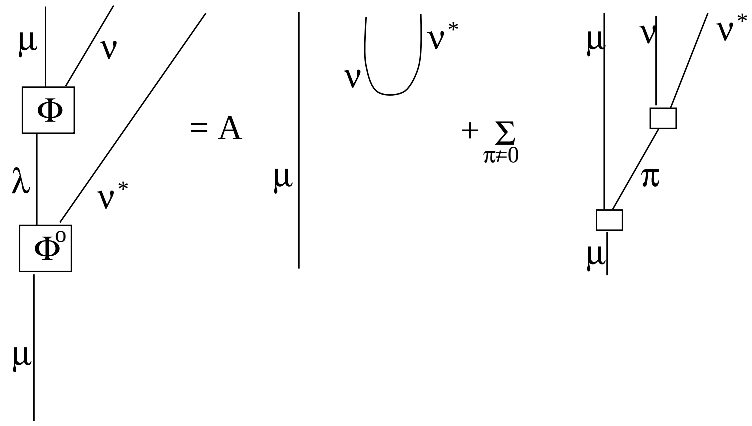
Theorem 11.3. *In the notations of Lemma* 11.1*, let us additionally assume that* V_1, V_2, V_3 *are irreducible:* $V_1 = L_{\lambda,k}, V_2 = L_{\mu,k}, V_3 = L_{\nu,k}$. *Let* $\Psi(z_1, z_2) = (\Phi(z_1) \otimes 1)\Phi^o(z_2)$ *be the flat section of the bundle of conformal blocks* $W(\mathcal{X}, V_i)$*, where* $\mathcal{X}$ *is* $\mathbb{C}P^1$ *with the marked points* $0, z_1, z_2, \infty$ *with the representations* V_2, V_3, DV_3, V_2 *assigned to these points respectively* (*cf. Proposition* 9.5). *Then*

$$\|\Phi\|^2 \operatorname{Id}_{L_\mu} = \left(\frac{\dim_\varepsilon L_\mu}{\dim_\varepsilon L_\lambda \dim_\varepsilon L_\nu}\right)^{1/2} \lim_{z_1, z_2 \to 1} (z_1 - z_2)^{2\Delta_\nu} (1 \otimes e_\nu) \langle \Psi(z_1, z_2) \rangle, \tag{11.2,}$$

where $e_\nu : L_\nu \otimes L_\nu^* \to \mathbb{C}$ *is the canonical map of* $\mathfrak{g}$*-modules.*

Proof.

It follows from existence of associativity isomorphism that Ψ can be rewritten as the sum



(11.3)

for some $A \in \mathbb{C}$ and some intertwiners in the boxes that are left blank. Substituting this in the expression for $\|\Phi\|^2$ given in Lemma 11.1, we see that

$$\|\Phi\|^2 = \left(\frac{\dim_\varepsilon L_\mu \ \dim_\varepsilon L_\nu}{\dim_\varepsilon L_\lambda}\right)^{1/2} A. \tag{11.4}$$

To calculate A, let us look at the asymptotics of $\langle\Psi\rangle$ as $z_1 \to z_2$. Then the first term in (11.3) gives the asymptotics $A(z_1-z_2)^{-2\Delta_\nu}\operatorname{Id}_{L_\mu}\otimes i_\nu$, where $i_\nu : \mathbb{C} \to L_\nu \otimes L_\nu^*$ is the canonical embedding and all other terms have highest term of the asymptotics $(z_1-z_2)^{-2\Delta_\nu+\Delta_\pi}$, which follows from Proposition 9.6. Since $\Delta_\pi > 0$ for $\pi \neq 0$, these terms give zero contribution to the limit, and therefore

$$\lim_{z_1,z_2\to 1}(z_1-z_2)^{2\Delta_\nu}\langle\Psi(z_1,z_2)\rangle = A\operatorname{Id}_{L_\mu}\otimes i_\nu.$$

Thus,

$$A = (\dim_\varepsilon L_\nu)^{-1}\lim_{z_1,z_2\to 1}(z_1-z_2)^{2\Delta_\mu}(1\otimes e_\nu)\langle\Psi(z,z')\rangle$$

Substituting this in (11.4), we get the statement of the theorem. $\square$

Thus, calculation of the norm reduces to computation of the limit (11.2). Since the asymptotics of $\langle\Psi\rangle$ in the limit $|z_2| \gg |z_1|$ is known, this is equivalent to calculation of Drinfeld's associator. Note that this shows that the structure of $L_{\lambda,k}$ as a module over $\hat{\mathfrak{g}}$ is not important here; in particular, this inner product can be as well calculated for $\varkappa \notin \mathbb{Q}$, and the answer depends analytically on $\varkappa$. Since there are integral formulas for the solutions of KZ equations, the answer can be always written in terms of asymptotics of some integrals. In the case of $\mathfrak{g} = \mathfrak{sl}_2$ these asymptotics reduce to Selberg integrals and can be calculated explicitly in terms of Γ-functions (see [V]); in general the answer can not be written down in terms of gamma-functions, but is written in terms of generalized hypergeometric functions in the sense of Varchenko (this is tautology).

Remark. In the case where all the spaces of conformal blocks are zero or one dimensional (as happens for $\mathfrak{sl}_2$), formula (11.2) for the inner product was suggested by Moore and Seiberg (see [MS2, Exercise 6.6]).

12. Example: $\mathfrak{g} = \mathfrak{sl}_2$.

In this section we calculate explicitly norms of vertex operators for $\mathfrak{sl}_2$. As was noted in the end of the last section, it suffices to calculate these norms for $\varkappa \notin \mathbb{Q}$. This can be done using integral formulas for the solutions, given by Schechtman and Varchenko [SV]. Moreover, in this case the asymptotics of the integrals can be calculated explicitly, using Selberg integral, and the answer is written in terms of Γ-functions (see [V]). This can be used to find the matrices which relate asymptotics of solutions in different asymptotic zones. Fortunately, this work has already been done in [V], so we just quote the results we need.

In this section we only consider $\mathfrak{g} = \mathfrak{sl}_2$.

$U_q\mathfrak{sl}_2$: notations.

As usual, we define the quantum group $U_q\mathfrak{sl}_2$ to be the algebra over $\mathbb{C}(q^{1/2})$ with generators $e, f, q^{\pm h/2}$ and the usual relations

$$
\begin{aligned}
q^h e &= q^2 e q^h,\\
q^h f &= q^{-2} f q^h,\\
[e,f] &= \frac{q^h - q^{-h}}{q-q^{-1}}.
\end{aligned}
\tag{12.1}
$$

This is a Hopf algebra with the following comultiplication and antipode:

$$
\begin{gathered}
\Delta e = e\otimes q^{h/2} + q^{-h/2}\otimes e,\\
\Delta f = f\otimes q^{h/2} + q^{-h/2}\otimes f,\\
\Delta q^h = q^h\otimes q^h,\\
Se = -qe, \qquad Sf = -q^{-1} f, \qquad Sq^h = q^{-h}.
\end{gathered}
\tag{12.2}
$$

Also, in this algebra one can define a universal R-matrix, which has the form $R = q^{\frac{1}{2}h\otimes h}(1+\dots)$.

To establish equivalence with definitions used by other authors, it is useful to introduce another generators of the same algebra:

$$
E = eq^{h/2}, \qquad F = q^{-h/2} f.
\tag{12.3}
$$

It is easy to see that E, F satisfy the same commutation relations (12.1) as e, f, and the comultiplication takes the form

$$
\begin{aligned}
\Delta E &= E\otimes q^h + 1\otimes E,\\
\Delta F &= F\otimes 1 + q^{-h}\otimes F.
\end{aligned}
$$

Note also the following useful identity:

$$
F^p = f^p q^{-\frac{ph}{2}+\frac{p(p+1)}{2}}.
$$

As usual, we denote

$$
\begin{gathered}
[n] = \frac{q^n - q^{-n}}{q-q^{-1}}, \qquad [n]! = [1]\dots[n]\\
\begin{bmatrix} n\\ i\end{bmatrix} = \frac{[n]!}{[i]![n-i]!},\\
f^{(p)} = \frac{f^p}{[p]!}.
\end{gathered}
$$

For every $j\in\frac{1}{2}\mathbb{Z}_+$ we denote by V_j the irreducible representation of $U_q\mathfrak{sl}_2$ spanned by the highest-weight vector v_j such that $q^h v_j = q^{2j} v_j$. It is a finite-dimensional representation of dimension $2j+1$. We fix a basis v^j_m in V_j, where $-j\le m\le j, m-j\in\mathbb{Z}$. Then the action of $U_q\mathfrak{sl}_2$ in this basis is given by

$$(12.4)\qquad \begin{aligned} q^h v^j_m &= q^{2m} v^j_m, \\ e v^j_m &= \sqrt{[j-m][j+m+1]}\, v^j_{m+1}, \\ f v^j_m &= \sqrt{[j+m][j-m+1]}\, v^j_{m-1}. \end{aligned}$$

We will sometimes denote the highest-weight vector $v^j_j \in V_j$ by v_j, omitting the superscript. Note that $f^{(p)} v^j_j = \begin{bmatrix} 2j \\ p \end{bmatrix}^{1/2} v^j_{j-p}$.

The quantum analogue Ω of the longest element of the Weyl group discussed in Section 7 acts in V_j by

$$\Omega(v_m) = (-1)^{j-m} q^{-j(j+1)-m} v_{-m};$$

thus, $\Omega^2|_{V_j} = (-1)^{2j} q^{-2j(j+1)} = (-1)^{2j}\theta_j^{-1}$ and Z acts in V_j by $(-1)^{2j}$.

For every j, define the morphisms $i_{V_j} : \mathbb{C} \to V_j \otimes V_j$ and $e_{V_j} : V_j \otimes V_j \to C$ by

$$(12.5)\qquad \begin{aligned} i_{V_j} &: 1 \mapsto \sum_{k=-j}^{j} (-1)^{j-k} q^k v^j_k \otimes v^j_{-k}, \\ e_{V_j} &: v^j_k \otimes v^j_l \mapsto \delta_{k+l,0} (-1)^{k-j} q^k. \end{aligned}$$

One can use these formulas to identify $V_j \simeq V_j^*$ so that i_{V_j} and e_{V_j} are identified with canonical morphisms $\mathbb{C} \to V_j \otimes V_j^*$ and $V_j^* \otimes V_j \to \mathbb{C}$ respectively. In particular, $e_{V_j} i_{V_j} = \dim_q V_j = [2j+1]$.

For every triple $j, j_1, j_2 \in \frac{1}{2}\mathbb{Z}_+$ such that $k = j_1 + j_2 - j \in \mathbb{Z}_+, k \le 2j_1, k \le 2j_2$ there exists a unique (up to a constant) $U_q\mathfrak{sl}_2$-homomorphism $g^{j_1 j_2}_j : V_j \to V_{j_1} \otimes V_{j_2}$. We fix the constant by requiring that

(12.6)

$$g^{j_1 j_2}_j v_j = \sum_{p=0}^{k} (-1)^p q^{-p(j+1)+k(k-2j_2+1)/2} \frac{[2j_1-p]![2j_2-k+p]!}{[2j_1]![2j_2-k]!} f^{(p)} v_{j_1} \otimes f^{(k-p)} v_{j_2}.$$

This normalization is chosen so that this coincides with the definition (1.2.1) of [V].

Examples.

(1) Let $j = j_1, j_2 = 0$. Then $g^{j0}_j : V_j \to V_j \otimes \mathbb{C} : v_j \mapsto v_j \otimes 1$.
(2) Let $j = 0, j_1 = j_2 = a$. Then $g^{aa}_0 = i_{V_a} : \mathbb{C} \to V_a \otimes V_a$.

All the constructions above can be also applied to representations of classical $\mathfrak{sl}_2$ by letting $q = 1$ in the formulas. In particular, we will use the notation $g^{j_1 j_2}_j (q = 1)$ for the classical counterpart of the intertwiner $g^{j_1 j_2}_j$.

Category $\mathcal{O}_\varkappa$ for $\mathfrak{sl}_2$ at generic level and equivalence of categories.

Let us assume that we are given $K \in \mathbb{C} \setminus \mathbb{Q}$, and we let $\varkappa = K + 2$. Let $\mathcal{O}_\varkappa$ be the category of $\widehat{\mathfrak{sl}_2}$-modules of level K such that:

1. They have weight decomposition
2. the action of $\mathfrak{sl}_2 \otimes t\mathbb{C}[t]$ is locally nilpotent
3. they are $\mathfrak{sl}_2$-integrable
4. they are of finite length.

It is known that this category is semisimple, and simple objects in this category are precisely irreducible modules $L_{\lambda,K}$, which for $\varkappa \notin \mathbb{Q}$ coincide with Weyl (induced) modules: $L_{\lambda,k} = V_{\lambda,K}, \lambda \in P^+ \simeq \frac{1}{2}\mathbb{Z}_+$. The same constructions as before define on this category a structure of a ribbon category, which is hermitian for $K \in \mathbb{R}$. Note that since we assumed $\varkappa \notin \mathbb{Q}$, we have an isomorphism $\mathrm{Hom}_{\mathcal{O}_\varkappa}(V_1, V_2 \dot{\otimes} V_3) \simeq \mathrm{Hom}_{\mathfrak{sl}_2}(V_1[0], V_2[0] \otimes V_3[0])$ (compare with (9.4)). Thus, this category could be described without any reference to affine Lie algebra at all, just in terms of finite-dimensional representations of $\mathfrak{sl}_2$ and Knizhnik-Zamolodchikov equation, as was done in [Dr3, Dr4]. We will use the notation

$$\Phi_j^{j_1 j_2} = \Phi^{g_j^{j_1 j_2}(q=1)}.$$

It follows from general theorem proved by Drinfeld for any finite-dimensional simple Lie algebra that this category is equivalent to the category *Rep* $U_q\mathfrak{sl}_2$ of finite-dimensional representations of $U_q\mathfrak{sl}_2$ with $q = q(\varkappa) = e^{\pi \mathrm{i}/\varkappa}$. Moreover, it turns out that in the case $\mathfrak{g} = \mathfrak{sl}_2$ the equivalence can be constructed explicitly, which is based on the fact that asymptotics of certain integrals can be calculated explicitly. This is the main result of the paper [V], which we briefly quote below, referring the reader to the original paper for the proofs. Another equivalence is constructed the paper [KL3] for any $\mathfrak{g}$, though writing it explicitly even for $\mathfrak{sl}_2$ requires more work.

Let us introduce the following notation

$$\llbracket x \rrbracket = \Gamma\left(-\frac{x}{\varkappa} + 1\right). \tag{12.8}$$

It follows from the known properties of Γ-function that

$$\llbracket 0 \rrbracket = 1, \qquad \llbracket x \rrbracket \llbracket -x \rrbracket = \frac{x}{[x]_q} \frac{2\pi \mathrm{i}}{\varkappa(q - q^{-1})}, \tag{12.9}$$

where $q = e^{\pi \mathrm{i}/\varkappa}$.

For a triple j, j_1, j_2 as before, let $k = j_1 + j_2 - j$ and define the normalizing constants $c_j^{j_1 j_2}$ by

$$c_j^{j_1 j_2} = q(\varkappa)^{-2j_2 k + 3k(k-1)/2} \prod_{i=0}^{k-1} \frac{\llbracket 2j_1 - i \rrbracket \llbracket -i - 1 \rrbracket}{\llbracket -2j_2 + i \rrbracket \llbracket 2j + i + 2 \rrbracket}. \tag{12.10}$$

Then the following theorem is an easy reformulation of Theorems 2.4.1, 2.4.2 from [V]:

Theorem 12.1. *Let the constants $c_j^{j_1j_2}$ be defined by* (12.10)*. Then the map*

$$\begin{aligned} \mathrm{Hom}_{U_q\mathfrak{sl}_2}(V_j, V_{j_1}\otimes V_{j_2}) &\to \mathrm{Hom}_{\mathcal{O}_\varkappa}(V_{j,K}, V_{j_1,K}\dot{\otimes} V_{j_2,K}) \\ g_j^{j_1j_2} &\mapsto c_j^{j_1j_2}\Phi_j^{j_1j_2} \end{aligned} \tag{12.11}$$

commutes with associativity and commutativity isomorphisms in Rep $U_q\mathfrak{sl}_2$ and $\mathcal{O}_\varkappa$ respectively if $q=e^{\pi \mathrm{i}/\varkappa}$.

Corollary 12.2. *Theorem 12.1 remains true if we replace $c_j^{j_1j_2}$ by*

$$\tilde{c}_j^{j_1j_2} = c_j^{j_1j_2}a_j a_{j_1}^{-1}a_{j_2}^{-1}, \quad a_j = q^{j^2-\frac{3}{2}j}.$$

From now on, we will use the equivalence in the form defined in this corollary. It is easy to see that there is a unique way to extend the map (12.11) to arbitrary homomorphisms $\mathrm{Hom}_{U_q\mathfrak{sl}_2}(V,W)$ so that this remains consistent with the associativity and commutativity isomorphisms. Thus defined map defines an equivalence of *Rep* $U_q\mathfrak{sl}_2$ and $\mathcal{O}_\varkappa$ as braided categories and thus, as ribbon (=monoidal rigid balanced braided) categories. Note that under this equivalence

$$\begin{aligned} i_{V_j} = g_0^{jj} &\mapsto \tilde{c}_0^{jj}\Phi^{i_{V_j}} = \frac{[\![1]\!]}{[\![2j+1]\!]}\Phi^{i_{V_j}}, \\ e_{V_j} &\mapsto \frac{[2j+1]}{2j+1}(\tilde{c}_0^{jj})^{-1}\Phi^{e_{V_j}} = \frac{[2j+1][\![2j+1]\!]}{(2j+1)[\![1]\!]}\Phi^{e_{V_j}} = \frac{[\![-1]\!]}{[\![-2j-1]\!]}\Phi^{i_{V_j}} \end{aligned} \tag{12.12}$$

which coincides with the definitions of the maps $V\dot{\otimes} DV \to L_{0,K}, L_{0,K}\to V\dot{\otimes} DV$ (see Theorem 9.1) if we choose the normalization constants c_λ used in Theorem 9.1 to be $c_j = q^{j^2-\frac{3}{2}j}\frac{[\![-1]\!]}{[\![-2j-1]\!]}$.

Explicit formulas for the inner product on intertwiners.

Let us assume that $\varkappa\in\mathbb{R}\setminus\mathbb{Q}$, so that $[\![x]\!]\in\mathbb{R}$ for $x\in\mathbb{R}$ and $|q|=1$. In this case we can define a hermitian structure on both $\mathcal{O}_\varkappa$ and *Rep* $U_q\mathfrak{sl}_2$ in the same way we have done for $\varkappa\in\mathbb{Z}_+$, and the equivalence of categories constructed above preserves this hermitian structure.

Since the definition of inner product (2.2) used only the structures of hermitian ribbon category, this shows that the map (12.11) preserves this inner product and thus,

$$\|\Phi_j^{j_1j_2}\|^2 = \left|c_j^{j_1j_2}\right|^{-2}\|g_j^{j_1j_2}\|^2, \tag{12.13}$$

Substituting in this formula the definition of $c_j^{j_1j_2}$ (formula (12.10)), we get the following answer (as before, we denote $k=j_1+j_2-j$):

$$\|\Phi^{g_j^{j_1j_2}(q=1)}\|^2 = \|g_j^{j_1j_2}\|^2\prod_{i=0}^{k-1}\frac{[\![-2j_2+i]\!]^2[\![2j+i+2]\!]^2}{[\![2j_1-i]\!]^2[\![-i-1]\!]^2}. \tag{12.14}$$

In particular, if $j = j_1$ then $k = j_2$ and

$$\|\Phi_j^{jk}\|^2 = \|g_j^{jk}\|^2 \prod_{i=1}^{k} \frac{[\![2j+1+i]\!]^2}{[\![2j+1-i]\!]^2} \frac{[\![-k-i]\!]^2}{[\![-k-1+i]\!]^2}. \tag{12.15}$$

Since formula (5.7) in our case gives

$$\|g_j^{jk} = c_k \prod_{i=1}^{k} \frac{[2j+1+i]}{[2j+1-i]},$$

we finally get

$$\|\Phi_j^{jk}\|^2 = c_k \prod_{i=1}^{k} \frac{[2j+1+i]}{[2j+1-i]} \frac{[\![2j+1+i]\!]^2}{[\![2j+1-i]\!]^2} \tag{12.16}$$

where the constant c_k does not depend on j.

These expressions are closely related to the expressions given by Gawędzki et al in [G, FGK]. In these papers they suggest a construction of the inner product on the spaces of conformal blocks based on the fact that these spaces are the state spaces of quantum Chern-Simons theory. The inner product is obtained by regularization of certain infinite-dimensional integrals, and the final answer is given by a finite-dimensional integral of the following type

$$\int_{\mathbb{C}^k} |\omega|^2,$$

where ω is the differential form which appears in the integral formulas of Schechtman and Varchenko (up to $\varkappa \mapsto -\varkappa$).

In the case $\mathfrak{g} = \mathfrak{sl}_2$ there is hope that integrals of these type can be calculated explicitly (see [A] for the simplest example of computation of integrals of this type), and the answer can be expressed in terms of Γ-functions similarly to Selberg integral. We expect that this procedure would give the same answer as the one given by formula (12.14) above.

Appendix

Here we give another description of the hermitian structure on $\mathcal{O}_\varkappa$, based on compact involution of $\mathfrak{g}$. This construction is parallel to the one for $U_q\mathfrak{g}$ (see Section 7).

Let ω_c be the compact involution on $\mathfrak{g}$, i.e. an antilinear Lie algebra automorphism given on generators by $\omega_c(e_i) = -f_i, \omega_c(f_i) = -e_i, \omega_c(h) = -h$. Let us extend it to an involution on $\hat{\mathfrak{g}}$ by letting $\omega_c(x[n]) = \omega_c(x)[n], \omega_c(c) = c$ (**Warning: this is different from the standard compact involution on $\hat{\mathfrak{g}}$!**). We define $V^{\omega_c}, \Phi^{\omega_c}$ similar to above. Note that for any λ, $L_{\lambda,k}^{\omega_c} \simeq L_{\lambda^*,k}$, and thus, from complete reducibility, we have $V^{\omega_c} \simeq DV$ for any $V \in \mathcal{O}_\varkappa$, though this isomorphisms are not canonical.

For any $\mathcal{X}$ as in the beginning of Section 8, let $\overline{\mathcal{X}}$ be the following collection of data:

(1) The curve $\overline{X}$ which is the same curve as X but with opposite complex structure (thus, if f is a meromorphic function on X then $\bar{f}$ is a meromorphic function on $\overline{X}$).
(2) The marked points z_i are the same as for $\mathcal{X}$.
(3) The new local parameters are $\overline{w_i}$ (here we consider w_i as holomorphic function on X vanisning at z_i).

Example. Let $\mathcal{X}$ be the Riemann sphere (8.8). Then $\overline{\mathcal{X}}$ is isomorphic to $\mathbb{C}P^1$ with global parameter w, marked points $\overline{z_i}, \infty$ and local parameters $w - \bar{z}_i, 1/w$ respectively.

Assume that we are given $\mathcal{X}$ as before and we have representations V_i assigned to the marked points. Let $\Phi \in W(\mathcal{X}; V_1, V_2, \ldots, V_n)$ be a conformal block, i.e. a map

$$\otimes_{In} V_i \to \hat{\otimes}_{Out} V_i$$

which commutes with the action of the algebra Γ of meromorphic $\mathfrak{g}$-valued functions on $\mathcal{X}$ (see Definition 8.1).

Let us denote by Φ^{ω_c} the same map considered as a map

$$\otimes_{In} V_i^{\omega_c} \to \otimes_{Out} V_i^{\omega_c}.$$

Lemma A.1. *The mapping $\Phi \mapsto \Phi^{\omega_c}$ is an involutive isomorphism $W(\mathcal{X}; V_1, \ldots, V_n) \to W(\overline{\mathcal{X}}; V_1^{\omega_c}, \ldots, V_n^{\omega_c})$.*

Proof. This is obvious from the fact that $f(z) \mapsto \omega_c(f(z))$ identifies the algebras Γ of meromorphic functions on $\mathcal{X}$ and $\overline{\mathcal{X}}$. $\square$

In particular, if $\mathcal{X}$ is the Riemann sphere (8.8) then Φ^{ω_c} will be a conformal block on the sphere with marked points $\bar{z}_1, \ldots, \bar{z}_n$. This shows that we have a natural isomorphism $(V \dot{\otimes} W)^{\omega_c} \simeq V^{\omega_c} \dot{\otimes} W^{\omega_c}$.

Now we can define the hermitian structure on $\mathcal{O}_\varkappa$ similarly to the definitions before (see Lemma 10.1); the only non-trivial part is that we define isomorphisms $(V \dot{\otimes} W)^{\omega_c} \simeq W^{\omega_c} \dot{\otimes} V^{\omega_c}$ to be given by composition of natural isomorphism $(V \dot{\otimes} W)^{\omega_c} \simeq V^{\omega_c} \dot{\otimes} W^{\omega_c}$ and commutativity morphism $Pe^{\pi \mathrm{i}\Omega/\varkappa}$. Also, we define the maps $\phi_\lambda : \mathbf{1} \to L_{\lambda,k} \dot{\otimes} L_{\lambda,k}^{\omega_c}$ by $\phi_\lambda(1) = e^{\pi \mathrm{i} \Delta_\lambda} v_{\lambda,k} \otimes v_{\lambda,k}^{\omega_c} + \ldots$, where Δ_c is defined by (9.3).

Lemma A.2. *So defined functor ω_c satisfies all the conditions of Lemma* 10.1 *and thus gives rise to a hermitian structure on $\mathcal{O}_\varkappa$.*

In fact, it turns out that this hermitian strucutre is equaivalent to the one described in Section 10. As in the classical case, this relies on the existence of an analogue of the longest element of Weyl group.

For every module $V \in \mathcal{O}_\varkappa$ define the map

$$\Omega = w_0 e^{-\pi \mathrm{i} L_0} : V \to V, \tag{10.2}$$

where w_0 is the longest element of Weyl group for $\mathfrak{g}$, which acts on every finite-dimensional representation of $\mathfrak{g}$ (see, e.g., [KL3, 19.4]); thus, it also acts on evry module in $\mathcal{O}_\varkappa$. Its q-analogue (which we denoted also by Ω) was discussed in Section 7.

One can easily check that

$$\begin{gathered}\omega(x)=\Omega^{-1}\omega_c(x)\Omega=\omega_c(\Omega^{-1}x\Omega),\quad x\in\hat{\mathfrak{g}}\\ \Omega^2=Ze^{-2\pi\mathrm{i}L_0}\end{gathered}\tag{10.3}$$

where $Z=w_0^2$ was discussed in Section 7. Recall that it is a central element which acts by ±1 in any irreducible $\mathfrak{g}$-module and satisfies $Z|_{V\otimes W}=Z_V\otimes Z_W$; thus, it also acts by constant in any simple $\hat{\mathfrak{g}}$-module from category $\mathcal{O}_\varkappa$ and satisfies $Z_{V\dot{\otimes}W}=Z_V\dot{\otimes}Z_W, Z^2=1$.

Thus, for every representation $V\in\mathcal{O}_\varkappa$ the map $\Omega: V\to V$, considered as a map $V^\omega\to V^{\omega_c}$, is an isomorphism of $\hat{\mathfrak{g}}$-modules; similarly, we can identify $\overline{\Phi}=\Omega^{-1}\Phi^{\omega_c}\Omega$.

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